COMBINATORIAL HOPF ALGEBRAS - ECCO'12
EXERCISES LECTURE 3

(1). Prove that $Sym^* \cong Sym$ by showing that the correspondance
$$h^*_\lambda \mapsto m_\lambda$$
is an isomorphism.

(2). Prove that
$$h_\lambda = \sum_\mu K_{\lambda,\mu} s_\mu$$
where $K_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and content (or filling) μ. Compare this coefficient with the ones from question (2) from yesterday.

(3). Compute a particular example of a product and coproduct in the M-basis of $Qsym$.

(4). Define $SSym := \bigoplus_{n\geq 0} kS_n$. A basis at degree n is given by $\{F_\sigma\}_{\sigma\in S_n}$. This basis multiplies and comultiplies as follows. For $\sigma \in S_n$:
$$F_\sigma F_\mu = \sum_{\nu=\sigma \sqcup\!\sqcup \mu\uparrow^n} F_\nu \qquad \Delta(F_\sigma) = \sum_{\sigma=\tau\cdot\pi} F_{st(\tau)} \otimes F_{st(\pi)}$$
where $\tau \cdot \pi$ is the concatenation of the permutations τ and π and $\sqcup\!\sqcup$ is the shuffle product of words. For example,
$$ab \sqcup\!\sqcup xy = abxy + axby + xaby + axyb + xayb + xyab$$
and $\uparrow^n \{1,\dots,m\} = \{1+n,\dots,m+n\}$.

Can you realize this (Hopf) algebra as a subspace of $k\langle\langle x_1, x_2, \dots\rangle\rangle$?

(5). Compute the dimension (as a vector space) of
$$k[x_1, x_2, \dots, x_n]/\langle QSym^+\rangle$$
for $n = 1, 2, 3, \dots$