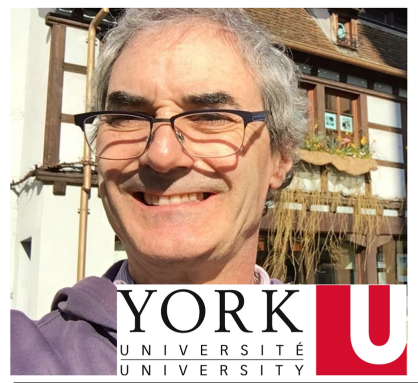

Antipode of linearized Hopf monoids

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SPECIES

- Informally, a *vector species* $\mathbf{H}$ is a "construction" $\mathbf{H}[I]$ on finite sets I that behaves well with bijections between finite sets.
- More formally, $\mathbf{H}$: {finite sets} $\to$ {vector spaces} is a functor between these two categories.

Example 1. (Hyper)Graphs.
Let $\mathbf{G}$ be the species such that $\mathbf{G}[I] :=$ span$_{\mathbb{C}}${simple (hyper)graphs on vertex set I}. Thus,

$$\mathbf{G}[\{a,b,c\}] = \text{span}\{\ a\ b\ c\ ,\ a\ b\ c\ , \dots,\ a\ b\ c\ , \dots\}.$$

Example 2. Linear orders.
Let $\mathbf{L}$ be the species such that $\mathbf{L}[I] :=$ span$_{\mathbb{C}}${linear orders on the set I}. Thus,

$$\mathbf{L}[\{a,b,c\}] = \text{span}\{abc, acb, bac, bca, cab, cba\}.$$

Hopf monoids

Let $\mathbf{H}$ be a species. Suppose we have maps

$$\mu_{I,J} : \mathbf{H}[I] \otimes \mathbf{H}[J] \to \mathbf{H}[I \uplus J]$$
$$\Delta_{I,J} : \mathbf{H}[I \uplus J] \to \mathbf{H}[I] \otimes \mathbf{H}[J]$$

on $\mathbf{H}$ satisfying some compatibilities. This extra structure on $\mathbf{H}$ turn it into a *Hopf monoid*.

Example 1. In the species $\mathbf{G}$ let $\mu_{I,J}$ and $\Delta_{I,J}$ be such that for graphs $\mathbf{x} \in \mathbf{G}[I]$, $\mathbf{y} \in \mathbf{G}[J]$ and $\mathbf{g} \in \mathbf{G}[I \uplus J]$ one has

$$\mu_{I,J}(\mathbf{x} \otimes \mathbf{y}) := \mathbf{x} \uplus \mathbf{y} \qquad \text{disjoint union}$$
$$b \otimes a\ c \mapsto b\ a\ c$$
$$\Delta_{I,J}(\mathbf{g}) := \mathbf{g}|_I \otimes \mathbf{g}|_J \qquad \text{restriction}$$
$$b\ a\ c \xrightarrow{\Delta_{b,ac}} b \otimes a\ c$$

Example 2. In the species $\mathbf{L}$ let $\mu_{I,J}$ and $\Delta_{I,J}$ be such that for linear orders $\alpha \in \mathbf{L}[I]$, $\beta \in \mathbf{L}[J]$ and $\pi \in \mathbf{L}[I \uplus J]$ one has

$$\mu_{I,J}(\alpha \otimes \alpha) := \alpha \cdot \alpha \qquad \text{concatenation}$$
$$b \otimes ca \mapsto bca$$
$$\Delta_{I,J}(\pi) := \pi|_I \otimes \pi|_J \qquad \text{deshuffle}$$
$$dcba \xrightarrow{\Delta_{b,acd}} b \otimes dca$$

ANTIPODE

Part of the hidden structure of these Hopf monoids is their *antipode* S which for each finite set I can be computed using Takeuchi-Sweedler's formula:

$$S_I = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \circ \Delta_A.$$

PROBLEM:

How to solve all these cancellations combinatorially?

BUILDING NEW MONOIDS FROM OLD ONES

Given a Hopf monoid $\mathbf{H}$ construct the monoid $\mathbf{L} \times \mathbf{H}$ such that

$$(\mathbf{L} \times \mathbf{H})[I] = \mathbf{L}[I] \times \mathbf{H}[I] \qquad \text{Hadamard product}$$

For instance, a typical element in $(\mathbf{L} \times \mathbf{G})[I]$ is an ordered pair (α, g) where g is a graph on I and α is an ordering of the vertex set I.

- the monoid $\mathbf{H}$ has a *basis* $\mathbf{h}$ if $\mathbf{H}[i] = \mathbb{K}\mathbf{h}[I]$ and $\mathbf{h}$ is a set valued species.
- the monoid $\mathbf{H}$ is *linearized* if it has a basis and the structure constants are in $\{0,1\}$ in the basis.

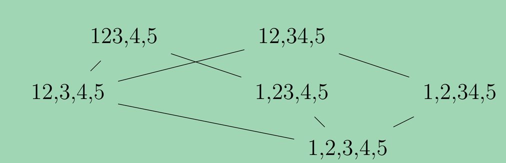
Theorem 1. [Antipode for linearized $\mathbf{L} \times \mathbf{H}$]*(B-B'16) Let $\mathbf{H}$ have basis $\mathbf{h}$ and let $(\alpha, x) \in (\ell \times \mathbf{h})[I]$.*

$$S_I((\alpha, x)) = \sum_{(\beta, y)} c_{\alpha x}^{\beta y} (\beta, y)$$

where $c_{\alpha x}^{\beta y} \in \{-1, 0, 1\}$.

EXAMPLE A: Let $\mathbf{HG}$ be the Hopf monoid of hypergraphs. Let $I = \{a,b,c,d,e,f,h\}$ and let

- Construct the most refined partition Λ of I such that $(\alpha_\Lambda, x_\Lambda) = (\beta, y)$, thus $\Lambda = (a, b, def, h, c)$.
- Identify Λ with $(1,2,3,4,5)$ since Λ has 5 parts.
- Get coarsenings Γ of Λ such that $(\alpha_\Gamma, x_\Gamma) = (\beta, y)$. Build a poset $\mathcal{C}_{\alpha,x}^{\beta,y}$



- The coefficient of (β, y) in $S_I((\alpha,x)) = c_{\alpha x}^{\beta y} = (-1)^5 + 3(1)^4 + 2(-1)^3 = 0$. This actually requires a *sign-reversing involution* with at most one fixed point.

- Motivation for wanting a formula as in Theorem 1, arises from the problem of understanding antipodes in Hopf algebras H obtained from $\mathbf{L} \times \mathbf{H}$ after applying the functor $\overline{\mathcal{K}}$ that "forgets" labels.

$$\overline{\mathcal{K}}(\mathbf{L} \times \mathbf{H}) = \mathcal{K}(\mathbf{H})$$

Commutative and cocommutative $\mathbf{H}$

Let $\mathbf{H}$ be commutative, cocommutative, linearized with basis $\mathbf{h}$. Let $x \in \mathbf{h}[I]$ and let G_x^y be the hypergraph that records the minimal Λ for which $\prod x_{\Lambda_i} \neq x_{\cup \Lambda_i}$.

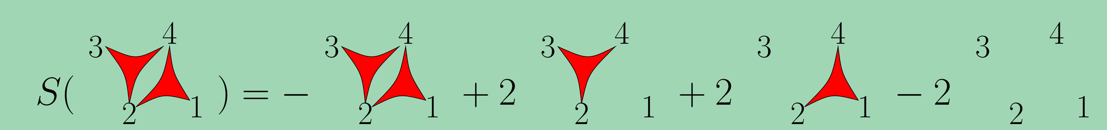
Theorem 2. [Antipode for co-cocom linearized $\mathbf{H}$]*(B-B'16) Under the conditions above*

$$S_I(x) = \sum_{y \in \mathbf{h}[I]} a(G_x^y) y,$$

where $a(G_x^y)$ is a signed sum of acyclic orientations of the hypergraph G_x^y.

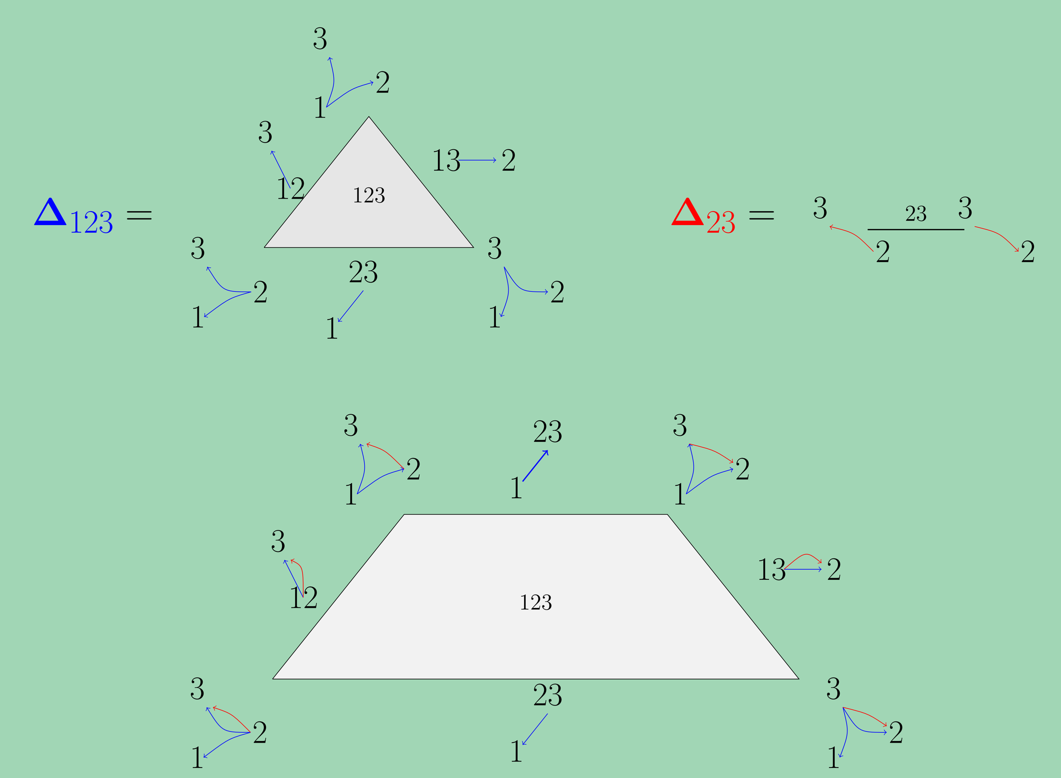
Theorem 3. [Reduction to Hypergraphs]*(B-B'16) Given a commutative and cocommutative linearized Hopf monoid $\mathbf{H}$, let $x, y \in \mathbf{h}[I]$. We have $c_x^y = c_{x/y}^\epsilon$ where ϵ is the hypergraph on $[m]$ with no edges and $x/y = G_x^y$*

EXAMPLE B1:



there are 20 acyclic orientations in $x/\emptyset$: $(4,3,2,1)$; $(3,4,2,1)$; $(34,2,1)$; $(3,2,4,1)$; $(2,4,3,1)$; $(23,4,1)$; $(1,4,3,2)$; $(3,1,4,2)$; $(1,2,4,3)$; $(1,23,4)$; $(1,24,3)$; $(1,34,2)$; $(3,12,4)$; $(12,4,3)$; $(123,4)$; $(14,3,2)$; $(3,14,2)$; $(134,2)$; $(3,24,1)$; $(24,3,1)$. There are 9 even length set compositions in this list and 11 odd length. The coefficient is indeed $9 - 11 = -2$. For $y = \{\{1,2,4\}\}$, x/y is a graph on two vertices with a single edge between the vertices. There are two orientations of such graph and each orientation is represented with a set composition having two parts. Hence the coefficient is 2. The same argument applies for $y' = \{\{2,3,4\}\}$

EXAMPLE B2: Consider the hypergraph $G =$ (3, 1, 2). The monoid $\mathbf{HG}$ does NOT embed into the Hopf monoid $\mathbf{GP}$ of generalized permutahedra studied by Ardila-Aguiar. Thus, we can not use their geometrical interpretation of antipode formulas in the case of hypergraphs. However, we can derive the antipode of G out of its *hypergraphical polytope* as follows.



The coefficient of the discrete graph is the sum of the six acyclic orientations that corresponding to the three faces on the left and the three faces on the right. We call these exterior faces as no contraction occurs. The total homology is 2 in this case. The coefficient $+2$ in $S(G)$ corresponds to the two horizontal faces in the picture (only $\{2,3\}$ is contracted). Finally the coefficient -1 corresponds the interior face of the polytope ($\{1,2,3\}$ is contracted). Thus, $S(G) = -G + 2(\,^{3}_{1}\,2\,) - 2(\,^{3}_{1}\,2\,)$.

For more info check us out: arXiv:1611.01657.

COMPLEMENT TO POSTER ON THE ANTIPODE OF LINEARIZED HOPF MONOIDS

CAROLINA BENEDETTI AND NANTEL BERGERON

Abstract. Many combinatorial Hopf algebras H in the literature are the functorial image of a linearized Hopf monoid $\mathbf{H}$. That is, $H = \mathcal{K}(\mathbf{H})$ or $H = \overline{\mathcal{K}}(\mathbf{H})$. Unlike the functor $\overline{\mathcal{K}}$, the functor $\mathcal{K}$ applied to $\mathbf{H}$ may not preserve the antipode of $\mathbf{H}$. In this case, one needs to consider the larger Hopf monoid $\mathbf{L}\times\mathbf{H}$ to get $H = \mathcal{K}(\mathbf{H}) = \overline{\mathcal{K}}(\mathbf{L}\times\mathbf{H})$ and study the antipode in $\mathbf{L}\times\mathbf{H}$. One of the main results in this paper provides a cancelation free and multiplicity free formula for the antipode of $\mathbf{L}\times\mathbf{H}$. From this formula we obtain a new antipode formula for H. We also explore the case when $\mathbf{H}$ is commutative and cocommutative. In this situation we get new antipode formulas that despite of not being cancelation free, can be used to obtain one for $\overline{\mathcal{K}}(\mathbf{H})$ in some cases. We recover as well many of the well-known cancelation free formulas in the literature. One of our formulas for computing the antipode in $\mathbf{H}$ involves acyclic orientations of hypergraphs as the central tool. In this vein, we obtain polynomials analogous to the chromatic polynomial of a graph, and also identities parallel to Stanley's (-1)-color theorem. One of our examples introduces a *chromatic* polynomial for permutations which counts increasing sequences of the permutation satisfying a pattern. We also study the statistic obtained after evaluating such polynomial at -1. Finally, we sketch q deformations and geometric interpretations of our results. This last part will appear in a sequel paper in joint work with J. Machacek.

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WHY COMPUTING ANTIPODE?

A motivation to find cancelation free formulas for antipode lies in their potential geometric interpretation (see poster or [1]), or in their use to derive information regarding combinatorial invariants of the discrete objects in play. The general principle is that antipode formulas provide interesting identities for the combinatorial invariants. One example of this is the Hopf algebra of graphs $\mathcal{G}$ (see, for instance [10]).

One of the key results in the theory of **Combinatorial Hopf algebras** (CHA) gives us a canonical way of constructing combinatorial invariants with values in the space $QSym$ of quasisymmetric functions (see [2]). Letting $H = \bigoplus_{n\geq 0} H_n$ be a CHA with *character* $\zeta\colon H \to \Bbbk$ we have a unique Hopf morphism

$$\Psi : H \to QSym$$

such that $\zeta = \phi_1 \circ \Psi$ where $\phi_1\big(f(x_1, x_2, \ldots)\big) = f(1, 0, 0, \ldots)$. Consider,

$$\begin{array}{rcl} \phi_t\colon QSym & \to & \Bbbk[t] \\ M_\alpha & \mapsto & \binom{t}{\ell(a)} \end{array}$$

and remark that

$$\phi_t\big(f(x_1, x_2, \ldots)\big)\Big|_{t=1} = \phi_1(f)\,.$$

In particular

$$\phi_t \circ \Psi\Big|_{t=1} = \big(\phi_t\big|_{t=1}\big) \circ \Psi = \phi_1 \circ \Psi = \zeta.$$

Polynomial Invariants

$$\begin{array}{rcl} \phi_t \circ \Psi\colon H & \to & \Bbbk[t] \\ x & \mapsto & \chi_x(t) \end{array}$$

At t = −1

$$\chi_G(-1) = S \circ \phi_t \circ \Psi(G)\Big|_{t=1} = \phi_t \circ \Psi \circ S(G)\Big|_{t=1} = \zeta \circ S(G)$$

FOR GRAPHS ($H = \mathcal{G}$)

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ is discrete graph,} \\ 0 & \text{otherwise.} \end{cases}$$

Stanley's (-1)-theorem

$$\boxed{\chi_G(-1) = \zeta \circ S(G) = (-1)^n a(G)}$$

Using the fact that the discrete graph has coefficient $(-1)^n a(G)$ in $S(G)$ (see [1, 7, 10]), where n is the number of vertices of G and $a(G)$ counts the acyclic orientations in it.

Hopf monoids

A **set composition** $(A_1, \ldots, A_k) \models I$ of a finite set I is a finite sequence of disjoint subsets of I whose union is I. A **Hopf monoid** consists of a vector species $\mathbf{H}$ equipped with two collections μ and Δ of linear maps

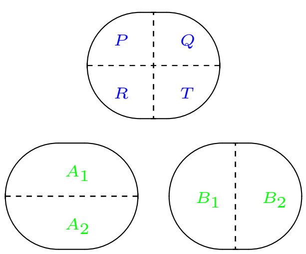
$$\mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \xrightarrow{\mu_{A_1,A_2}} \mathbf{H}[I] \quad \text{and} \quad \mathbf{H}[I] \xrightarrow{\Delta_{A_1,A_2}} \mathbf{H}[A_1] \otimes \mathbf{H}[A_2]$$

such that

Associativity: For each set composition $(A_1, A_2, A_3) \models I$

$$\begin{array}{ccc} \mathbf{H}[A_1]\otimes\mathbf{H}[A_2]\otimes\mathbf{H}[A_3] & \xrightarrow{\mathrm{id}\otimes\mu_{A_1,A_2}} & \mathbf{H}[A_1]\otimes\mathbf{H}[A_2\cup A_3] \\ \downarrow{\scriptstyle \mu_{A_1,A_2}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mu_{A_1,A_2\cup A_3}} \\ \mathbf{H}[A_1\cup A_2]\otimes\mathbf{H}[A_3] & \xrightarrow[\mu_{A_1\cup A_2,A_3}]{} & \mathbf{H}[I] \end{array} \qquad \begin{array}{ccc} \mathbf{H}[I] & \xrightarrow{\Delta_{A_1\cup A_2,A_3}} & \mathbf{H}[A_1\cup A_2]\otimes\mathbf{H}[A_3] \\ \downarrow{\scriptstyle \Delta_{A_1,A_2\cup A_3}} & & \downarrow{\scriptstyle \Delta_{A_1,A_2}\otimes\mathrm{id}} \\ \mathbf{H}[A_1]\otimes\mathbf{H}[A_2\cup A_3] & \xrightarrow[\mathrm{id}\otimes\Delta_{A_1,A_2}]{} & \mathbf{H}[A_1]\otimes\mathbf{H}[A_2]\otimes\mathbf{H}[A_3] \end{array}$$

Compatibility: Given two decomposition of I: $(A_1, A_2) \models I$ and $(B_1, B_2) \models I$



$$\begin{array}{ccccc} \mathbf{H}[P]\otimes\mathbf{H}[Q]\otimes\mathbf{H}[R]\otimes\mathbf{H}[T] & & \xrightarrow{\cong} & & \mathbf{H}[P]\otimes\mathbf{H}[R]\otimes\mathbf{H}[Q]\otimes\mathbf{H}[T] \\ \uparrow{\scriptstyle \Delta_{P,Q}\otimes\Delta_{R,T}} & & & & \downarrow{\scriptstyle \mu_{P,R}\otimes\mu_{Q,T}} \\ \mathbf{H}[A_1]\otimes\mathbf{H}[A_2] & \xrightarrow[\mu_{A_1,A_2}]{} & \mathbf{H}[I] & \xrightarrow[\Delta_{B_1,B_2}]{} & \mathbf{H}[B_1]\otimes\mathbf{H}[B_2] \end{array}$$

Unity: $u_\emptyset \colon \Bbbk \to H[\emptyset]$ and **Counity:** $\epsilon_\emptyset \colon H[\emptyset] \to \Bbbk$.

Connected: $\mathbf{H}[\emptyset] = \Bbbk$ with $\mathbf{H}[I]\otimes\mathbf{H}[\emptyset] \underset{\Delta_{I,\emptyset}}{\overset{\mu_{I,\emptyset}}{\rightleftarrows}} \mathbf{H}[I]$ and $\mathbf{H}[\emptyset]\otimes\mathbf{H}[I] \underset{\Delta_{\emptyset,I}}{\overset{\mu_{\emptyset,I}}{\rightleftarrows}} \mathbf{H}[I]$.

Antipode (Takeuchi-Sweedler's formula) $S_I = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \Delta_A : \mathbf{H}[I] \to \mathbf{H}[I]$

(Co)Commutative?

$$\begin{array}{ccc} \mathbf{H}[A_1]\otimes\mathbf{H}[A_2] & \xrightarrow{\tau_{A_1,A_2}} & \mathbf{H}[A_2]\otimes\mathbf{H}[A_1] \\ & {\scriptstyle \mu_{A_1,A_2}} \searrow \quad \swarrow {\scriptstyle \mu_{A_2,A_1}} & \\ & \mathbf{H}[I] & \end{array} \qquad \begin{array}{ccc} \mathbf{H}[A_1]\otimes\mathbf{H}[A_2] & \xrightarrow{\tau_{A_1,A_2}} & \mathbf{H}[A_2]\otimes\mathbf{H}[A_1] \\ & {\scriptstyle \Delta_{A_1,A_2}} \nwarrow \quad \nearrow {\scriptstyle \Delta_{A_2,A_1}} & \\ & \mathbf{H}[I] & \end{array}$$

Linearized Hopf monoids

A **set species** $\mathbf{h}$ is a collection of sets $\mathbf{h}[I]$, one for each finite set I, equivariant with respect to bijections $I \cong J$. We say that $\mathbf{h}$ is a **basis** for a Hopf monoid $\mathbf{H}$ if for every finite set I we have that

$$\mathbf{H}[I] = \Bbbk\mathbf{h}[I]$$

The monoid $\mathbf{H}$ is **linearized** in the basis $\mathbf{h}$ if

$$\mu_{A_1,A_2} : \mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \to \mathbf{H}[I] \qquad \text{and} \qquad \Delta_{A_1,A_2} : \mathbf{H}[I] \to \mathbf{H}[A_1] \otimes \mathbf{H}[A_2]$$

are the linearization of a maps

$$\mu_{A_1,A_2} : \mathbf{h}[A_1] \otimes \mathbf{h}[A_2] \to \mathbf{h}[I] \qquad \text{and} \qquad \Delta_{A_1,A_2} : \mathbf{h}[I] \to (\mathbf{h}[A_1] \otimes \mathbf{h}[A_2]) \cup \{0\}\,.$$

Form Hopf monoids to Hopf algebras

The functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ [4] $[n] := \{1, 2, \dots, n\}$ and $\mathrm{char}(\Bbbk) = 0$. S_n acts on $\mathbf{H}[n]$.

$$\mathcal{K}(\mathbf{H}) = \bigoplus_{n\geq 0} \mathbf{H}[n] \qquad\qquad \overline{\mathcal{K}}(\mathbf{H}) = \bigoplus_{n\geq 0} \mathbf{H}[n]_{S_n}$$

where

$$\mathbf{H}[n]_{S_n} = \mathbf{H}[n] \Big/ \langle x - \mathbf{H}[\sigma](x) \mid \ \sigma \in S_n;\ x \in \mathbf{H}[n] \rangle$$

The spaces $\mathcal{K}(\mathbf{H})$ and $\overline{\mathcal{K}}(\mathbf{H})$ are **graded Hopf algebras** with (co)multiplication

$$\begin{aligned} \mu_{m,n}\colon H[m] \otimes H[n] &\to H[m+n] \\ x \otimes y &\mapsto \mu_{[m],\{m+1,\dots,m+n\}}(x \otimes \uparrow^m y) \end{aligned}$$

and

$$\begin{aligned} \Delta_{m,n}\colon H[m+n] &\to H[m] \otimes H[n] \\ x &\mapsto \sum_{\substack{(A,B)\models[m+n] \\ |A|=m, |B|=n}} (St \otimes St)\Delta_{A,B}(x) \end{aligned}$$

$\mathcal{K}$ **does not** preserve antipode but $\overline{\mathcal{K}}$ **does**.

Fortunately [4] we have

$$\boxed{\overline{\mathcal{K}}(\mathbf{L} \times \mathbf{H}) \cong \mathcal{K}(\mathbf{H})}$$

1. (Theorem 1) Antipode for linearized Hopf Monoid $\mathbf{L} \times \mathbf{H}$

Let $(\alpha, x) \in (\mathbf{l} \times \mathbf{h})[I]$, where $\mathbf{l} \times \mathbf{h}$ linearize $\mathbf{L} \times \mathbf{H}$. Takeuchi-Sweedler:

$$S_I(\alpha, x) = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \Delta_A(\alpha, x) = \sum_{\substack{A \models I \\ \Delta_A(x) \neq 0}} (-1)^{\ell(A)} (\alpha_A, x_A)$$

Each composition A gives rise to **single elements** α_A and x_A. The coefficient of (β, y) in $S_I(\alpha, x)$ is a signed sum of the element in

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \{A \models I : (\alpha_A, x_A) = (\beta, y)\}$$

we order this set using **refinement**.

Lemma 1.1. *If $\mathcal{C}_{\alpha,x}^{\beta,y} \neq \emptyset$, then there is a unique minimal element Λ in $(\mathcal{C}_{\alpha,x}^{\beta,y}, \leq)$.*

Lemma 1.2. *If $\mathcal{C}_{\alpha,x}^{\beta,y} \neq \emptyset$, then for any $A \in \mathcal{C}_{\alpha,x}^{\beta,y}$ we have that $[\Lambda, A] \subseteq \mathcal{C}_{\alpha,x}^{\beta,y}$.*

Lemma 1.3. *For $\mathcal{C}_{\alpha,x}^{\beta,y} \neq \emptyset$, the minimal elements of $[\Lambda, (I)] \setminus \mathcal{C}_{\alpha,x}^{\beta,y}$ are all of the form*

$$(\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1} \cup \dots \cup \Lambda_j, \Lambda_{j+1}, \dots, \Lambda_m)$$

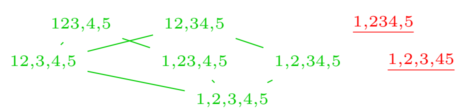
for some $1 \leq i < j \leq m$.

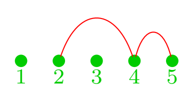
The minimal (i, j) given by lemma 1.3 define an oriented linear graph $G_{\alpha,x}^{\beta,y}$

Example A (of poster): For

$$x = \{\{a, c\}, \{b, h\}, \{d, e, f\}\}, \quad \alpha = abcdefh, \quad y = \{\{d, e, f\}\} \quad \text{and} \quad \beta = abdefhc,$$

We have $\Lambda = (a, b, def, h, c)$. The poset $\mathcal{C}_{\alpha,x}^{\beta,y}$ is



The graph $G_{\alpha,x}^{\beta,y}$ is then given by 

defines $\mathcal{C}_{\alpha,x}^{\beta,y}$

Lemma 1.4. *If $G_{\alpha,x}^{\beta,y}$ is disconnected, then $c(G_{\alpha,x}^{\beta,y})_{\alpha,x}^{\beta,y} = \sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = 0$.*

Lemma 1.5. *If $(i, i+1) \in G = G_{\alpha,x}^{\beta,y}$, then $c(G) = c\big(G|_{\{1,\dots,i\}}\big) \cdot c\big(G|_{\{i+1,\dots,m\}}\big)$.*

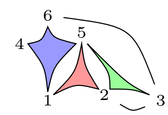
When $G_{\alpha,x}^{\beta,y}$ is non-nested, connected, with no short egdes, we construct a sign reversing involution on $\mathcal{C}_{\alpha,x}^{\beta,y}$ with **at most** two fixed points of opposite sign. The involution depends only on the structure of the graph $G_{\alpha,x}^{\beta,y}$. This shows that in all cases The coefficient of (β, y) in $S_I(\alpha, x)$ is

$$c(G_{\alpha,x}^{\beta,y}) = \sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = 0, 1 \text{ or } -1.$$

Orientations of Hypergraphs

A **hypergraph** on the vertex set $[m]$ is $G \subset 2^{[m]}$ such that $U \in G \implies U \geq 2$.

$$G = \big\{\{2,3\}, \{1,2,5\}, \{1,4,5,f\}, \{2,3,5\}, \{3,6\}\big\} \;=$$

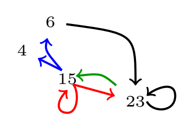


An **orientation** $(\mathfrak{a},\mathfrak{b})$ of a hyperedge $U \in G$ is a partition $(\mathfrak{a},\mathfrak{b}) \models U$. We represent it as a **flow** on U from a single vertex of $\mathfrak{a}$ to the vertices in $\mathfrak{b}$. $\mathfrak{a}$ is the **head** of the orientation. If $|U| = n$, then there are a total of $2^n - 2$ possible orientations.

An **orientation** $\mathcal{O}$ of G is an orientation of all its hyperedges.

We construct an **oriented graph** $G/\mathcal{O}$ on $[m]/\mathcal{O}$ be the finest equivalence class of elements of V defined by the heads of $\mathcal{O}$. For each oriented hyperedge $(\mathfrak{a},\mathfrak{b})$ of $\mathcal{O}$, we have $|\mathfrak{b}|$ oriented edges $([\mathfrak{a}],[b])$ in $G/\mathcal{O}$ where $[\mathfrak{a}],[b] \in [m]/\mathcal{O}$.

$\mathcal{O}$={({2},{3}),({1},{2,5}),({1,5},{4,6}),({2,3},{5}),({6},{3})} $\longrightarrow$ $G/\mathcal{O} =$



An orientation $\mathcal{O}$ of G is **acyclic** if the oriented graph $G/\mathcal{O}$ has no cycles.

Let $G = \big\{\{1,2,4\},\{2,3,4\}\big\}$. There are $(2^3-2)(2^3-2) = 36$ possible orientations. The list of all 20 possible acyclic orientation is

{({4},{1,2}),({4},{2,3})}; {({4},{1,2}),({3},{2,4})}; {({4},{1,2}),({3,4},{2})}; {({2},{1,4}),({3},{2,4})};
{({2},{1,4}),({2},{3,4})}; {({2},{1,4}),({2,3},{4})}; {({1},{2,4}),({4},{2,3})}; {({1},{2,4}),({3},{2,4})};
{({1},{2,4}),({2},{3,4})}; {({1},{2,4}),({2,3},{4})}; {({1},{2,4}),({2,4},{3})}; {({1},{2,4}),({3,4},{2})};
{({1,2},{4}),({3},{2,4})}; {({1,2},{4}),({2},{3,4})}; {({1,2},{4}),({2,3},{4})}; {({1,4},{2}),({4},{2,3})};
{({1,4},{2}),({3},{2,4})}; {({1,4},{2}),({3,4},{2})}; {({2,4},{1}),({3},{2,4})}; {({2,4},{1}),({2,4},{3})}.

2. (Theorem 2 and 3) Antipode for commutative linearized Hopf monoid

Let $x \in \mathbf{h}[I]$, where $\mathbf{h}$ linearize $\mathbf{H}$ co-comutative. Takeuchi-Sweedler:.

$$S_I(x) = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \Delta_A(x) = \sum_{\substack{A \models I \\ \Delta_A(x) \neq 0}} (-1)^{\ell(A)} x_A.$$

The coefficient of y in $S_I(x)$ is a signed sum of the elements of

$$\mathcal{C}_x^y = \{A \models I : x_A = y\}$$

Lemma 2.1. *Let $\Lambda \in \mathcal{C}_x^y$ minimal. Any $A \in \mathcal{C}_x^y$ is minimal iff $A = \sigma(\Lambda)$.*

Lemma 2.2. *If $\mathcal{C}_x^y \neq \emptyset$, then for any $A \leq B \in \mathcal{C}_x^y$ we have that $[A, B] \subseteq \mathcal{C}_x^y$.*

Lemma 2.3. *The minimal elements of $\left(\bigcup_{\sigma \in S_m} [\sigma\Lambda, (I)]\right) \setminus \mathcal{C}_x^y$ are of the form*

$$\sigma(\bigcup_{i \in U} \Lambda_i, \Lambda_{v_1}, \Lambda_{v_2}, \dots, \Lambda_{v_r})$$

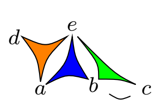
for some $U \subseteq \{1, 2, \dots, m\}$, $\sigma \in S_{r+1}$, where $r = m - |U|$ and $\{v_1, \dots, v_r\} = I \setminus U$.

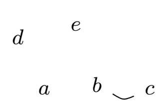
This gives us an hypergraph $G_x^y = \{U \subseteq I : \ U \text{ minimal}, \ \prod_{i \in U} x_{\Lambda_i} \neq x_{\bigcup_{i \in U} \Lambda_i}\}$

Lemma 2.4. *There is a surjective map $\Omega \colon \mathcal{C}_x^y \to \mathfrak{O}_x^y$ (acyclic orientation of G_x^y).*

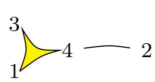
(a) *$\Omega(A) \in \mathfrak{O}_x^y$ is such that for $U \in G_x^y$ the orientation of U is given by $(U \cap A_i, U \setminus A_i)$ for i minimal. Furthermore $V/\Omega(A)$ is a refinement of $\{A_1, A_2, \dots, A_\ell\}$.*

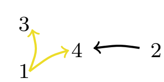
(b) *there is a unique $\Omega(A_{\mathcal{O}}) = \mathcal{O}$ such that $A_{\mathcal{O}} = (A_1, A_2, \dots, A_\ell)$, $A = V/\mathcal{O}$ and A_i is the unique maximal source of $G/\mathcal{O}_{i,\ell}$.*

EXAMPLE $x =$ 

$y =$ 

The minimum is $\Lambda = (a, bc, d, e) \in \mathcal{C}_x^y$.

$G_x^y =$ 

exclude $\{1, 3, 4\}$ and $\{2, 4\}$ in parts of $\mathcal{C}_x^y$. $(12, 34) \overset{\Omega}{\mapsto}$ 

An signed reversing involution on the fibers of Ω has **unique fixed point** $A_{\mathcal{O}}$

Theorem $c_x^y = \sum_{\mathcal{O} \in \mathfrak{O}_x^y} (-1)^{\ell(A_{\mathcal{O}})} = a(G_y^x)$.

Corollary $c_x^y = c_{G_y^x}^{\epsilon}$ (Here the RHS is always in **hypergraphs monoid**)

3. Some examples of applications

Theorem 3.1. *For $\alpha \in \mathbf{l}[n]$, the chromatic polynomial $\chi_\alpha(t)$ counts the number of ways to color increasing sequences of α with at most t distinct colors. We have the identity*

$$\chi_\alpha(-1) = (-1)^n d_{\alpha,\epsilon},$$

where $d_{\alpha,\epsilon}$ is the number of α-decreasing order.

Example 3.2. Let $\zeta_{21} \colon PR \to \Bbbk$ defined by $\zeta_{21}(x) = 1$ if $x = 2143\dots(2n)(2n-1)$, zero otherwise. This defines a symmetric functions $\Psi_{21} \colon PR \to QSym$:

$$\Psi_{21}(\alpha) = \sum_{a \models n} c'_a(\alpha) M_a,$$

where

$$c'_a(\alpha) = \big|\{A \models [n] : \text{for } 1 \le i \le \ell,\ |A_i| = 2a_i \text{ and } st(\alpha|_{A_i}) = 2143\dots(2a_i)(2a_i-1)\}\big|.$$

The *chromatic* polynomial $\chi^{21}_\alpha(t)$ is then

$$\chi^{21}_\alpha(t) = \sum_{a \models n} c'_a(\alpha) \binom{t}{\ell(a)}.$$

We get the identity

Theorem 3.3.

$$\sum_{a \models n} (-1)^{\ell(a)} c'_a(\alpha) = (-1)^{n/2} d_{\alpha,2143\dots(2n)(2n-1)}.$$

Conjecture 3.4. $(-1)^{n/2}\Psi_{21}(\alpha)(-h_1,-h_2,\dots)$ *is h-positive for any α. So far, our computer evidence suggests that this fact seams to be true as well for* **any kind** *of chromatic symmetric function.*

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