Affine Crystal Structures and Promotion Operators

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Outline

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1. Type *A* Crystals
   - Classical Crystals
   - Affine Crystals

2. A Question

3. Outline of Proof

What is a crystal?

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I will say: for $\mathfrak{g}$ a Kac-Moody algebra, a *crystal* is a collection of data which encodes the structure of an integrable, highest-weight $U_q(\mathfrak{g})$-module. For this talk $\mathfrak{g}$ will always be $\mathfrak{sl}_n$ or $\widehat{\mathfrak{sl}}_n$.

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Applications:

- Representation theory
- Topology
- Mathematical physics

Combinatorics of $\mathfrak{sl}_n$ crystals

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Combinatorics of $\mathfrak{sl}_n$ crystals

A *combinatorial* model:

The $\mathfrak{sl}_n$ crystal indexed by the partition λ is

- A directed, edge-colored graph
- Vertex set: all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$
- Edges: An i-colored edge changes a single tableau entry $i \to i + 1$.

Edges in the crystal graph

1. Consider i's and $i+1$'s in the reading word of the tableau
2. "Bracket" pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

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Example

The $\mathfrak{sl}_2$ crystal indexed by the partition $(3,3)$:

$$
\begin{tikzcd}
\begin{array}{|c|c|c|}\hline 2&2&2\\\hline 1&1&1\\\hline\end{array} \arrow[d, "2"'] & & & \\
\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 1&1&1\\\hline\end{array} \arrow[r, "1"] \arrow[d, "2"'] & \begin{array}{|c|c|c|}\hline 2&2&3\\\hline 1&1&2\\\hline\end{array} \arrow[d, "2"'] & & \\
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\end{tikzcd}
$$

Tensor products

The tensor product $B(\lambda) \otimes B(\mu)$

- Vertex set: $v \otimes w$ with $v \in B(\lambda), w \in B(\mu)$
- Edges: Described exactly as before (concatenate the reading word of v with that of w).

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$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|cc}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{2}$$

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- Finite dimensional, irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$ modules classified by Chari-Pressley (Drinfeld polynomials)
- Kirillov-Reshetikhin modules: A subset of these *with* a crystal basis

Kirillov-Reshetikhin Crystals

A combinatorial description of the KR crystal for $(\widehat{\mathfrak{sl}}_n)$, $B^{r,s}$

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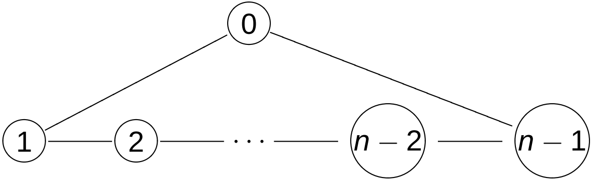
The 0-edges have a wonderful combinatorial description due to Mark Shimozono.

A Dynkin diagram automorphism

The Dynkin diagram for $(\widehat{\mathfrak{sl}}_n)$ is



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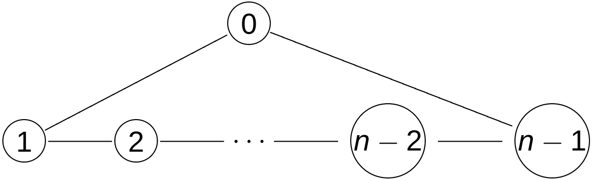

There is an automorphism of this graph (call it Ω) given by rotating all entries one unit counter-clockwise.

The KR crystals have a corresponding automorphism.

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- pr shifts content: If $wt(b) = (m_1, \ldots, m_n)$ is the content of the crystal element $b \in B$, then $wt(pr(b)) = (m_n, m_1, \ldots, m_{n-1})$;
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Properties can be verified from classical data.

Example

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- Remove all n's.
- Play *jeu de taquin.*
- *Increase all entries by 1.*
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Theorem (Shimozono)

This operation satisfies the definition of a promotion operator.

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Back to crystals

A definition of 0-edges:

The edge $b \xrightarrow{0} b'$ exists if and only if the edge $\mathrm{pr}(b) \xrightarrow{1} \mathrm{pr}(b')$ exists.

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therefore

$$\begin{array}{|c|c|c|}\hline 3&4&4\\\hline 2&3&3\\\hline 1&2&2\\\hline\end{array} \xrightarrow{0} \begin{array}{|c|c|c|}\hline 3&3&4\\\hline 2&2&3\\\hline 1&1&2\\\hline\end{array}$$

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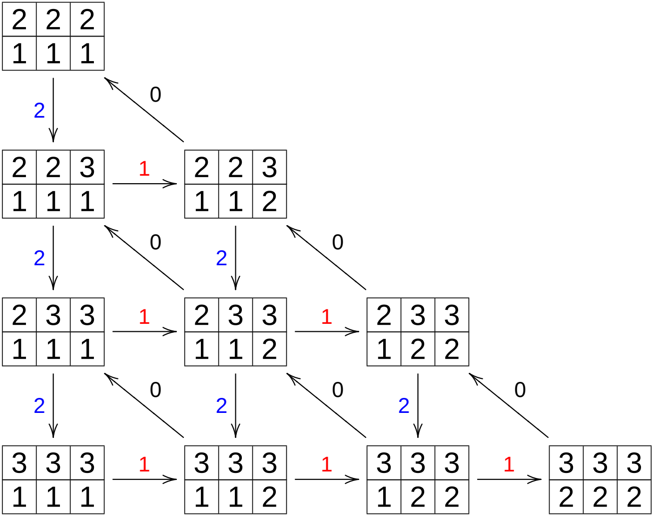
2 2 2 / 1 1 1
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2 2 3 / 1 1 2
2 3 3 / 1 1 1
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2 3 3 / 1 2 2
3 3 3 / 1 1 1
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- Under the assumption that any affine crystal has a promotion operator, this shows that there is a unique affine structure on rectangles.
- What can we say about tensor products?

Theorem (BST)

There is a **unique** *promotion operator on the tensor product of two distinct irreducible $\mathfrak{sl}_n$ crystals.*

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We want to better understand how finite-dimensional affine crystal structures arise.

- It is conjectured that the only finite-dimensional affine crystals are the KR crystals $B^{r,s}$, and tensor products of these.
- Understand the correspondence with rigged configurations.

Main Ingredients

- Content consideration/“bracketing” arguments
- Reduction to special elements
- Induction
- Duality

Bracketing Arguments

In $B^{1,2} \otimes B^{1,1}$ (as $\mathfrak{sl}_3$ crystals) consider:

$$23 \otimes 1 \xrightarrow{pr} 13 \otimes 2$$

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$$13 \otimes 2 \longrightarrow ? \longrightarrow 23 \otimes 1$$

So we must have

$$13 \otimes 2 \longrightarrow 12 \otimes 3$$

Reduction to special elements

Lemma

If $pr(b)$ is fixed and there is a path from $b \to b'$ without using $(n-1)$-colored edges, then $pr(b')$ is fixed.

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Example:

$$\begin{tikzcd} 13 \otimes 2 \arrow[r, "pr"] \arrow[d, "1"'] & 12 \otimes 3 \arrow[d, "2"] \\ 23 \otimes 2 \arrow[r, dotted, "pr"] & 13 \otimes 3 \end{tikzcd}$$

Induction

Lemma

In the crystal $B^{r,s} \otimes B^{r',s'}$, there is a only one possibility for promotion on any element with $s + s'$ occurences of 1.

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Set $C_i =$ set of elements with $s + s'$ copies of i, along with all “internal” edges (those edges not colored i or $i - 1$).

Induction

Set $D = B^{r-1,s} \otimes B^{r'-1,s'}$ as an $\mathfrak{sl}_{n-1}$ crystal.

Induction

Set $D = B^{r-1,s} \otimes B^{r'-1,s'}$ as an $\mathfrak{sl}_{n-1}$ crystal. There is a *unique* crystal map $\phi_1 : C_1 \to D$.

Given a proposed promotion p, define $\phi_2 : C_2 \to D$ by

$$\phi_2 = \mathrm{pr}_D \circ \phi_1 \circ \mathrm{pr}_C^{-1}$$

$$\begin{tikzcd}
C_1 \arrow[r, "p"] \arrow[d, "\phi_1"'] & C_2 \arrow[r, "p"] \arrow[d, "\phi_2"'] & C_3 \arrow[r, "p"] \arrow[d, "\phi_3"'] & \cdots \arrow[r, "p"] & C_n \arrow[r, "p=\tau"] \arrow[d, "\phi_n"'] & C_1 \arrow[dl, "\phi_1"'] \\
D \arrow[r, "\rho_1"] & D \arrow[r, "\rho_1"] & D \arrow[r, "\rho_1"] & \cdots \arrow[r, "\rho_1"] & D &
\end{tikzcd}$$

Duality

There is an isomorphism of $\mathfrak{sl}_n$ crystals $B^{r,s} \cong B^{n-r,s}$. Thus for a fixed tensor product $B^{r,s} \otimes B^{r',s'}$, it is sufficient to consider $n \leq r + r'$.