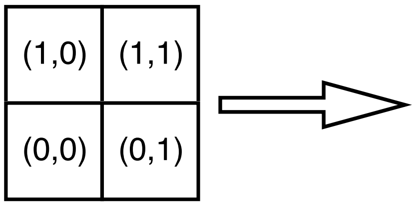
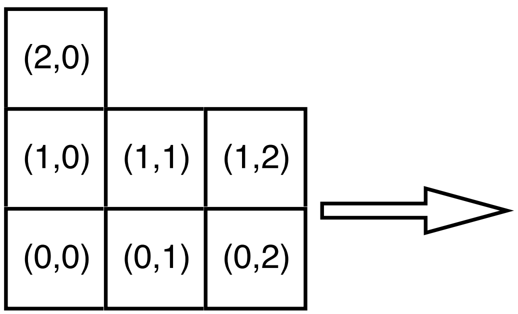
Open Problems in the Theory of Macdonald Polynomials

next

A remarkable determinant



$$\Delta_{2,2}(X,Y) = \det \begin{pmatrix} x_1^0 y_1^0 & x_2^0 y_2^0 & x_3^0 y_3^0 & x_4^0 y_4^0 \\ x_1^0 y_1^1 & x_2^0 y_2^1 & x_3^0 y_3^1 & x_4^0 y_4^1 \\ x_1^1 y_1^0 & x_2^1 y_2^0 & x_3^1 y_3^0 & x_4^1 y_4^0 \\ x_1^1 y_1^1 & x_2^1 y_2^1 & x_3^1 y_3^1 & x_4^1 y_4^1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 \end{pmatrix}$$



$$\Delta_{331}(X,Y) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 & x_6 y_6 & x_7 y_7 \\ x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 & x_4 y_4^2 & x_5 y_5^2 & x_6 y_6^2 & x_7 y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

If $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(X,Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^{n}$$

next

In case we forgot .....

Theorem (easy)

For any $\mu \vdash n$ the dimension of the linear span of the derivatives of $\Delta_\mu(X,Y)$ is at most $n!$

In symbols

$$\dim \mathbf{M}_\mu[\mathbf{X},\mathbf{Y}] \leq \mathbf{n}!$$

where

$$M_\mu[X,Y] \;=\; L\left[\delta_{x_1}^{r_1}\delta_{y_1}^{s_1}\delta_{x_2}^{r_2}\delta_{y_2}^{s_2}\cdots\delta_{x_n}^{r_n}\delta_{y_n}^{s_n}\Delta_\mu(X,Y)\;:\;r_i,s_i\geq 0\right]$$

For example (using MAPLE)

```
DDmu([2,1]);
```

$$\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}$$

```
D21:=det(");
```

$$D21 := y2\,x3 - y3\,x2 - y1\,x3 + y1\,x2 + x1\,y3 - x1\,y2$$

```
diff(D21,x1);
```

$y3 - y2$

```
diff(D21,x3);
```

$y2 - y1$

```
diff(D21,y1);
```

$-x3 + x2$

```
diff(D21,y3);
```

$-x2 + x1$

```
diff(D21,x3,y2);
```

1

6 independent derivatives!

how do we get these n! derivatives?

next

The n! Conjecture

(1990 - 2000)

For $\mu \vdash \mathbf{n}$

$$\dim M_\mu[X,Y] = n!$$

Proved by Mark Haiman using algebraic geometry

1,000$

Reward

for an “elementary” proof

“Elementary” means:

By calculus or/and combinatorics

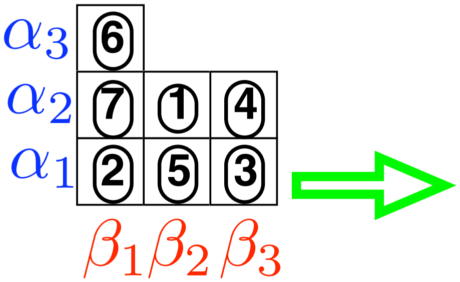
give an algorithm

that produces a “triangular” set of n! derivatives

next

The basic construction

from a tableau to a point in 2n dimensional space

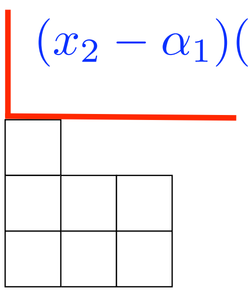


$$\left(\frac{\alpha_2\ \alpha_1\ \alpha_1\ \alpha_2\ \alpha_1\ \alpha_3\ \alpha_2\ \beta_2\ \beta_1\ \beta_3\ \beta_3\ \beta_2\ \beta_1\ \beta_1}{x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7} \right)$$

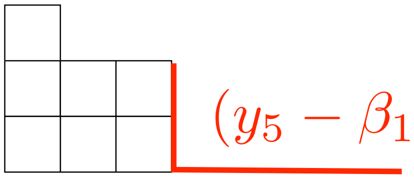
some polynomials vanishing at this orbit point

$x_1 - \alpha_2 \qquad x_4 - \alpha_2 \qquad x_7 - \alpha_2 \qquad y_2 - \beta_1 \qquad y_5 - \beta_2$

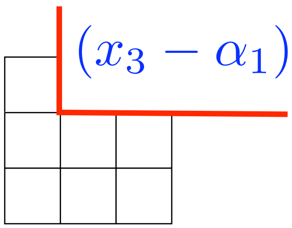
Polynomials vanishing at all these tableau points:



$(x_2 - \alpha_1)(x_2 - \alpha_2)(x_2 - \alpha_3)$



$(y_5 - \beta_1)(y_5 - \beta_2)(y_5 - \beta_3)$

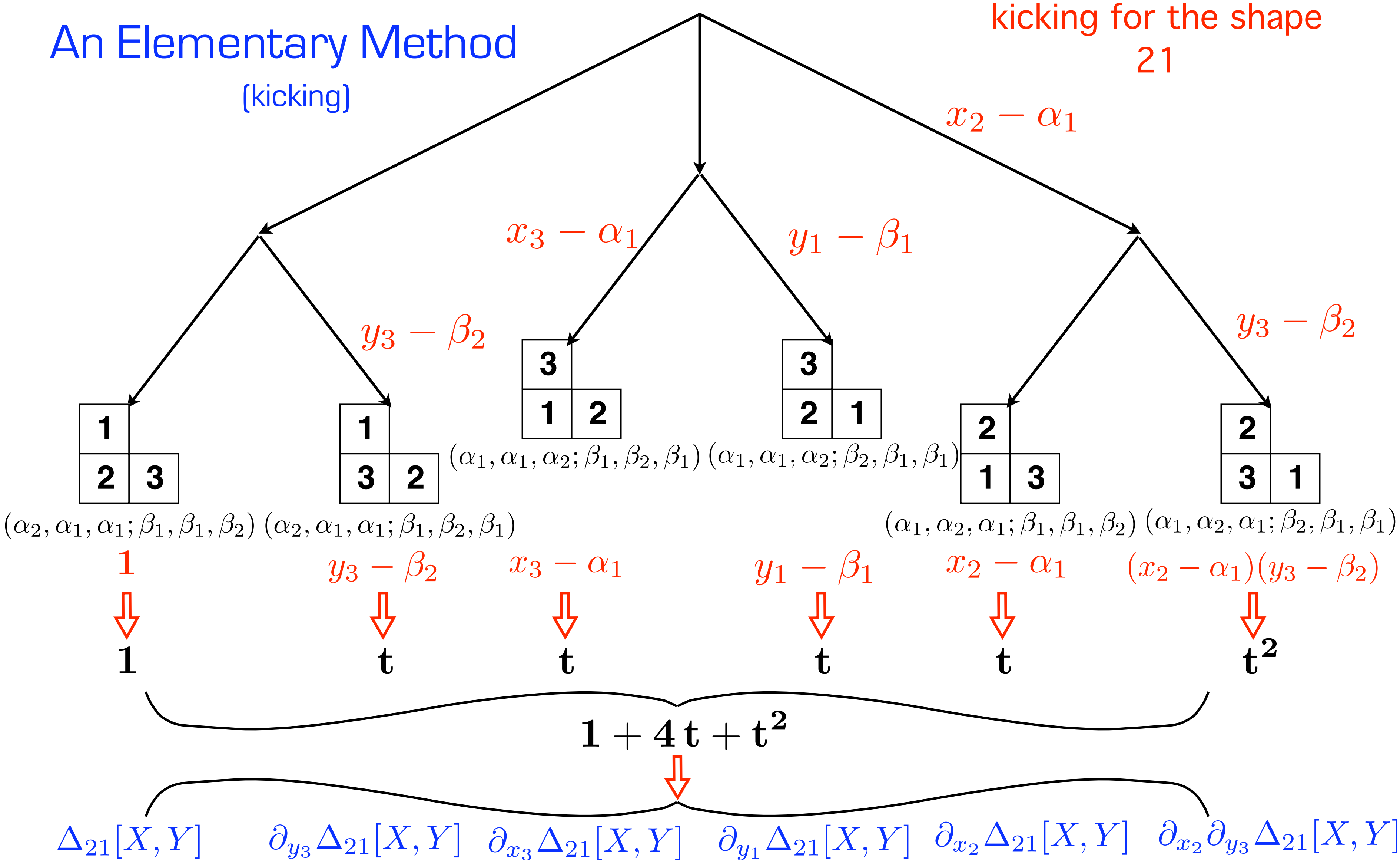


$(x_3 - \alpha_1)(x_3 - \alpha_2)(y_3 - \beta_1)$

An Elementary Method

(kicking)

kicking for the shape 21



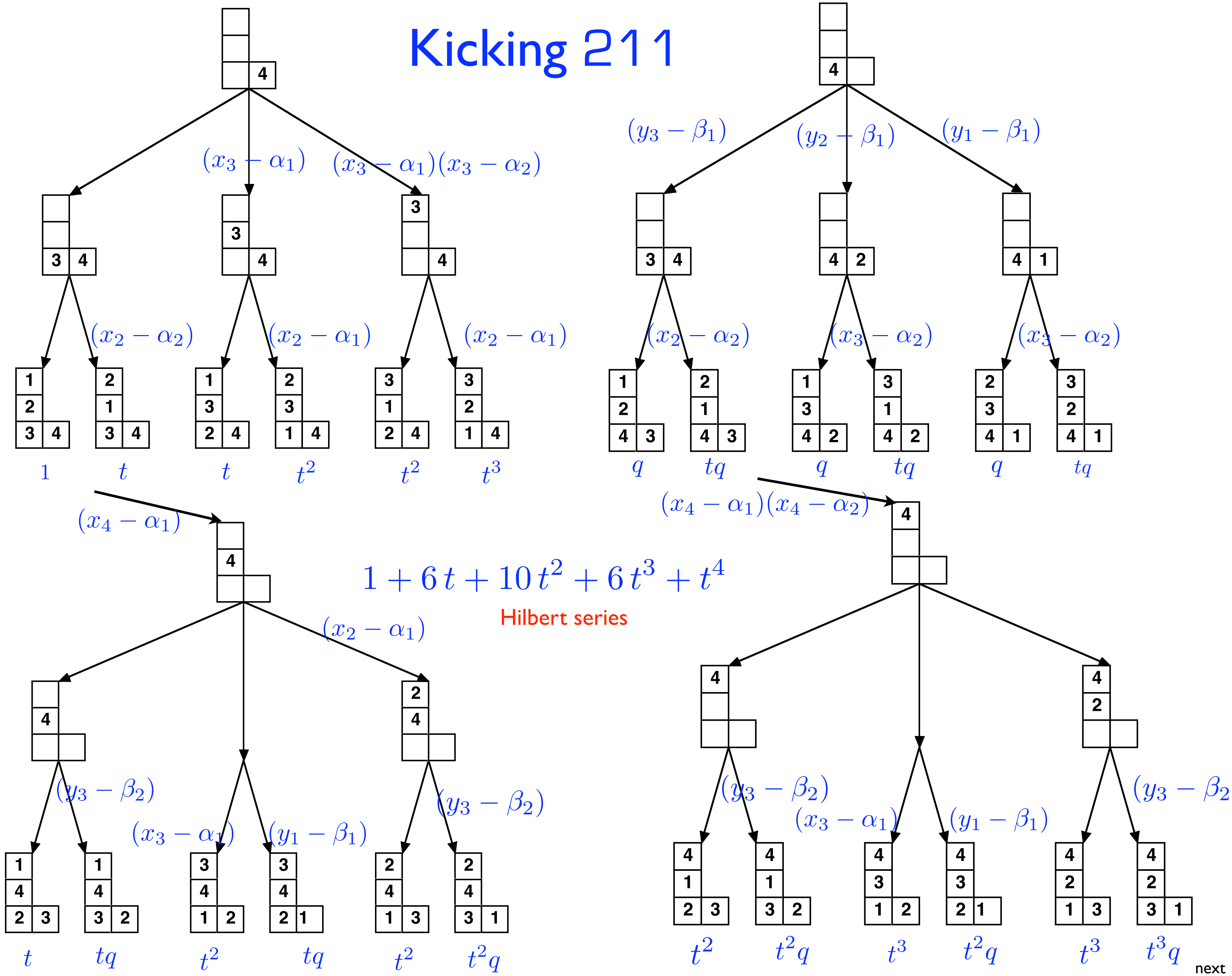
$$\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}$$

$-x2 + x1$ | $y2 - y1$ | $-x3 + x2$ | $-y_3 + y_1$ | 1

AN INDEPENDENT SET WITH 6=3! ELEMENTS

next

Kicking 211



$$1+6\,t+10\,t^2+6\,t^3+t^4$$

Hilbert series

next

The Algorithm

Step 1

Fill the Ferrers diagram of μ *with* $1, 2, \ldots, \mathrm{n}$ *in all possible ways to get the* $\mathrm{n}!$ *tableaux.*

Step 2

Construct the corresponding orbit points

5040
In this case

3		
6	1	2
4	7	5

$$\begin{bmatrix} 1 \\ 12 \\ 75 \\ 306 \\ 807 \\ 1319 \\ 1319 \\ 807 \\ 306 \\ 75 \\ 12 \\ 1 \end{bmatrix}$$

Step 3

Order the orbit points and construct the kicking polynomials.

Step 4

Compute the kicking statistics.

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials give the $\mathrm{n}!$ *independent derivatives of* $\Delta_\mu(X,Y) = \det \| x_j^{p_i} y_j^{q_i} \|_{i,j=1}^{n}$

I am planning to advertise kicking

as a replacement

for SUDOKU

next

Recalling Hilbert series

A vector space V is called "*graded*" if and only if

$$V = H_o(V) \oplus H_1(V) \oplus H_2(V) \oplus \cdots \oplus H_m(V) \oplus \cdots$$

If $\dim H_m(V) < \infty$ for all m, we set

$$F_V(t) = \sum_{m \geq 0} t^m \dim H_m(V)$$

the "*Hilbert series*" of V

Our spaces $M_\mu[X,Y]$ are "*bigraded* that is we have the double decomposition

$$M_\mu[X,Y] = \bigoplus_{r=0}^{n(\mu)} \bigoplus_{s=0}^{n(\mu')} H_{r,s}(M_\mu[X,Y])$$

With $H_{r,s}(M_\mu[X,Y])$ the linear span of derivatives of $\Delta_\mu(x,y)$

that are homogeneous of degree r in $x_1, x_2, \ldots, x_n$ and degree s in $y_1, y_2, \ldots, y_n$

Here and after we set

$$F_\mu(q,t) = \sum_{r=0}^{n(\mu)} \sum_{s=0}^{n(\mu')} t^r q^s \dim H_{r,s}(M_\mu[X,Y])$$

Using Maple

```
hilb([3,2]);
```

$$q^4 t^2 + 4 q^4 t + 4 q^3 t^2 + 5 q^4 + 15 q^3 t + 9 q^2 t^2 + 11 q^3 + 22 q^2 t + 11 q t^2 + 9 q^2 + 15 q t + 5 t^2 + 4 q + 4 t + 1$$

$$\begin{array}{c|ccccc} 2 \rightarrow & 5 & 11 & 9 & 4 & 1 \\ 1 \rightarrow & 4 & 15 & 22 & 15 & 4 \\ 0 \rightarrow & 1 & 4 & 9 & 11 & 5 \\ \hline & \uparrow 0 & \uparrow 1 & \uparrow 2 & \uparrow 3 & \uparrow 4 \end{array}$$

The dimension of $H_{2,1}(M_{3,2}[X,Y])$

The dimension of $H_{1,3}(M_{3,2}[X,Y])$

next

Flip

Definition

We say that a vector space $\mathbf{V}$ *of polynomials is a "cone" if it is the linear span of derivatives of a single homogeneous polynomial* $\Delta(x)$.

In this case $\mathbf{V}$ *has an automorphism " Flip " defined by*

$$\text{flip}\, P(x) \;=\; P(\partial_x)\Delta(x)$$

Note this is a non-singular linear map of $\mathbf{V}$ onto $\mathbf{V}$ since if

$$P(x) \;=\; Q(\partial_x)\Delta(x) \qquad \text{and} \qquad P(\partial_x)\Delta(x) = 0$$

then

$$P(\partial_x)P(x) \;= P(\partial_x)Q(\partial_x)\Delta(x) \;= Q(\partial_x)P(\partial_x)\Delta(x) \;= 0$$

Thus

$$\text{flip}\, P(x) = 0 \;\Longrightarrow\; P(\partial_x)P(x) = 0 \;\Longrightarrow\; P(x) = 0$$

Note that this implies that if degree $\Delta(x) = n_o$ then

$$F_V(t) \;=\; t^{n_o}\, F_V(1/t)$$

The same is true if $\mathbf{V}$ is a cone with summit a bihomogeneous polynomial $\Delta(x,y)$ of bi-degree n_x, n_y In this case

$$\boxed{F_V(q,t) \;=\; t^{n_x} q^{n_y} F_V(1/q,1/t)}$$

This explains the symmetry

$$F_{32}(q,t) \;=\; \begin{bmatrix} \boxed{5} & 11 & \boxed{9} & \boxed{4} & 1 \\ 4 & 15 & 22 & 15 & 4 \\ 1 & \boxed{4} & \boxed{9} & 11 & \boxed{5} \end{bmatrix}$$

next

ORBITS

Let G be a group of permutations of $1, 2, \ldots, n$. Given an element $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in G$ and a vector $a = (a_1, a_2, \ldots, a_n)$ we set

$$\sigma\, a \;=\; (a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_n})$$

This given, we let

$$[a]_G \;=\; \{b = \sigma a \,:\, \sigma \in G\}$$

and call it the "*orbit*" of a under G. The subgroup

$$G_a \;=\; \{\sigma \in G \,:\, \sigma a = a\}$$

is called the "*stabilizer*" of a.

If G_a is trivial then a is called "*regular*" and the orbit $[a]_G$ has cardinality $|G|$.

In general we have

$$|[a]_G| \;=\; |G|/|G_a|$$

In any case we set

$$J_{[a]_G} \;=\; (P \,:\, P(b) = 0 \quad \forall \quad b \in [a]_G) \qquad gr\,\mathbf{J}_{[a]_G} = \big(h(P) \,:\, P \in \mathbf{J}_{[a]_G}\big)$$

and

$$R_{[a]_G} = \mathbb{Q}[x_1, x_2, \ldots, x_n]/J_{[a]_G}\,, \qquad gr\,R_{[a]_G} = \mathbb{Q}[x_1, x_2, \ldots, x_n]/gr\,J_{[a]_G}$$

We also set

$$H_{[a]_G} \;=\; (gr\,J_{[a]_G})^{\perp}$$

and call the elements of this space the "*orbit harmonics*".

Now it follows that

$$\dim R_{[a]_G} = \dim gr\,R_{[a]_G} = \dim H_{[a]_G} \;=\; |G|/|G_a|$$

next

Conical Orbit Harmonics

(This is why symmetry proves the n! result)

Theorem

Let a *be a regular point. Set*

$$n_o = \text{maxdegree}\ R_{[a]_G}$$

Suppose that $\mathcal{B}$ *is a basis for* $R_{[a]_G}$ *with*

$$\mathcal{B}^{\leq n_o} = \mathcal{B} \qquad \text{and} \qquad |\mathcal{B}^{\leq i}| = d_o + d_1 + \cdots + d_i \qquad (\, i = 0, 1, \ldots, n_o \,)$$

Then the equalities

$$d_i \;=\; d_{n_o - i} \qquad\qquad (\, i = 0, 1, \ldots, n_o \,)$$

imply

(i) $\quad h(\mathcal{B}) \;=\; \{\, h(b) \,:\, b \in \mathcal{B} \,\}$ is a basis for $\text{gr}\, R_{[a]}$

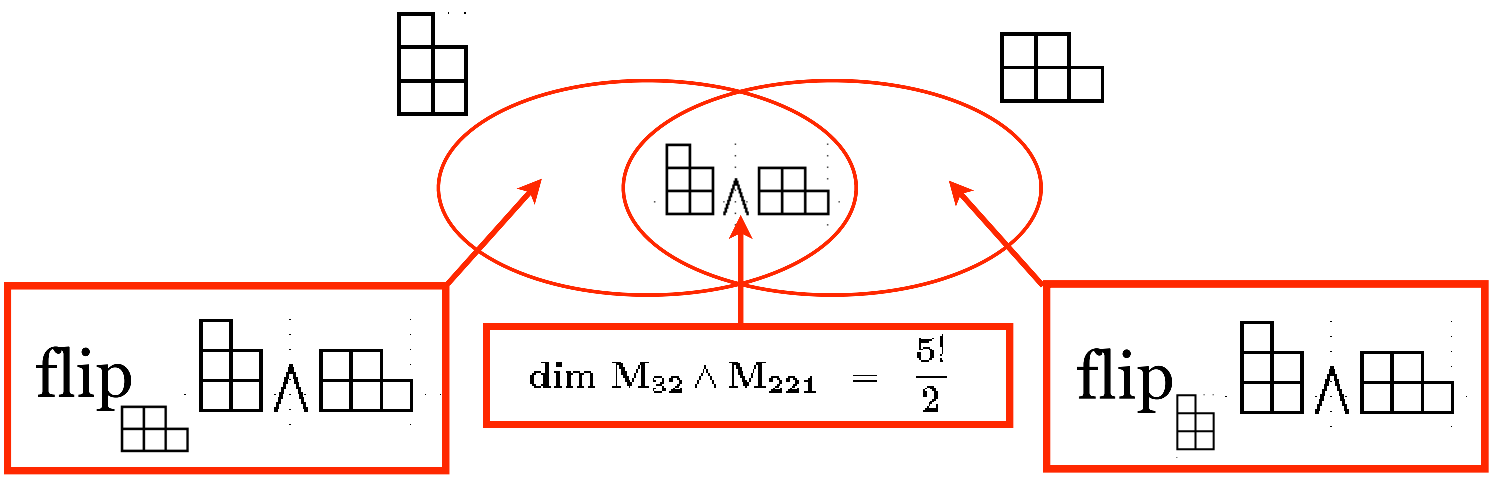
(ii) $\quad F_{\text{gr}R_{[a]_G}}(q) \;=\; F_{H_{[a]_G}}(q) \;=\; \sum_{i=0}^{n_o} d_i\, q^i$,

(iii) $\quad H_{[a]_G}$ is a cone.

(iv) *The elements of* $h(\mathcal{B})$ *yield the desired n! derivatives*

next

Now the miracles



$M_{221} \;=\; M_{32} \wedge M_{221} \;\oplus\; \text{flip}_{221} M_{32} \wedge M_{221}$ $\qquad$ $M_{32} \;=\; M_{221} \wedge M_{32} \;\oplus\; \text{flip}_{32} M_{221} \wedge M_{32}$

$\frac{n!}{2}$ **Conjecture** (still open!)

If $\alpha, \beta \vdash n$ differ only by the position of one corner cell then $\quad \dim M_\alpha \wedge M_\beta \;=\; \frac{n!}{2}$

SF Conjecture (still open!)

If $\alpha, \beta \vdash n$ differ only by the position of one corner cell then

$$M_\alpha = M_\alpha \wedge M_\beta \;\oplus\; \text{flip}_\alpha M_\alpha \wedge M_\beta \quad , \qquad M_\beta = M_\alpha \wedge M_\beta \;\oplus\; \text{flip}_\beta M_\alpha \wedge M_\beta$$

This implies

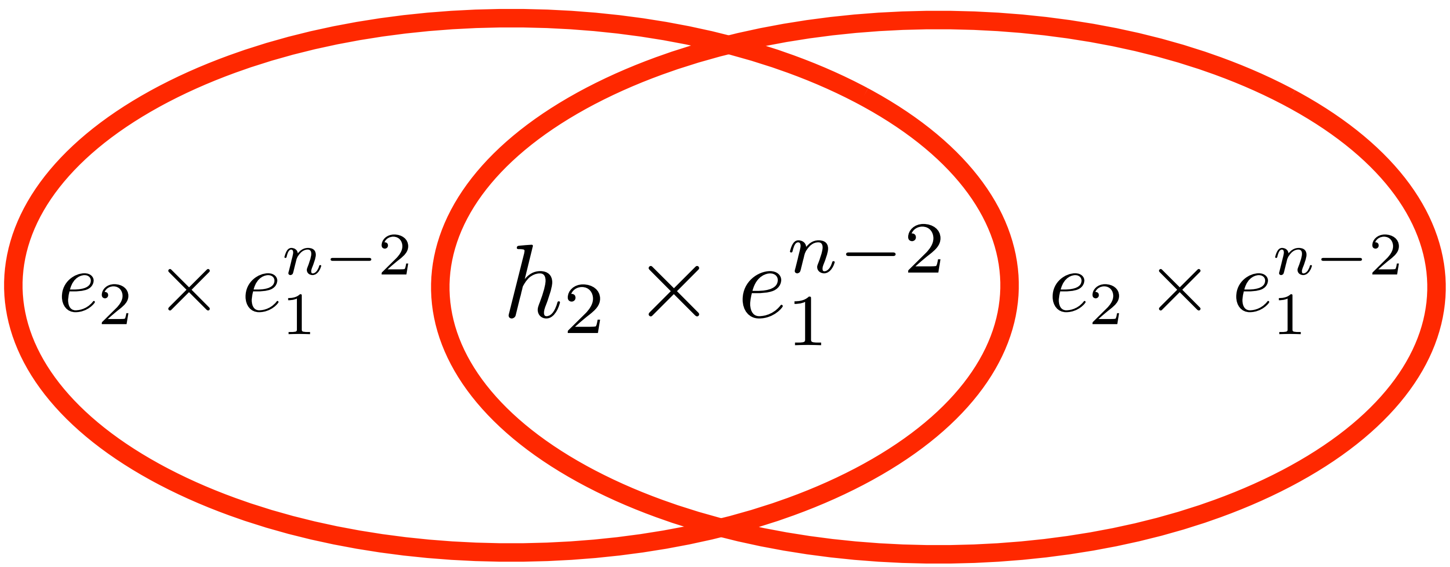
$$F_{M_\alpha \wedge M_\beta}(q,t) \;=\; \frac{T_\beta F_{M_\alpha}(q,t) \;-\; T_\alpha F_{M_\beta}(q,t)}{T_\alpha \;-\; T_\beta}$$

with $T_\mu = t^{n(\mu)} q^{n(\mu')}$ *and* $n(\mu) = \sum_i (i-1)\mu_i$

next

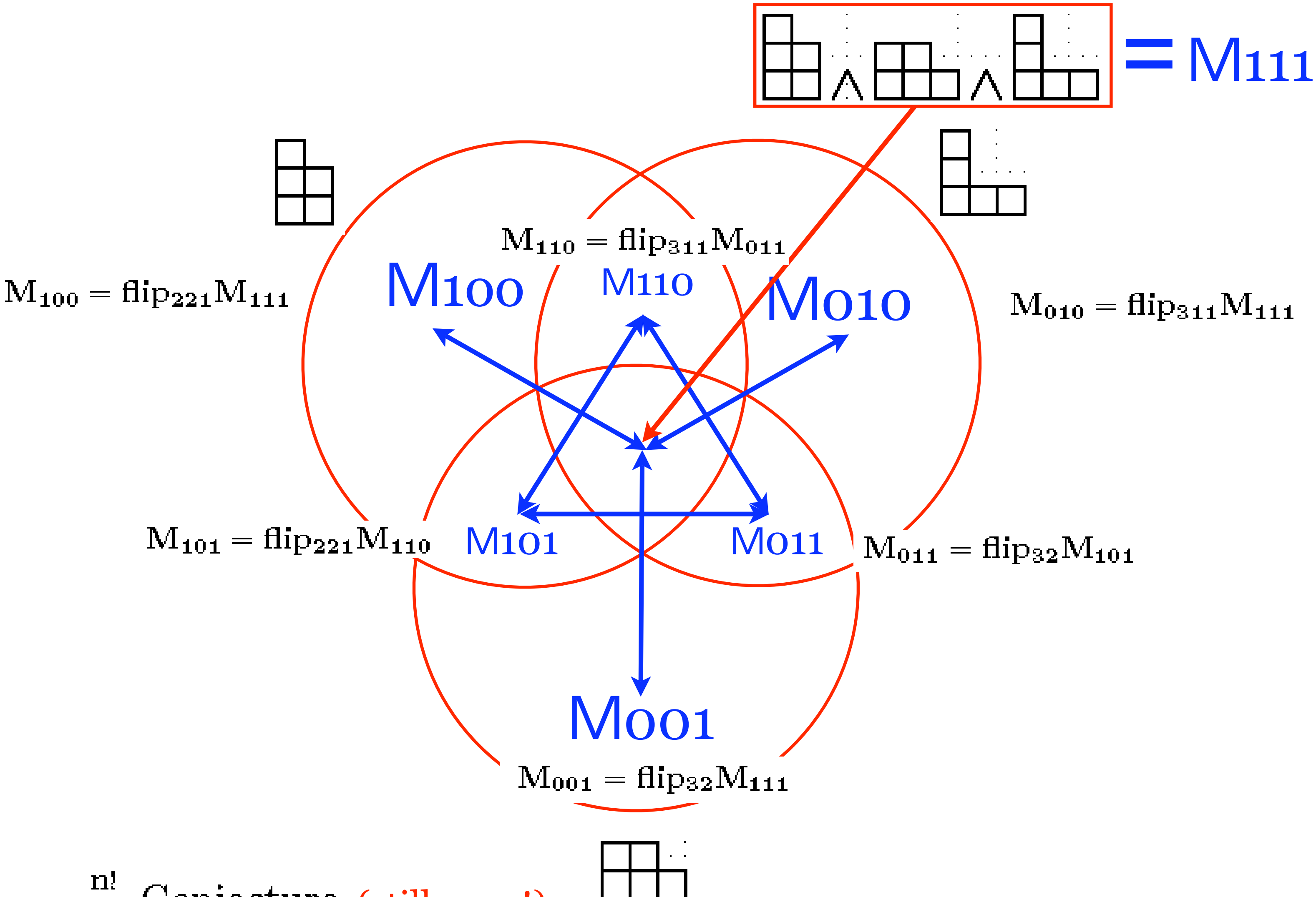
How would you split in one half
a
Left regular representation ?

$$e_1^n = e_1^2 \times e_1^{n-2}$$
$$= (h_2 + e_2) \times e_1^{n-2}$$



next

More miracles



$\frac{n!}{k!}$ **Conjecture** (still open!)

If $\alpha_1, \alpha_2, \ldots, \alpha_k \vdash n$ *differ only by one cell from each other then*

$$\dim M_{\alpha_1} \wedge M_{\alpha_2} \wedge \cdots \wedge M_{\alpha_k} = \frac{n!}{k}$$

Etc Etc Etc

next

ENTER LASCOUX-LAPOINTE-MORSE k-SCHUR

The 3-Schur for 3 bounded partitions of 6

[1, 1, 1, 1, 1, 1], --> , $t^3 s_{1,1,1,1,1,1} + t^2 s_{2,1,1,1,1} + t\, s_{2,2,1,1} + s_{2,2,2}$

[2, 1, 1, 1, 1], --> , $s_{3,2,1} + (s_{3,1,1,1} + s_{2,2,1,1})\, t + t^2 s_{2,1,1,1,1}$

[2, 2, 1, 1], --> , $t^2 s_{2,2,1,1} + t\, s_{3,2,1} + s_{3,3}$

[3, 1, 1, 1], --> , $s_{4,1,1} + t\, s_{3,1,1,1}$

[2, 2, 2], --> , $t^2 s_{2,2,2} + t\, s_{3,2,1} + s_{4,2}$

[3, 2, 1], --> , $s_{5,1} + (s_{4,2} + s_{4,1,1})\, t + t^2 s_{3,2,1}$

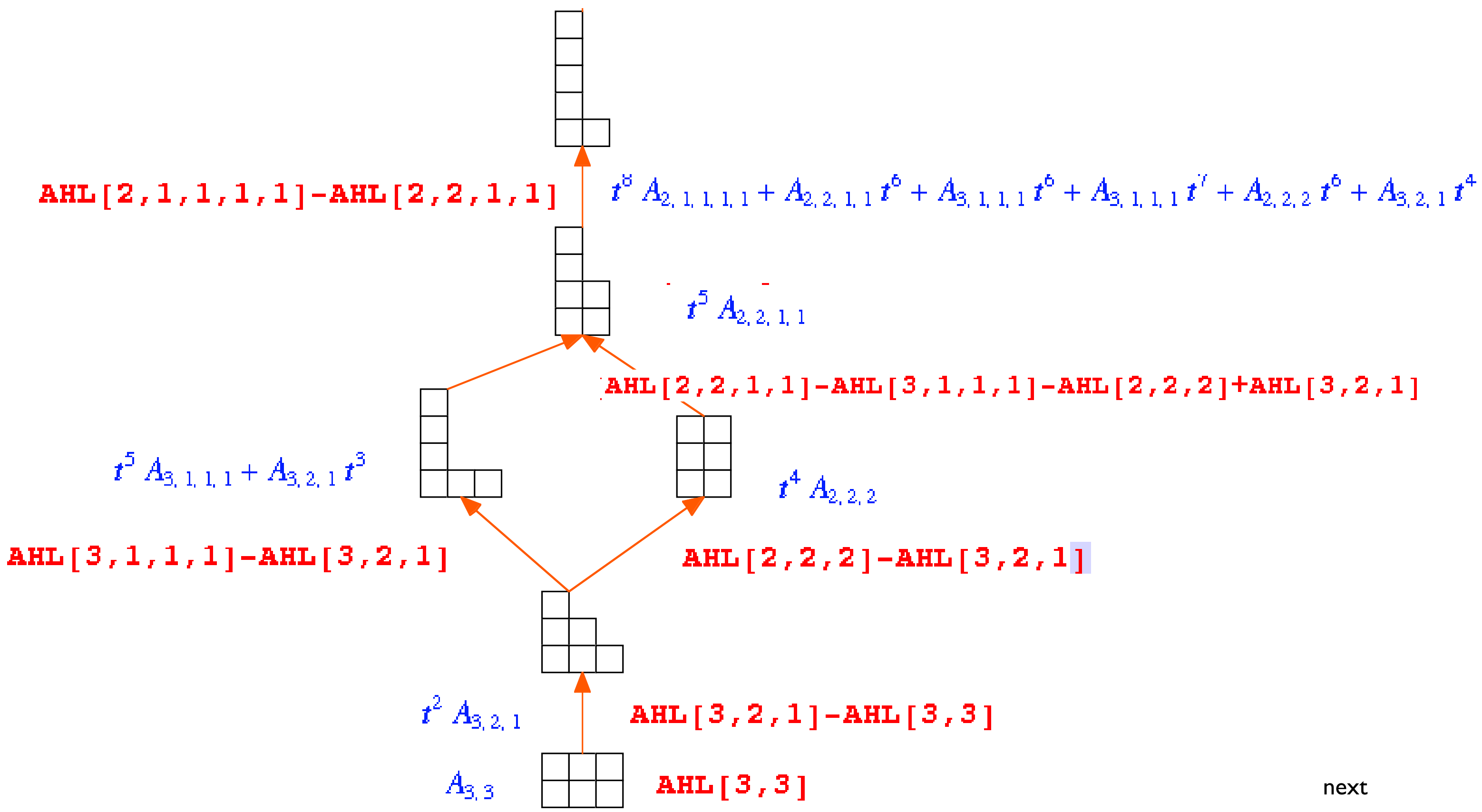
[3, 3], --> , $s_6 + s_{5,1}\, t + s_{4,2}\, t^2 + s_{3,3}\, t^3$

next

Atomic Decomposition of 3-Littlewoods

(for 3-bounded partitions of 6)

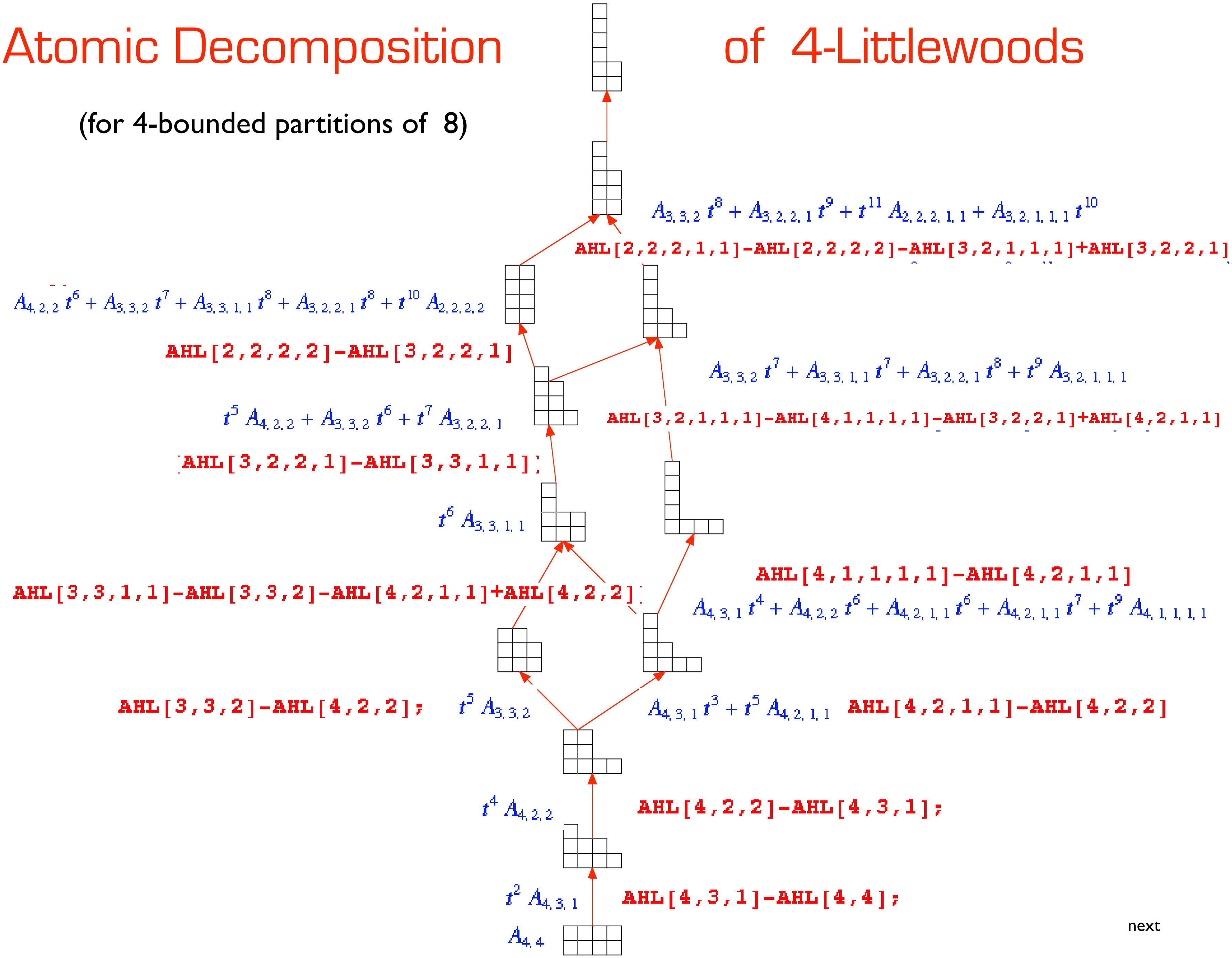
(Lascoux-LaPointe-Morse 3-Schur)



next

Atomic Decomposition of 4-Littlewoods

(for 4-bounded partitions of 8)



next

k-Schur expansion of Macdonald Polynomials

(for 2 bounded partitions of 4)

H([2,2]) $q^2 A_{1,1,1,1} + (q + t\,q)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} 0 & A_{2,1,1} & 0 \\ A_{2,2} & A_{2,1,1} & A_{1,1,1,1} \end{bmatrix}$

H([2,1,1] $t\,q\, A_{1,1,1,1} + (q + t^2)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{2,1,1} & 0 \\ 0 & A_{1,1,1,1} \\ A_{2,2} & A_{2,1,1} \end{bmatrix}$

H([1,1,1,1]) $t^4 A_{1,1,1,1} + (t^2 + t^3)\, A_{2,1,1} + A_{2,2}$ $\begin{bmatrix} A_{1,1,1,1} \\ A_{2,1,1} \\ A_{2,1,1} \\ 0 \\ A_{2,2} \end{bmatrix}$

(for 2 bounded partitions of 6)

$H_{2,2,2}$, --> , , $\begin{bmatrix} 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} & A_{1,1,1,1,1,1} \end{bmatrix}$

$H_{2,2,1,1}$, --> , , $\begin{bmatrix} A_{2,2,1,1} & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,1,1,1,1} & 0 \\ 0 & A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,2} & A_{2,2,1,1} & A_{2,1,1,1,1} \end{bmatrix}$

$H_{2,1,1,1,1}$, --> , , $\begin{bmatrix} A_{2,1,1,1,1} & 0 \\ 0 & 0 \\ A_{2,2,1,1} & A_{1,1,1,1,1,1} \\ A_{2,2,1,1} & A_{2,1,1,1,1} \\ 0 & A_{2,1,1,1,1} \\ 0 & 0 \\ A_{2,2,2} & A_{2,2,1,1} \end{bmatrix}$

$H_{1,1,1,1,1,1}$, --> , , $\begin{bmatrix} A_{1,1,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,1,1,1,1} \\ A_{2,2,1,1} \\ A_{2,2,1,1} \\ A_{2,2,1,1} \\ 0 \\ 0 \\ A_{2,2,2} \end{bmatrix}$

next

k-Schur function compatibility with Science Fiction

(for pairs of 3 bounded partitions of 5)

$$[2,2,1],[3,1,1], \quad \text{-->}, \begin{bmatrix} A_{3,1,1} & A_{2,1,1,1} & 0 \\ 0 & A_{2,2,1} & 0 \\ A_{3,2} & A_{3,1,1} & A_{2,2,1} \end{bmatrix}$$

$$[3,2],[3,1,1], \quad \text{-->}, \begin{bmatrix} 0 & A_{2,2,1} & 0 & 0 \\ A_{3,2} & A_{3,1,1} & A_{2,2,1}+A_{3,1,1} & A_{2,1,1,1} \end{bmatrix}$$

(for pairs of 3 bounded partitions of 6)

$$[3,2,1],[3,1,1,1], \quad \text{-->}, \begin{bmatrix} A_{3,2,1} & A_{2,2,1,1}+A_{3,1,1,1}+A_{2,2,2} & A_{3,1,1,1} & 0 \\ 0 & 0 & A_{2,2,1,1} & A_{2,1,1,1,1} \\ A_{3,3} & A_{3,2,1} & A_{2,2,2}+A_{3,2,1} & A_{3,1,1,1} \end{bmatrix}$$

$$[2,2,2],[2,2,1,1], \quad \text{-->}, \begin{bmatrix} A_{2,2,2} & 0 & 0 \\ 0 & A_{2,2,1,1}+A_{3,1,1,1} & 0 \\ A_{3,2,1} & A_{2,2,1,1}+A_{3,1,1,1} & A_{2,1,1,1,1} \\ 0 & A_{3,2,1} & A_{3,1,1,1} \\ A_{3,3} & A_{3,2,1} & A_{2,2,2} \end{bmatrix}$$

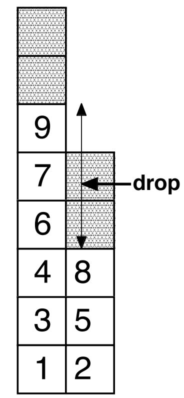
$$[2,1,1,1,1],[2,2,1,1], \quad \text{-->}, \begin{bmatrix} A_{2,2,1,1}+A_{3,1,1,1} & 0 \\ A_{2,2,2} & A_{2,1,1,1,1} \\ A_{3,2,1} & A_{3,1,1,1} \\ A_{3,2,1} & A_{2,2,1,1}+A_{3,1,1,1}+A_{2,2,2} \\ 0 & 0 \\ A_{3,3} & A_{3,2,1} \end{bmatrix}$$

next

A new Problem

Recall that from the Haglund-Haiman-Loehr result we now have statistics $a(T), b(T)$ giving

$$F_\mu(q,t) \;=\; \sum_{T\in INJ(\mu)} t^{a(T)} q^{b(T)}$$

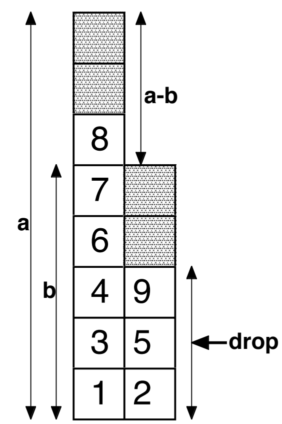


Jim Haglund asked for a new formula of the form

$$F_\mu(q,t) = \sum_{T\in ST(\mu)} P_T(q,t) \qquad\qquad (P_T(q,t)\in \mathbb{N}[q,t])$$

In a new burst of genius Jim conjectured that

$$F_{2^b 1^{a-b}}(q,t) \;=\; \sum_{T\in ST(2^b1^{a-b})} \prod_{i\in T} \left(1+t^{drop_T(i)}\right) \prod_{i\in col_2(T)} \left(t^{NW>i}+q\right) \qquad (*)$$



With "$drop_T(i)$" is the number of rows in $T_{\leq i}$ of the same length as the row that contains i, and "$NW > i$" gives the number of entries in T stritly NW of i that are larger than i.

This is equivalent to the recursion

$$F_{2^b1^{a-b}}(q,t) \;=\; (1+q)[b]_t F_{2^{b-1}1^{a-b+1}}(q,t) + [a-b]_t t^b F_{2^b1^{a-b-1}}(q/t,t) \qquad (**)$$

This is quite remarkable since in the 20 years since the Macdonald conjectures, we never suspected that we could have a purely combinatorial recursion for $F_\mu(q,t)$.

It is a challenging problem to find the general form of (*) and (**)

next

Proof of the Haglund conjecture

Science Fiction suggested that the recursion

$$F_{2^b1^{a-b}}(q,t) \;=\; (1+q)[b]_t F_{2^{b-1}1^{a-b+1}}(q,t) + [a-b]_t t^b F_{2^b1^{a-b-1}}(q/t,t)$$

is none other than the result of applying $\partial_{p_1}^{n-1}$ to the Macdonald polynomial identity

$$\partial_{p_1}\tilde{H}_{2^b1^{a-b}}(X;q,t) \;=\; (1+q)[b]_t\tilde{H}_{2^{b-1}1^{a-b+1}}(X;q,t) + [a-b]_t t^b\tilde{H}_{2^b1^{a-b-1}}(X;q/t,t)$$

for instance when $a=3$ and $b=2$ this is

$$\partial_{p_1}\square_{221}(q,t) \;=\; (1+t)(1+q)\square_{211}(q,t) + t^2\boxplus(q/t,t)$$

Now Science Fiction suggests writing

$$\square_{211} = \square_{211}\wedge\boxplus + t^3q\downarrow\square_{211}\wedge\boxplus \qquad\qquad \boxplus = \square_{211}\wedge\boxplus + t^2q^2\downarrow\square_{211}\wedge\boxplus$$

thus

$$\square_{211}\wedge\boxplus = \frac{q\square_{211} - t\boxplus}{q-t} =_{def} \phi_{32}$$

$$t^3q\downarrow\square_{211}\wedge\boxplus \;=\; t\frac{-\square_{211}+\boxplus}{q-t} =_{def} t\psi_{32} \qquad\qquad t^2q^2\downarrow\square_{211}\wedge\boxplus \;=\; q\frac{\boxplus-\square_{211}}{q-t} =_{def} q\psi_{32}$$

and thus

$$\square_{211} \;=\; \phi_{32}+t\psi_{32} \qquad\qquad \boxplus \;=\; \phi_{32}+q\psi_{32}$$

Now in terms of 2-Schur we get

$$\phi_{32} \;=\; qA_{211}+A_{22}\,, \qquad q\psi_{32} \;=\; q^2A_{1111}+qtA_{211}$$

this gives

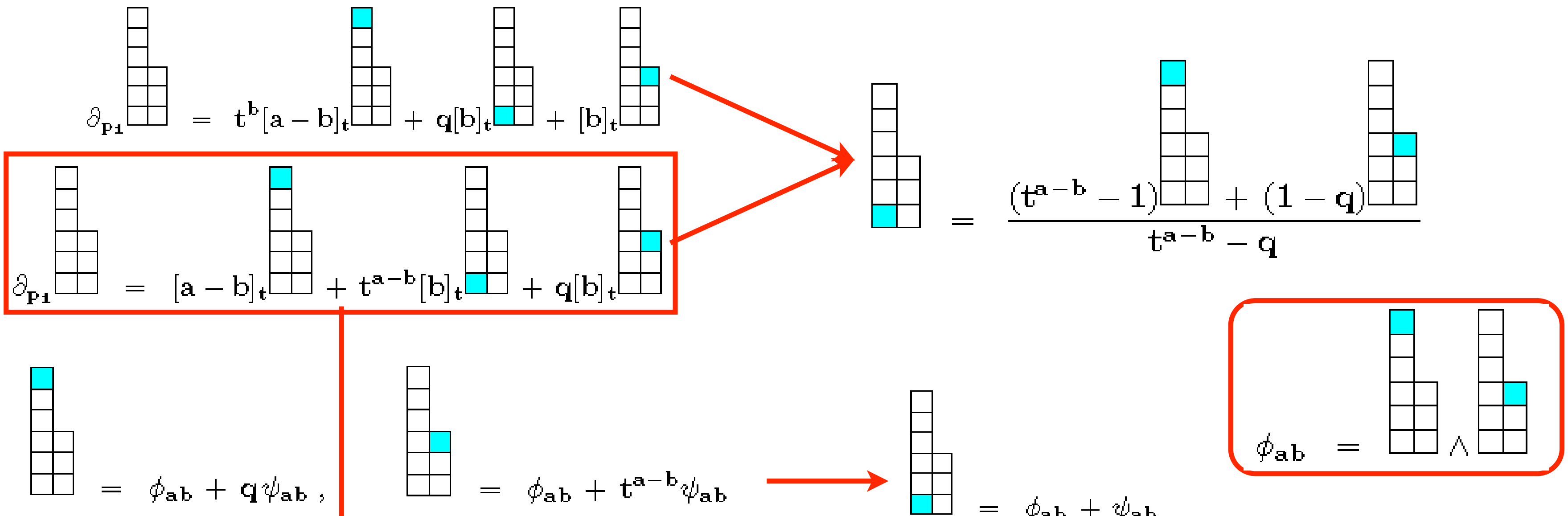
$$t^2\boxplus(q/t,t) \;=\; \begin{bmatrix} A_{22} & A_{211} & 0 \\ 0 & A_{211} & 0 \\ 0 & 0 & A_{1111} \end{bmatrix} \;=\; t^2\phi_{32}+q\psi_{32}$$

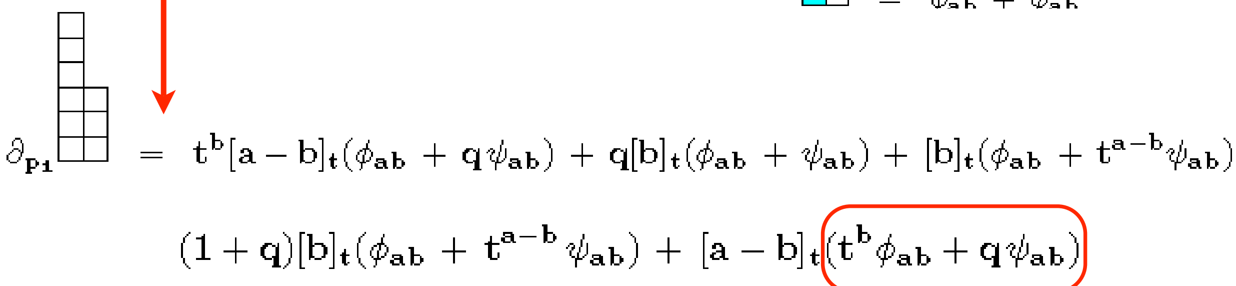
and using this in (*) we are reduced to show that

$$\partial_{p_1}\square_{221}(q,t) \;=\; (1+t)(1+q)\big(\phi_{32}+t\psi_{32}\big) + t^2\phi_{32}+q\psi_{32}$$

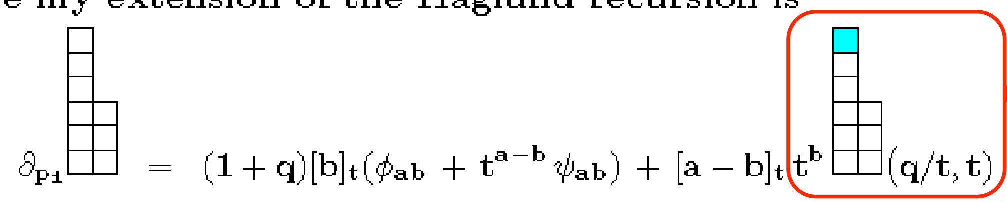
next

Using “pistol” Macdonalds

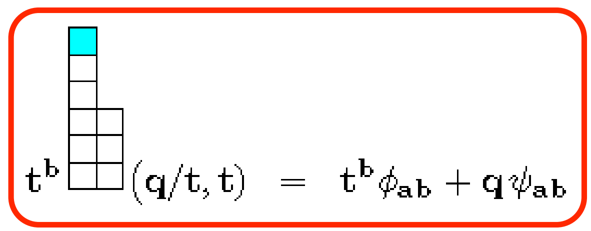




while my extension of the Haglund recursion is



so we are reduced to proving the identity



we are still Searching for "gistol" Macdonalds

(1) A "gistol" is a lattice diagram that can be transformed to a skew diagram by row and column interchanges

(2) We postulate the existence of a family of polynomials indexed by gistols with the following basic properties:

$$
\begin{cases}
(0) \quad G_D(x;q,t) = \tilde{H}_\mu(x;q,t) & \text{if } D \text{ is the diagram of } \mu \\
(1) \quad G_{D_1}(x;q,t) = G_{D_2}(x;q,t) & \text{if } D_1 \approx D_2 \\
(2) \quad G_{D_1}(x;q,t) = G_{D_2}(x;t,q) & \text{if } D_2 \approx D_1' \\
(3) \quad G_D(x;q,t) = G_{D_1}(x;q,t)G_{D_2}(x;q,t) & \text{if } D \approx D_1 \times D_2 \\
(4) \quad \partial_{p_1} G_D(x;q,t) = \sum_{s \in D} w_{s,D}(q,t)\; G_{D/s}(x;q,t) \;,
\end{cases}
$$

Representation theoretical reasons suggest that, in the case that D *is a skew diagram,*

$$\text{a)} \quad w_1[s,D] = t^{l_D(s)} q^{a'_D(s)} \qquad \text{and} \qquad \text{b)} \quad w_2[s,D] = t^{l'_D(s)} q^{a_D(s)}$$

Note: these properties overdetermine the family $\{G_D(x;q,t)\}_D$*,*

existence is by no means guaranteed.

next

THE END