COMMUTATION RELATIONS BETWEEN MULTIPLICATION OPERATIONS IN $NSym$ AND $QSym^*$

MIKE ZABROCKI

To begin we will need to review the product and coproduct rules of the main bases of algebras $NSym$ and $QSym$. I state them here without proof (some of them would usually be taken as definition). I will use bold face letters to indicate elements of $NSym$ and capital letters to indicate elements of $QSym$.

A few notational remarks: If a composition has only one entry (r), then I may simplify notation by just r.

Proposition 1. *The product/coproduct rules of $NSym$. Let $\mathbf{h}_\alpha := \mathbf{h}_{\alpha_1}\mathbf{h}_{\alpha_2}\cdots\mathbf{h}_{\alpha_{\ell(\alpha)}}$, then*

$$\mathbf{h}_\alpha \mathbf{h}_\beta = \mathbf{h}_{(\alpha,\beta)}$$

Since, $\Delta(\mathbf{h}_n) = \sum_{i=0}^{n} \mathbf{h}_i \otimes \mathbf{h}_{n-i}$, then

$$\Delta(\mathbf{h}_\alpha) = \sum_{\beta+\gamma=\alpha} \mathbf{h}_\beta \otimes \mathbf{h}_\gamma$$

where in this last sum the sum is over all weak compositions β and γ of length $\ell(\alpha)$.

Proposition 2. *The product rules in $QSym$. Let $\{M_\alpha\}_\alpha$ be the dual basis to $\{\mathbf{h}_\alpha\}_\alpha$. Then*

$$M_\alpha M_\beta = \sum_{\gamma \in \alpha \widetilde{\sqcup\!\sqcup} \beta} M_\gamma$$

where for $a, b \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^r$ and $\beta \in \mathbb{Z}^s$ for $r, s \geq 0$,

$$(a,\alpha)\widetilde{\sqcup\!\sqcup}(b,\beta) = a\cdot(\alpha\widetilde{\sqcup\!\sqcup}(b,\beta)) \uplus b\cdot((a,\alpha)\widetilde{\sqcup\!\sqcup}\beta) + \uplus(a+b)\cdot(\alpha\widetilde{\sqcup\!\sqcup}\beta)$$

where $a \cdot S$ represents the operation of concatenating an entry a in front of each of the elements of S and $\alpha\widetilde{\sqcup\!\sqcup}() = ()\widetilde{\sqcup\!\sqcup}\alpha = \{\alpha\}$. The coproduct rule is

$$\Delta(M_\alpha) = \sum_{(\beta,\gamma)=\alpha} M_\beta \otimes M_\gamma \ .$$

The dual pairing between $QSym$ and $NSym$ will be denoted $\langle \cdot, \cdot \rangle : NSym \otimes QSym \to \mathbb{Q}$. The dual pairing is defined on the basis $\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$. Note that the product and the coproduct and the pairing are defined so that

$$\langle \mathbf{fg}, H \rangle = \langle \mathbf{f} \otimes \mathbf{g}, \Delta(H) \rangle$$

and

$$\langle \mathbf{g}, GH \rangle = \langle \Delta(\mathbf{f}), G \otimes H \rangle \ .$$

Now the operators which are dual to multiplication by an element of $QSym$ will be denoted $G^\perp$ where $G \in QSym$ and is defined to be

$$G^\perp(\mathbf{f}) = \sum_\alpha \langle \mathbf{f}, GM_\alpha \rangle \mathbf{h}_\alpha \ .$$

The elements which are dual to the left multiplication by an element of $NSym$ will be denoted

$${}^L\mathbf{f}^\perp(G) = \sum_\alpha \langle \mathbf{f}\mathbf{h}_\alpha, G \rangle M_\alpha$$

and the dual to right multiplication will be denoted

$${}^R\mathbf{f}^\perp(G) = \sum_\alpha \langle \mathbf{h}_\alpha \mathbf{f}, G \rangle M_\alpha \ .$$

In order to derive the commutation relations, we need the following general formula for computing the action of an element dual to multiplication by its action on a product. This result is more general (the proof works for any pair of dual graded Hopf algebras, but I choose to state it for $QSym$ in particular.

Proposition 3. *For $G \in QSym$ and $\mathbf{f}, \mathbf{g} \in NSym$, if $\Delta(G) = \sum_i G^{(i)} \otimes G_{(i)}$, then*

$$G^\perp(\mathbf{f}\mathbf{g}) = \sum_i G^{(i)\perp}(\mathbf{f}) G^\perp_{(i)}(\mathbf{g}) \ .$$

Proof.

$$\begin{aligned}
G^\perp(\mathbf{f}\mathbf{g}) &= \sum_\beta \langle \mathbf{f}\mathbf{g}, GM_\alpha \rangle \mathbf{h}_b eta \\
&= \sum_\beta \langle \mathbf{f} \otimes \mathbf{g}, \Delta(GM_\alpha) \rangle \mathbf{h}_b eta \\
&= \sum_\beta \langle \mathbf{f} \otimes \mathbf{g}, \Delta(G)\Delta(M_\alpha) \rangle \mathbf{h}_b eta \\
&= \sum_\beta \sum_i \langle \mathbf{f} \otimes \mathbf{g}, (G^{(i)} \otimes G_{(i)})\Delta(M_\alpha) \rangle \mathbf{h}_b eta \\
&= \sum_\beta \sum_i \langle G^{(i)\perp}(\mathbf{f}) \otimes G^\perp_{(i)}(\mathbf{g}), \Delta(M_\alpha) \rangle \mathbf{h}_b eta \\
&= \sum_\beta \sum_i \langle G^{(i)\perp}(\mathbf{f}) G^\perp_{(i)}(\mathbf{g}), M_\alpha \rangle \mathbf{h}_b eta \\
&= \sum_i G^{(i)\perp}(\mathbf{f}) G^\perp_{(i)}(\mathbf{g})
\end{aligned}$$

□

A special case of Proposition 2 is $\Delta(M_n) = M_n \otimes 1 + 1 \otimes M_n$. Therefore we have as a corollary

Corollary 4. *Using the notation* $[A,B] = AB - BA$,

$$[M_n^\perp, {}^L\mathbf{f}] = {}^L(M_n^\perp \mathbf{f}) \ .$$

Proof. Proposition 3 says that

$$M_n^\perp(\mathbf{fg}) = M_n^\perp(\mathbf{f})\mathbf{g} + \mathbf{f}M_n^\perp(\mathbf{g}) \ .$$

Cast in terms of operators this says

$$M_n^\perp({}^L\mathbf{f}(\mathbf{g})) - {}^L\mathbf{f}(M_n^\perp(\mathbf{g})) = {}^L(M_n^\perp(\mathbf{f}))(\mathbf{g}) \ .$$

In terms of the bracket notation, this equation can be written as

$$[M_n^\perp, {}^L\mathbf{f}](\mathbf{g}) = {}^L(M_n^\perp(\mathbf{f}))(\mathbf{g}) \ .$$

□

Since the elements M_n do not generate $QSym$, this isn't enough to generate the algebra. In fact, we need to know the commutation of M_α for any composition α. What we will do is give the commutation with those elements with a set of generators of $NSym$. Technically, we only need to know how the $M_\alpha^\perp$ commute for the generators of the algebra (those indexed by the Lyndon compositions), but the formula we will present here indicates the properties of the indexing composition do not play a significant role in the formula.

Proposition 5.

$$[M_\alpha^\perp, {}^L\mathbf{h}_n] = {}^L\mathbf{h}_{n-\alpha_1} M^\perp_{(\alpha_2,\alpha_3,\dots,\alpha_{\ell(\alpha)})}$$

and

$$[M_\alpha^\perp, {}^R\mathbf{h}_n] = {}^R\mathbf{h}_{n-\alpha_{\ell(\alpha)}} M^\perp_{(\alpha_1,\alpha_2,\dots,\alpha_{\ell(\alpha)-1})}$$

In order to show this result we need to know the action of $M_\alpha^\perp$ on $\mathbf{h}_n$.

Lemma 6.

$$M_\alpha^\perp(\mathbf{h}_n) = \begin{cases} 0 & \text{if } \ell(\alpha) > 1 \\ \mathbf{h}_{n-r} & \text{if } \alpha = (r) \end{cases}$$

with the convention that $\mathbf{h}_{n-r} = 0$ *if* $r > n$.

Proof.

$$M_\alpha^\perp(\mathbf{h}_n) = \sum_\beta \langle \mathbf{h}_n, M_\alpha M_\beta \rangle \mathbf{h}_\alpha$$

and since the terms of $M_\alpha M_\beta$ are those in the quasi-shuffle of the compositions α and β, and since if $\gamma \in \alpha \widetilde{\sqcup\!\sqcup} \beta$ then $\ell(\gamma) \geq max(\ell(\alpha), \ell(\beta))$, then it must be that the only terms $\langle \mathbf{h}_n, M_\alpha M_\beta \rangle$ that are non-zero are those where $\ell(\alpha) = \ell(\beta) = 1$. If $\alpha = (r)$, then it must be that $\beta = (n-r)$ and

$$M_\alpha^\perp(\mathbf{h}_n) = \langle \mathbf{h}_n, M_r M_{n-r} \rangle \mathbf{h}_{n-r} = \langle \mathbf{h}_n, M_{(r,n-r)} + M_{(n-r,r)} + M_n \rangle \mathbf{h}_{n-r} = \mathbf{h}_{n-r}.$$

□

Proof. (of Proposition 5) Using Proposition 3, we have that

$$M_\alpha^{\perp L}\mathbf{h}_n(\mathbf{g}) = M_\alpha^\perp(\mathbf{h}_n\mathbf{g}) = \sum_{(\beta,\gamma)=\alpha} M_\beta^\perp(\mathbf{h}_n)M_\gamma^\perp(\mathbf{g}) \ .$$

Now by Lemma 6, the only terms that are non-zero in this sum are those where $\beta = ()$, $\gamma = \alpha$ and $\beta = \alpha_1$, $\gamma = (\alpha_2, \ldots, \alpha_\ell)$ and so this sum is equal to

$$M_\alpha^\perp({}^L\mathbf{h}_n(\mathbf{g})) = \mathbf{h}_n M_\alpha^\perp(\mathbf{g}) + \mathbf{h}_{n-\alpha_1} M_{(\alpha_2,\ldots,\alpha_\ell)}^\perp(\mathbf{g}) = {}^L\mathbf{h}_n(M_\alpha^\perp(\mathbf{g})) + {}^L\mathbf{h}_{n-\alpha_1}(M_{(\alpha_2,\ldots,\alpha_\ell)}^\perp(\mathbf{g}))$$

Therefore,

$$[M_\alpha^\perp, {}^L\mathbf{h}_n](\mathbf{g}) = M_\alpha^\perp({}^L\mathbf{h}_n(\mathbf{g})) - {}^L\mathbf{h}_n(M_\alpha^\perp(\mathbf{g})) = {}^L\mathbf{h}_{n-\alpha_1}(M_{(\alpha_2,\ldots,\alpha_\ell)}^\perp(\mathbf{g})) \ .$$

The proof that $[M_\alpha^\perp, {}^R\mathbf{h}_n] = {}^R\mathbf{h}_{n-\alpha_{\ell(\alpha)}} M_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell(\alpha)-1})}^\perp$ is similar and uses exactly the same identities. □

If we consider the action of ${}^L\mathbf{h}_n$ and ${}^R\mathbf{h}_m$ on a basis of $NSym$, then we see that they commute

$${}^L\mathbf{h}_n({}^R\mathbf{h}_m(\mathbf{h}_\alpha)) = {}^L\mathbf{h}_n(\mathbf{h}_{(\alpha,m)}) = \mathbf{h}_{(n,\alpha,m)} = {}^R\mathbf{h}_m({}^L\mathbf{h}_n(\mathbf{h}_\alpha))$$

hence

$$[{}^L\mathbf{h}_n, {}^R\mathbf{h}_m] = 0 \ .$$

Also, since $QSym$ is commutative,

$$[M_\alpha^\perp, M_\beta^\perp] = 0 \ .$$

I will not prove here, but the arguments are again analogous (and probably can be derived from those above), that we have as elements of $End(QSym)$,

$$[{}^L\mathbf{h}_n^\perp, M_\alpha] = M_{(\alpha_2,\alpha_3,\ldots,\alpha_{\ell(\alpha)})}{}^L\mathbf{h}_{n-\alpha_1}^\perp$$

$$[{}^R\mathbf{h}_n^\perp, M_\alpha] = M_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell(\alpha)-1})}{}^R\mathbf{h}_{n-\alpha_{\ell(\alpha)}}^\perp$$

$$[{}^L\mathbf{h}_n^\perp, {}^R\mathbf{h}_m^\perp] = 0$$

$$[M_\alpha, M_\beta] = 0$$