FROM LOCAL TO GLOBAL VECTOR FIELDS ON THE SPHERE

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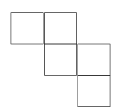
1. TRYING TO DEFINE KERNEL

(in discussing with Serdar Sozubek) Let $\sigma = \mathrm{Id}_5 = 12345$. I will describe the local flow of the kernel (open region) for each composition $a \models 5$. That is the set of (b,τ) such that $\langle (b,\tau),(a,\sigma)\rangle = 1$ (otherwise it is zero). The first observation is that we must have that τ is such that the descent composition $D(\tau^{-1}\sigma) = a$. This happens to be an interval in the weak order, in particular it is a connected set of simplexes. The interval is described as follow. Fix a composition $a = 221$. Draw the ribbon shape associated to it (that is draw a_i box in each row; The first row being at the top, draw the next row starting bellow the last box of the previous row). For $a = 221$ this is



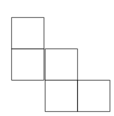
Now we fill this shape with the number $1, 2, 3, \ldots, n$ in each column, from left to right, bottom to top. In the case above this gives

$$\begin{array}{ccc}1&3&\\&2&5\\&&4\end{array}$$

Now read the permutation you get reading as usual, row by row, top to bottom, left to right:

$$\tau_0 = 13254$$

This is the bottom of the interval we want. The top element is obtain as follow. Rotate the shape of a by 180°



Now, fill the number $1, 2, 3, \ldots, n$ row by row, starting at the bottom, from left to right

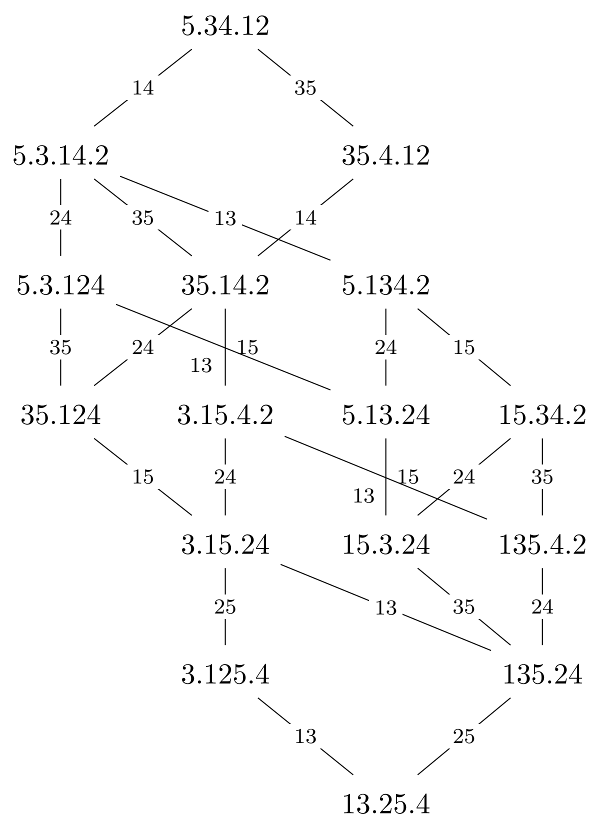
$$\begin{array}{ccc}5&&\\3&4&\\&1&2\end{array}$$

and read top to bottom, row by row

$$\tau_1 = 53412$$

The permutation $\tau \in [\tau_0, \tau_1]$ defines the open region of interest. [What I describe is a Theorem of Thibon and Krob in the paper about projective rep. of $H_n(0)$] Its fairly easy to construct the interval $[\tau_0, \tau_1]$. One start with τ_0 and transpose only adjacent entries $\tau = \ldots xy \ldots$ of the permutation $x < y$ such that $\{x, y\}$ are not in the same block of σ cut by a. In our running example $\sigma = 12345$ and $a = 221$, so we are not allowed to transpose $\{1, 2\}$ and $\{3, 4\}$. We start with $\tau_0 = 13254$ and go up transposing any $x < y$ up in the order. That gives

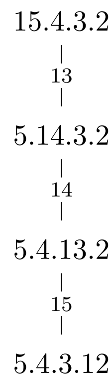
$\mathbf{a = 221}$



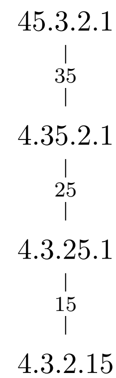
In this picture, we read of the composition $b = D(\sigma^{-1}\tau) = D(\tau)$ easily. I put the dot to indicate the descents. The set of all intervals are as follow. Here I fix $\sigma = 12345$.

$\mathbf{a = 11111}$ 5.4.3.2.1;

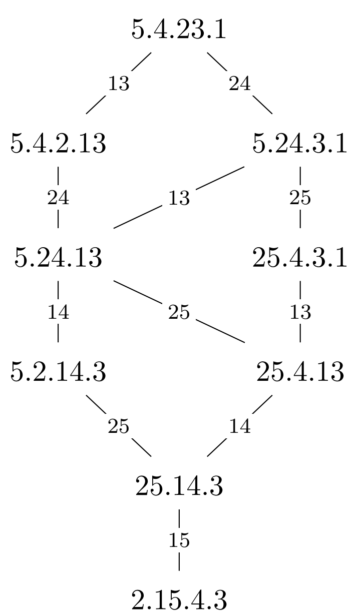
$\mathbf{a = 2111}$



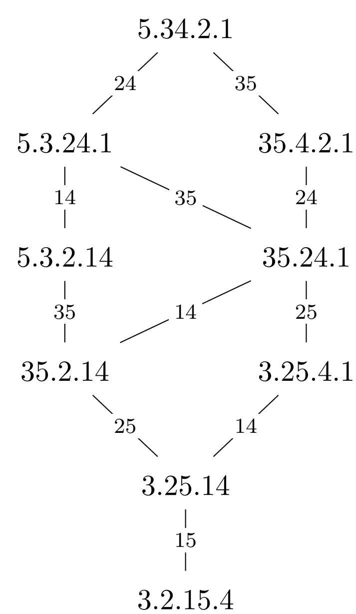
$\mathbf{a = 1112}$



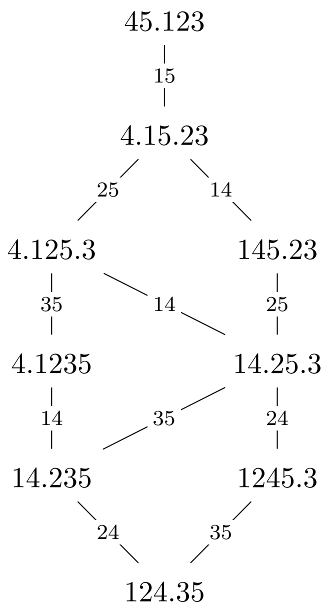
$\mathbf{a = 1211}$



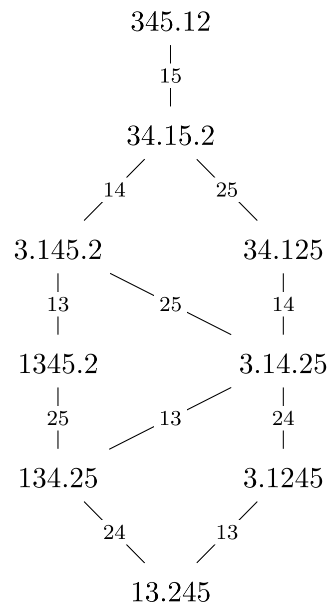
$\mathbf{a = 1121}$



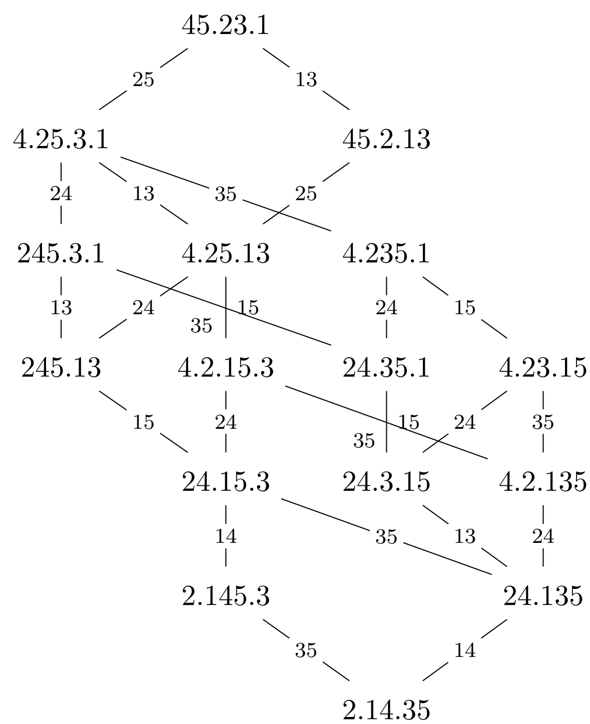
a = 32


45.123
15
4.15.23
25
14
4.125.3
145.23
35
14
25
4.1235
14.25.3
14
35
24
14.235
1245.3
24
35
124.35

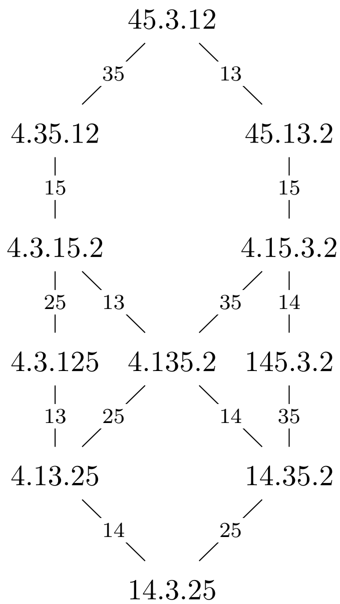
a = 23


345.12
15
34.15.2
14
25
3.145.2
34.125
13
25
14
1345.2
3.14.25
25
13
24
134.25
3.1245
24
13
13.245

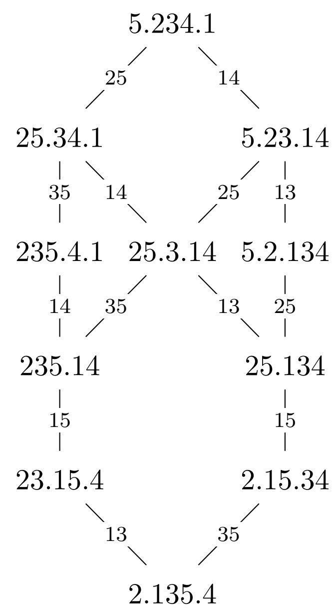
a = 122


45.23.1
25
13
4.25.3.1
45.2.13
24
13
35
25
245.3.1
4.25.13
4.235.1
13
24
35
15
24
15
245.13
4.2.15.3
24.35.1
4.23.15
15
24
35
15
24
35
24.15.3
24.3.15
4.2.135
14
35
13
24
2.145.3
24.135
35
14
2.14.35

$\mathbf{a = 212}$



$\mathbf{a = 131}$



$\mathbf{a = 113}$

$$\begin{tikzcd}[column sep=small]
 & 345.2.1 \arrow[d, dash, "25"] & \\
 & 34.25.1 \arrow[dl, dash, "15"'] \arrow[dr, dash, "24"] & \\
34.2.15 \arrow[dr, dash, "24"'] & & 3.245.1 \arrow[dl, dash, "15"] \\
 & 3.24.15 \arrow[d, dash, "14"] & \\
 & 3.2.145 &
\end{tikzcd}$$

$\mathbf{a = 311}$

$$\begin{tikzcd}[column sep=small]
 & 5.4.123 \arrow[d, dash, "14"] & \\
 & 5.14.23 \arrow[dl, dash, "15"'] \arrow[dr, dash, "24"] & \\
15.4.23 \arrow[dr, dash, "24"'] & & 5.124.3 \arrow[dl, dash, "15"] \\
 & 15.24.3 \arrow[d, dash, "25"] & \\
 & 125.4.3 &
\end{tikzcd}$$

$\mathbf{a = 5}$ 12345;

$\mathbf{a = 41}$

$$\begin{tikzcd}
5.1234 \arrow[d, dash, "15"] \\
15.234 \arrow[d, dash, "25"] \\
125.34 \arrow[d, dash, "35"] \\
1235.4
\end{tikzcd}$$

$\mathbf{a = 14}$

$$\begin{tikzcd}
2345.1 \arrow[d, dash, "15"] \\
234.15 \arrow[d, dash, "14"] \\
23.145 \arrow[d, dash, "13"] \\
2.1345
\end{tikzcd}$$

Remark that the region for (a, σ) for $a \models n$ and the region for $(\overleftarrow{a}, \overleftarrow{\sigma})$ are isomorphic with the antipodal map. There are other symmetries as well. I have some thoughts on how to construct possible kernel element. I can generate some of them using local rules... but I don't have yet a full understanding of how to get all kernel elements. [I still think the "flows" will be particularly interesting elements, and the technics I present here does not give particularly nice flows.] For $n = 5$ only a of the form 122, 221, 131 (each with only two points), 23, 32 (with six points) and 41, 14 (with 4 points), involve points.

Remark that in each region, we always have the same number of points of type 23 and 32. same thing happen with 14 and 41. What if we start with $a = 131$ and choose $235.14 - 25.134$. I will always choose to match point of type 23 with points of types 32 (and the same with 14 and 41). I will always put $+1$ for 32 and -1 for 23 (for now I can choose to ignore the point of type 41 and 14). Can I patch the sphere with the region with this? So 235.14 appears in six types of regions. Each region do the following

$a = 131$ gives: 2**3**5.14 − 25.1**3**4
$a = 122$ gives: 24**5**.13 − 24.13**5**
$a = 221$ gives: **1**35.24 − 35.**1**24

$a = 41$ gives: $\mathbf{1}\mathbf{2}5.34 - 15.\mathbf{2}34$
$a = 14$ gives: $23\mathbf{4}.15 - 23.1\mathbf{4}5$

Region for a of type 23 and 32, each cover 235.14 in two different ways in this gives more choices (I would just add both points of type 32 in the two different ways and subtract 32). From 235.14 I get

$$235.14 - 25.134 - 23.145 - 35.214 - 25.314 - 23.154 - .... \tag{1.1}$$

Now I want to propagate this with each point of type 23 that I obtained (each will more new points and so on). With -25.134 I get

$$-25.134 + 125.13 + 254.13 + 235.14 + 215.34 + 253.14 + ...$$

We already have 235.14 so we are adding less new elements. I continue with the other elements of type 23 in the expression (**??**). For 23.145 I get

$$-23.145 + 243.15 + 235.14 + 123.45 + 213.45 + 234.15 + ...$$

What happen if we continue [...]

Theorem 1.1 (for general n)**.** *Every open region has the same number of points of type α and $\overleftarrow{\alpha}$. We can thus start with an arrow of type α and propagate (using open regions) to a global element of the kernel* **assuming** $\alpha \neq \overleftarrow{\alpha}$*. If $\alpha = \overleftarrow{\alpha}$ the procedure fail so that this does not define element of the kernel for points of this type. This is to be expected.*

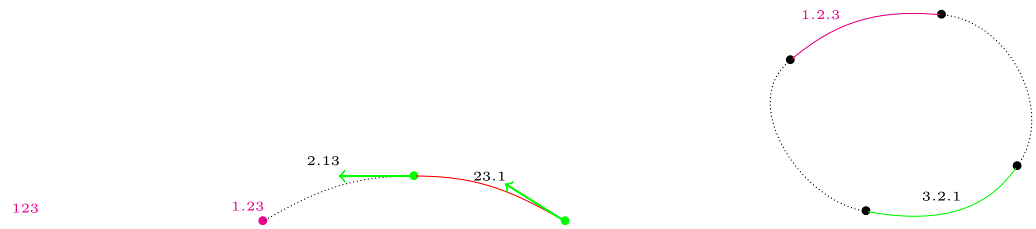
Observations/Questions

This requires only to prove some property about the open regions and the elements we pick in them. These regions are studied as projective modules of $H_n(0)$. This should give use many elements of the kernel. but it does not give us all. For $n = 4$ I get dim $= 18$ linearly independent element. [THANKS Franco to set up the notebook with full access to your code] I still need the "special" flow. I am trying to see if using the region I can come up with "rules" to generate flows. That is, instead of matching points according to their type, I what to match point inside region according to their location (living on paralell hyperplane)

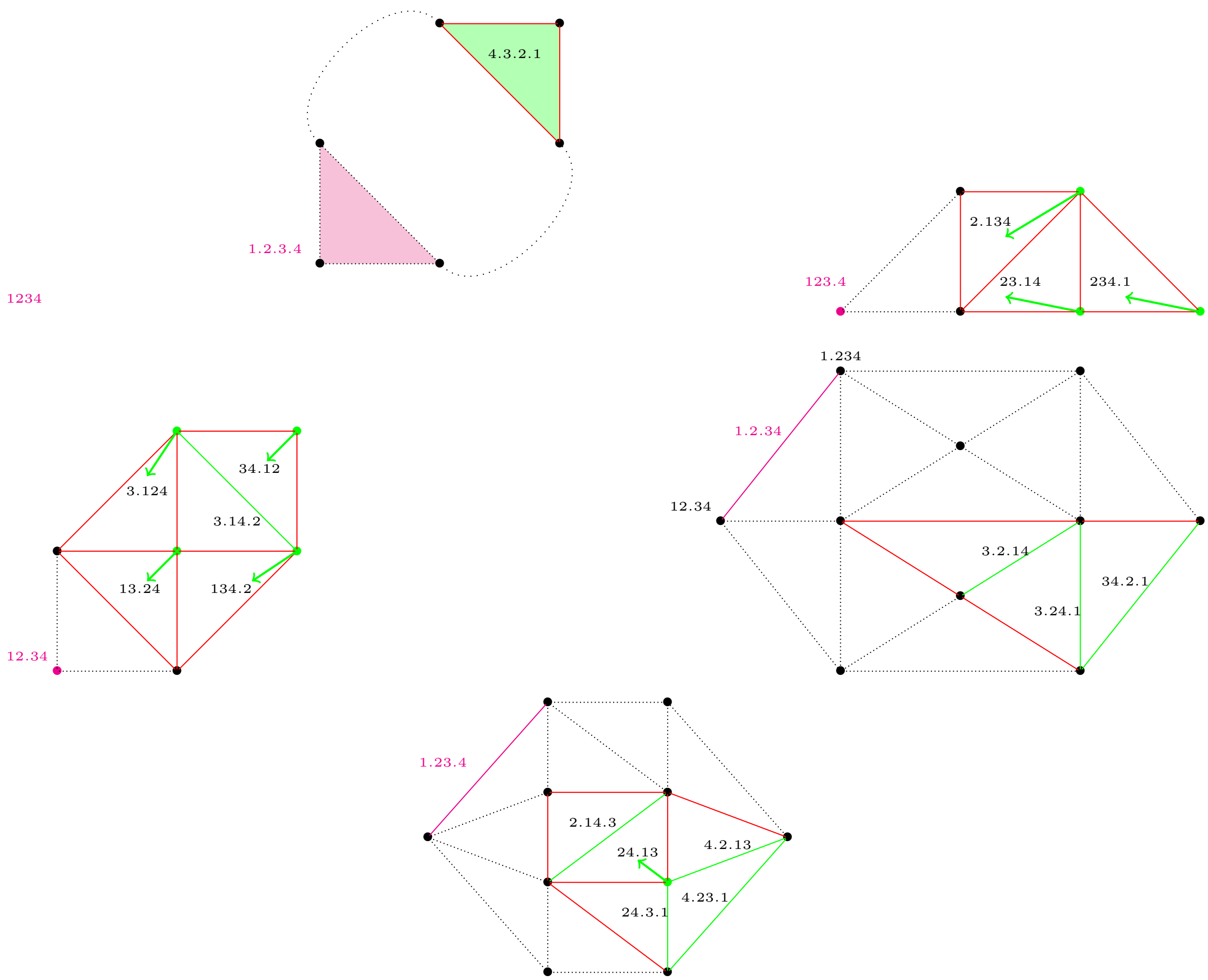
2. Region for n=3,4,5

I started drawing the open regions that are not too big [learning slowly how to use TikZ... it is fun]. In red is the *open* region with the vector drawn (for points only). The element defining the region is in magenta and the elements in the region are in green (with a vector for single points).

For $n = 3$ we have

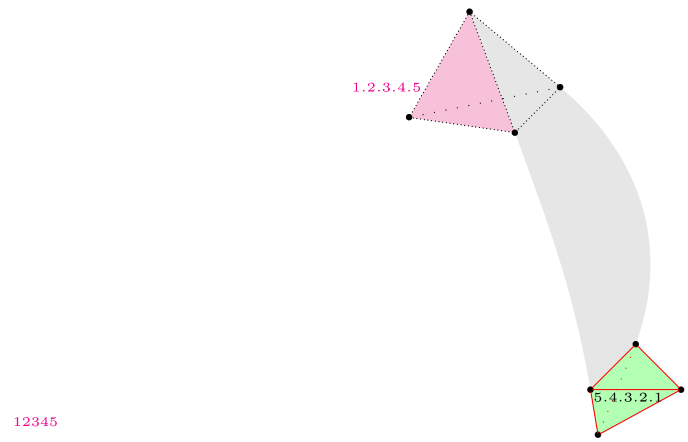


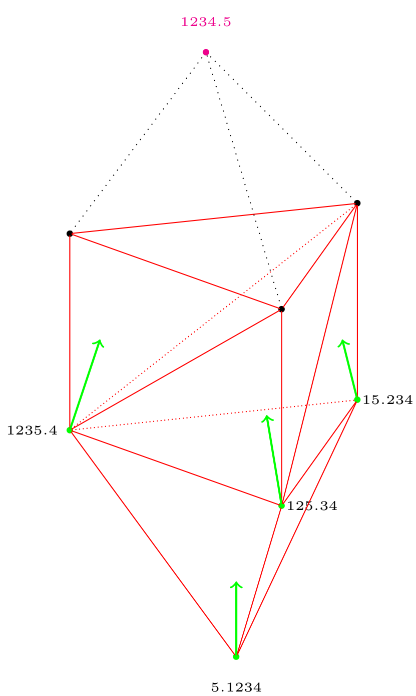
For $n = 4$ we have

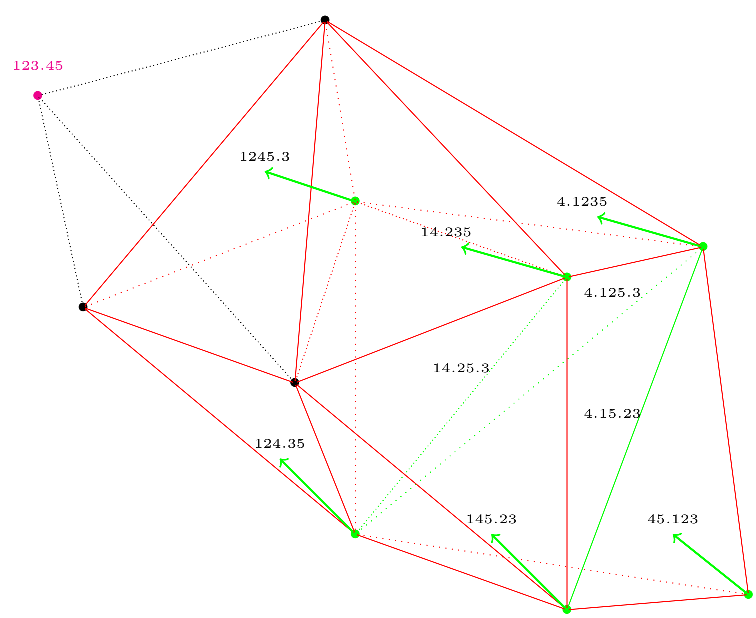


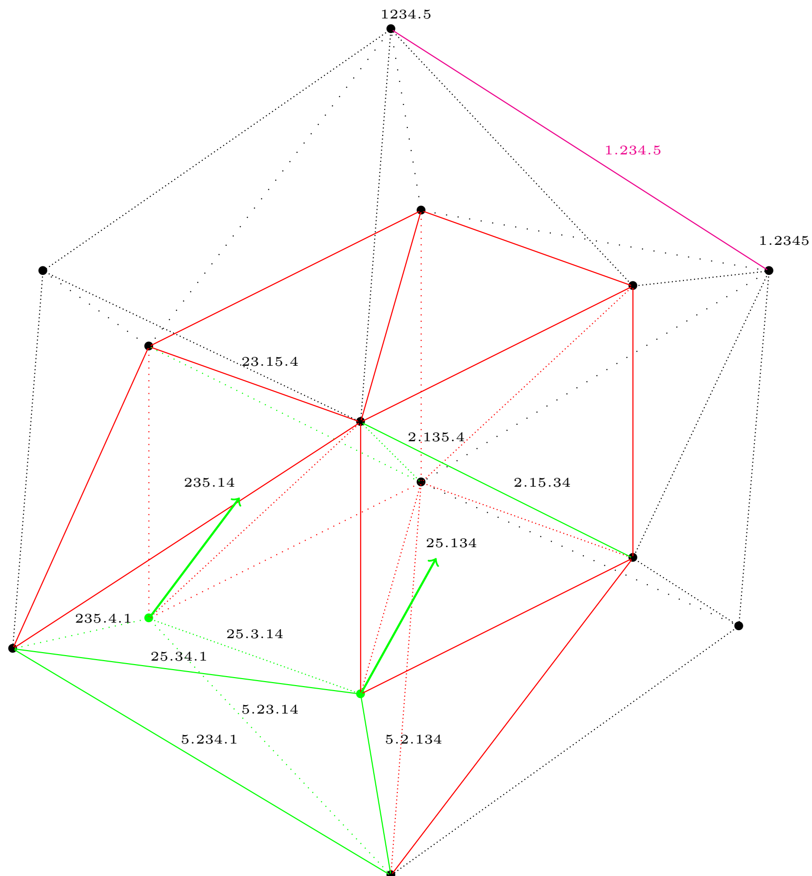
You see now that with the rule of the Theorem, the kernel MUST vanish on point of type 22 and 121. But we can define kernel element starting with a point of type 41 or 211.

Now, n=5


1.2.3.4.5
5.4.3.2.1
12345


1234.5
1235.4
15.234
125.34
5.1234


123.45
1245.3
4.1235
14.235
4.125.3
14.25.3
4.15.23
124.35
145.23
45.123


1234.5
1.234.5
1.2345
23.15.4
2.135.4
235.14
2.15.34
25.134
235.4.1
25.3.14
25.34.1
5.23.14
5.234.1
5.2.134