$\mathfrak{sl}_2$ - trace 0 2×2 matrices over $\mathbb{Q}$

The elements $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

form a basis for $\mathfrak{sl}_2$ satisfying

$[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$.

The universal enveloping algebra of $\mathfrak{sl}_2$ is

the associative algebra over $\mathbb{Q}$ with 1 given by

$$\mathcal{U}(\mathfrak{sl}_2) = \langle e, f, h \mid he - eh = 2e,\ hf - fh = -2f,\ ef - fe = h \rangle$$

The quantized universal enveloping algebra

of $\mathfrak{sl}_2$ is the associative algebra over $\mathbb{Q}(q)$

with 1 given by

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle e, f, t, t^{-1} \mid tet^{-1} = q^2 e,\ tft^{-1} = q^{-2} f,\ [e,f] = \frac{t - t^{-1}}{q - q^{-1}},\ tt^{-1} = t^{-1}t = 1 \rangle$$

Let $A_1 := \{ f/g \mid f, g \in \mathbb{Q}[q] \text{ and } g(1) \neq 0 \}$

Let U_{A_1} be the A_1-subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ generated by $e, f, \frac{t-1}{q-1}, t, t^{-1}$.

Let $U_1 = U_{A_1} / (q-1)U_{A_1}$

U_1 is isomorphic to $\mathcal{U}(\mathfrak{sl}_2)$:

Define $\phi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow U_1$ by

$$\begin{aligned} e &\longmapsto \bar{e} := e + (q-1)U_{A_1} \\ f &\longmapsto \bar{f} := f + (q-1)U_{A_1} \\ h &\longmapsto \bar{h} := \frac{t-1}{q-1} + (q-1)U_{A_1} \end{aligned}$$

We have $t-1 = (q-1)\frac{t-1}{q-1} \in (q-1)U_{A_1}$.

Hence $t = 1 \bmod (q-1)U_{A_1}$ (also $t^{-1} = 1 \bmod (q-1)U_{A_1}$)

So ϕ is onto.

We now show that ϕ is well defined:

$$\bar{e}\bar{f} - \bar{f}\bar{e} = \frac{t-t^{-1}}{q-q^{-1}} + (q-1)U_{A_1} = \frac{t^{-1}}{q^{-1}}\,\frac{t+1}{q+1}\,\bar{h} = \bar{h}$$

$$\frac{t-1}{q-1}e - e\frac{t-1}{q-1} = e\,\frac{q^2t-1}{q-1} - e\,\frac{t-1}{q-1}$$

$$= \frac{q^2-1}{q-1}\,et = (q+1)et$$

Hence $[\bar{h},\bar{e}] = 2\bar{e}$. $\Box$

Similarly $[\bar{h},\bar{f}] = -2\bar{f}$

Finite dimensional representations of $\mathfrak{sl}_2$ (and $\mathcal{U}(\mathfrak{sl}_2)$)

For each $m \in \mathbb{N}$, define an $(m+1)$-dimensional $\mathfrak{sl}_2$-module by $V(m) = \bigoplus_{k=0}^{m} \mathbb{Q}\, u_k^{(m)}$

$$h \cdot u_k^{(m)} = (m-2k)\, u_k^{(m)}$$

$$f \cdot u_k^{(m)} = (k+1)\, u_{k+1}^{(m)}$$

$$e \cdot u_k^{(m)} = (m-k+1)\, u_{k-1}^{(m)}$$

(where $u_{-1}^{(m)} = u_{m+1}^{(m)} = 0$)

- Facts:
  1. Every f.d. irreducible $\mathfrak{sl}_2$-module is equal to $V(m)$ for some $m \in \mathbb{N}$.
  2. Every f.d. $\mathfrak{sl}_2$-module is completely reducible.

Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$

For $m \in \mathbb{N}$, define an $(m+1)$-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module by $V^q(m) = \bigoplus_{k=0}^{m} \mathbb{Q}(q)\, u_k^{(m)}$

$$t\, u_k^{(m)} = q^{m-2k}\, u_k^{(m)}$$

$$f\, u_k^{(m)} = [k+1]\, u_{k+1}^{(m)}$$

$$e\, u_k^{(m)} = [m-k+1]\, u_{k-1}^{(m)}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

Let $V_{A_1}^q(m) := \bigoplus_{k=0}^{m} A_1 u_k^{(m)}$.

Note: $U_{A_1} / (q-1) U_{A_1}$ acts on $V_{A_1}^q(m) / (q-1) V_{A_1}^q(m)$

$$\begin{array}{ccc} U_{A_1} / (q-1) U_{A_1} & \text{acts on} & V_{A_1}^q(m) / (q-1) V_{A_1}^q(m) \\ \cong & & \cong \\ \mathcal{U}(\mathfrak{sl}_2) & & V(m) \end{array}$$

The weight of $u_k^{(m)} := \mathrm{wt}(u_k^{(m)}) = m - 2k$

Crystal Base of $V^q(m)$

Let $A_0 := \{ f/g \mid f, g \in \mathbb{Q}[q] \text{ and } g(0) \neq 0 \}$

Let $L(m) = \bigoplus_{k=0}^{m} A_0 u_k^{(m)}$ and

$$B(m) = \{ u_k^{(m)} \bmod q L(m) \mid k = 0, \dots, m \}$$

$(L(m), B(m))$ is called the crystal base of $V^q(m)$

Define $\tilde{f}$ on $B(m)$ by $\tilde{f}\,\overline{u}_k^{(m)} = \overline{u}_{k+1}^{(m)}$,

and $\tilde{e}$ by $\tilde{e}\,\overline{u}_k^{(m)} = \overline{u}_{k-1}^{(m)}$.

The crystal graph of $V^q(m)$ is the graph

$$\begin{tikzcd} \overline{u}_0^{(m)} \arrow[d] \\ \overline{u}_1^{(m)} \arrow[d, dotted] \\ \overline{u}_{m-1}^{(m)} \arrow[d] \\ \overline{u}_m^{(m)} \end{tikzcd}$$

If $V \cong \bigoplus_{i=1}^{N} V^q(m_i)$, for $m_i \in \mathbb{N}$,

the crystal base of V is a pair (L, B) such that

- L is a free A_0-module with $V \cong \mathbb{Q}(q) \otimes_{A_0} L$
- B is a basis of the $\mathbb{Q}$ v.s. L/qL
- $L \cong \bigoplus_{i=1}^{N} L(m_i)$ and $B \cong \dot{\bigsqcup}_{i=1}^{N} B(m_i)$

V is integrable if $V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu$

$V_\mu := \{ v \in V \mid t v = q^\mu v \}$

Comultiplication on $\mathcal{U}_q(\mathfrak{sl}_2)$

$$\Delta : \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$$

$$t \longmapsto t \otimes t$$

$$e \longmapsto e \otimes t^{-1} + 1 \otimes e$$

$$f \longmapsto f \otimes 1 + t \otimes f$$

Example

Let $\boxed{1} := u_0^{(1)}$ and $\boxed{2} := u_1^{(1)}$.

So that

$$t\,\boxed{1} = q\,\boxed{1}\,,\quad e\,\boxed{1} = 0\,,\quad f\,\boxed{1} = \boxed{2}$$

$$t\,\boxed{2} = q^{-1}\,\boxed{2}\,,\quad e\,\boxed{2} = \boxed{1}\,,\quad f\,\boxed{2} = 0$$

$$V(1) = \mathbb{Q}(q)\,\boxed{1} \oplus \mathbb{Q}(q)\,\boxed{2}$$

$$\underbrace{V(1)\otimes V(1)}_{\text{Decomposes into irreducibles}} = \mathbb{Q}(q)\,\boxed{1}\otimes\boxed{1} \oplus \mathbb{Q}(q)\,\boxed{1}\otimes\boxed{2} \oplus \mathbb{Q}(q)\,\boxed{2}\otimes\boxed{1} \oplus \mathbb{Q}(q)\,\boxed{2}\otimes\boxed{2}$$

$$f(\boxed{1}\otimes\boxed{1}) = (f\otimes 1 + t\otimes f)(\boxed{1}\otimes\boxed{1}) = \boxed{2}\otimes\boxed{1} + q\,\boxed{1}\otimes\boxed{2}$$

$$e(\boxed{1}\otimes\boxed{1}) = 0$$

$$f(\boxed{1}\otimes\boxed{2}) = \boxed{2}\otimes\boxed{2}$$

$$e(\boxed{1}\otimes\boxed{2}) = \boxed{1}\otimes\boxed{1}$$

$$f(\boxed{2}\otimes\boxed{1}) = q^{-1}\,\boxed{2}\otimes\boxed{2}$$

$$e(\boxed{2}\otimes\boxed{1}) = q^{-1}\,\boxed{1}\otimes\boxed{1}$$

$$f(\boxed{2}\otimes\boxed{2}) = 0$$

$$e(\boxed{2}\otimes\boxed{2}) = q\,\boxed{1}\otimes\boxed{2} + \boxed{2}\otimes\boxed{1}$$

$$\boxed{1} \otimes \boxed{1} = u_0$$

$$\Big\downarrow f$$

$$\boxed{2} \otimes \boxed{1} + q\, \boxed{1} \otimes \boxed{2} = u_1 \qquad \boxed{1} \otimes \boxed{2} - q\, \boxed{2} \otimes \boxed{1} = v_0$$

$$\Big\downarrow \frac{1}{q+q^{-1}} f$$

$$\boxed{2} \otimes \boxed{2} = u_2$$

$$L = \underbrace{A_0 u_0 + A_0 u_1 + A_0 u_2}_{\substack{\cong \\ L(2)}} + \underbrace{A_0 v_0}_{\substack{\cong \\ L(0)}}$$

Crystal graph

$$\boxed{1} \otimes \boxed{1}$$

$$\Big\downarrow \tilde{f}$$

$$\boxed{2} \otimes \boxed{1} \qquad\qquad \boxed{1} \otimes \boxed{2}$$

$$\Big\downarrow \tilde{f}$$

$$\boxed{2} \otimes \boxed{2}$$