

\mathfrak{sl}_2 - trace 0 2×2 matrices over \mathbb{Q}

The elements $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

form a basis for \mathfrak{sl}_2 satisfying

$$[h, e] = 2e, [h, f] = -2f \text{ and } [e, f] = h.$$

The universal enveloping algebra of \mathfrak{sl}_2 is the associative algebra over \mathbb{Q} with 1 given by

$$U(\mathfrak{sl}_2) = \langle e, f, h \mid he - eh = 2e, hf - fh = -2f, ef - fe = h \rangle$$

The quantized universal enveloping algebra of \mathfrak{sl}_2 is the associative algebra over $\mathbb{Q}(q)$ with 1 given by

$$U_q(\mathfrak{sl}_2) = \langle e, f, t, t^{-1} \mid tet^{-1} = q^2e, tft^{-1} = q^{-2}f, [e, f] = \frac{t - t^{-1}}{q - q^{-1}}, tt^{-1} = t^{-1}t = 1 \rangle$$

Let $A_1 := \{ f/g \mid f, g \in \mathbb{Q}[q] \text{ and } g(1) \neq 0 \}$

Let U_{A_1} be the A_1 -subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $e, f, \frac{t-1}{q-1}, t, t^{-1}$.

Let $U_1 = U_{A_1} / (q-1)U_{A_1}$

U_1 is isomorphic to $U(\mathfrak{sl}_2)$:

Define $\phi: U(\mathfrak{sl}_2) \rightarrow U_1$ by

$$\begin{aligned} e &\longmapsto \bar{e} := e + (q-1)U_{A_1} \\ f &\longmapsto \bar{f} := f + (q-1)U_{A_1} \\ h &\longmapsto \bar{h} := \frac{t-1}{q-1} + (q-1)U_{A_1} \end{aligned}$$

We have $t-1 = (q-1) \frac{t-1}{q-1} \in (q-1)U_{A_1}$.

Hence $t = 1 \pmod{(q-1)U_{A_1}}$, (also $t^{-1} = 1 \pmod{(q-1)U_{A_1}}$)

So ϕ is onto.

We now show that ϕ is well defined:

$$\bar{e}\bar{f} - \bar{f}\bar{e} = \frac{t-t^{-1}}{q-q^{-1}} + (q-1)U_{A_1} = \frac{t^{-1}}{q^{-1}} \frac{t+1}{q+1} \bar{h} = \bar{h}$$

$$\begin{aligned} \frac{t-1}{q-1} e - e \frac{t-1}{q-1} &= e \frac{q^2 t-1}{q-1} - e \frac{t-1}{q-1} \\ &= \frac{q^2-1}{q-1} e t = (q+1) e t \end{aligned}$$

Hence $[\bar{h}, \bar{e}] = 2\bar{e}$. \square

Similarly $[\bar{h}, \bar{f}] = -2\bar{f}$

Finite dimensional representations of \mathfrak{sl}_2 (and $U(\mathfrak{sl}_2)$)

For each $m \in \mathbb{N}$, define an $(m+1)$ -dimensional \mathfrak{sl}_2 -module by $V(m) = \bigoplus_{k=0}^m \mathbb{Q} U_k^{(m)}$

$$h \cdot U_k^{(m)} = (m - 2k) U_k^{(m)}$$

$$f \cdot U_k^{(m)} = (k+1) U_{k+1}^{(m)}$$

$$e \cdot U_k^{(m)} = (m - k + 1) U_{k-1}^{(m)}$$

(where $U_{-1}^{(m)} = U_{m+1}^{(m)} = 0$)

• Facts:

1. Every f.d. irreducible \mathfrak{sl}_2 -module is equal to $V(m)$ for some $m \in \mathbb{N}$.

2. Every f.d. \mathfrak{sl}_2 -module is completely reducible.

Representations of $U_q(\mathfrak{sl}_2)$

For $m \in \mathbb{N}$, define an $(m+1)$ -dimensional $U_q(\mathfrak{sl}_2)$ -module by $V^q(m) = \bigoplus_{k=0}^m \mathbb{Q}(q) U_k^{(m)}$

$$t U_k^{(m)} = q^{m-2k} U_k^{(m)}$$

$$f U_k^{(m)} = [k+1] U_{k+1}^{(m)}$$

$$e U_k^{(m)} = [m-k+1] U_{k-1}^{(m)}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$

Let $V_{A_1}^q(m) := \bigoplus_{k=0}^m A_1 U_k^{(m)}$.

Note: $U_{A_1} / (q-1)U_{A_1} \stackrel{||\mathbb{Z}}{\cong} U(\mathfrak{sl}_2)$ acts on $V_{A_1}^q(m) / (q-1)V_{A_1}^q(m) \stackrel{||\mathbb{Z}}{\cong} V(m)$

The weight of $U_k^{(m)} := \text{wt}(U_k^{(m)}) = m - 2k$

Crystal Base of $V^q(m)$

Let $A_0 := \{ f/g \mid f, g \in \mathbb{Q}[q] \text{ and } g(0) \neq 0 \}$

Let $L(m) = \bigoplus_{k=0}^m A_0 U_k^{(m)}$ and

$$B(m) = \{ U_k^{(m)} \bmod q L(m) \mid k=0, \dots, m \}$$

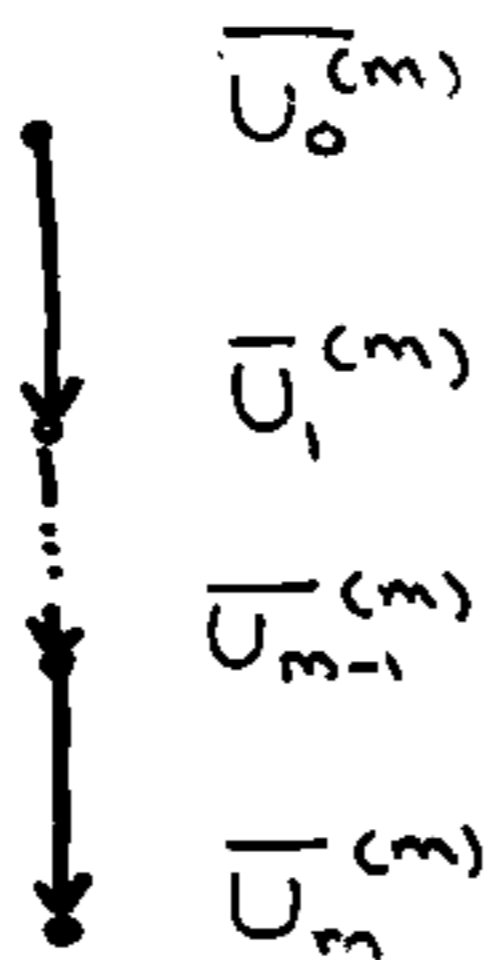
$(L(m), B(m))$ is called the crystal base of $V^q(m)$

Define \tilde{f} on $B(m)$ by $\tilde{f} \overline{U}_k^{(m)} = \overline{U}_{k+1}^{(m)}$,

and \tilde{e} by $\tilde{e} \overline{U}_k^{(m)} = \overline{U}_{k-1}^{(m)}$.

The crystal graph of $V^q(m)$ is the

graph



If $V \cong \bigoplus_{i=1}^{\infty} V^q(m_i)$, for $m_i \in \mathbb{N}$,

the crystal base of V is a pair (L, \mathcal{B}) such that

- L is a free A_0 -module with $V \cong \mathbb{Q}(q) \otimes_{A_0} L$
- \mathcal{B} is a basis of the \mathbb{Q} v.s. L/qL
- $L \cong \bigoplus_{i=1}^{\infty} L(m_i)$ and $\mathcal{B} \cong \bigsqcup_{i=1}^{\infty} \mathcal{B}(m_i)$

V is integrable if $V = \bigoplus_{\mu \in \mathbb{Z}} V_{\mu}$

$$V_{\mu} := \{ v \in V \mid tv = q^{\mu} v \}$$

Comultiplication on $U_q(\mathfrak{sl}_2)$

$$\Delta: U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$$

$$t \mapsto t \otimes t$$

$$e \mapsto e \otimes t^{-1} + 1 \otimes e$$

$$f \mapsto f \otimes 1 + t \otimes f$$

Example

Let $\boxed{1} := U_0^{(1)}$ and $\boxed{2} := U_1^{(1)}$.

So that

$$t \boxed{1} = q \boxed{1}, \quad e \boxed{1} = 0, \quad f \boxed{1} = \boxed{2}$$

$$t \boxed{2} = q^{-1} \boxed{2}, \quad e \boxed{2} = \boxed{1}, \quad f \boxed{2} = 0$$

$$V(1) = \mathbb{Q}(q) \boxed{1} \oplus \mathbb{Q}(q) \boxed{2}$$

$$V(1) \otimes V(1) = \mathbb{Q}(q) \boxed{1} \otimes \boxed{1} \oplus \mathbb{Q}(q) \boxed{1} \otimes \boxed{2} \oplus \mathbb{Q}(q) \boxed{2} \otimes \boxed{1} \oplus \mathbb{Q}(q) \boxed{2} \otimes \boxed{2}$$

Decomposes
into irreducibles

$$f(\boxed{1} \otimes \boxed{1}) = (f \otimes 1 + t \otimes f)(\boxed{1} \otimes \boxed{1}) = \boxed{2} \otimes \boxed{1} + q \boxed{1} \otimes \boxed{2}$$

$$e(\boxed{1} \otimes \boxed{1}) = 0$$

$$f(\boxed{1} \otimes \boxed{2}) = \boxed{2} \otimes \boxed{2}$$

$$e(\boxed{1} \otimes \boxed{2}) = \boxed{1} \otimes \boxed{1}$$

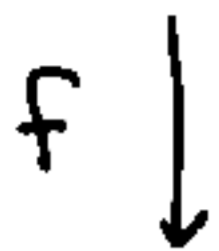
$$f(\boxed{2} \otimes \boxed{1}) = q^{-1} \boxed{2} \otimes \boxed{2}$$

$$e(\boxed{2} \otimes \boxed{1}) = q^{-1} \boxed{1} \otimes \boxed{1}$$

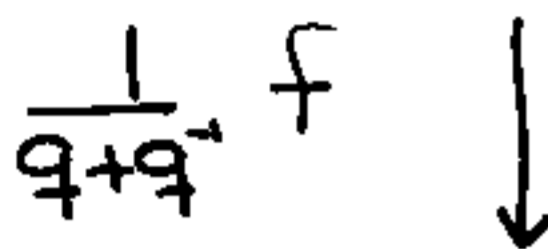
$$f(\boxed{2} \otimes \boxed{2}) = 0$$

$$e(\boxed{2} \otimes \boxed{2}) = q \boxed{1} \otimes \boxed{2} + \boxed{2} \otimes \boxed{1}$$

$$\boxed{1} \otimes \boxed{1} = u_0$$



$$\boxed{2} \otimes \boxed{1} + q \boxed{1} \otimes \boxed{2} = u_1 \quad \boxed{1} \otimes \boxed{2} - q \boxed{2} \otimes \boxed{1} = v_0$$

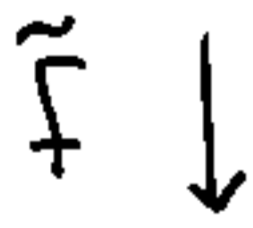


$$\boxed{2} \otimes \boxed{2} = u_2$$

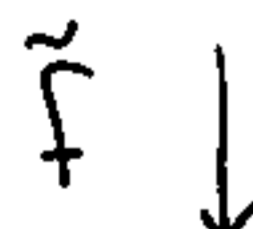
$$L = \underbrace{A_0 u_0 + A_0 u_1 + A_0 u_2}_{\substack{||2 \\ L(2)}} + \underbrace{A_0 v_0}_{\substack{||2 \\ L(0)}}$$

Crystal graph

$$\boxed{1} \otimes \boxed{1}$$



$$\boxed{2} \otimes \boxed{1}$$



$$\boxed{2} \otimes \boxed{2}$$

$$\boxed{1} \otimes \boxed{2}$$