Facts: (1) If V is a **finite dimensional** $\mathcal{U}_q(\mathfrak{sl}_2)$-module, then V is **completely reducible**.

(2) Any **irreducible** $(\ell+1)$-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module is isomorphic to $\mathbf{V(\ell)}$ or $\mathbf{V(\ell)\otimes V_-}$ (where $V_- = \mathbb{Q}(q)\,1_-$, $t\,1_- = -1_-$, $e\cdot 1_- = f\cdot 1_- = 0$)

(3) If V is a finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module such that

$$V = \bigoplus_{k\in\mathbb{Z}} V_k \text{ , then } V \cong \bigoplus_{\ell_i \in S \subseteq \mathbb{Z}} V(\ell_i),$$

where the weight space of V of weight k is $V_k := \{x \in V \mid tv = q^k v\}$.
(V is called integrable)

Note: If V and W decompose into the direct sum of their weight spaces, so does $V\otimes W$.

Define operators $\tilde{e}$ and $\tilde{f}$ on $V(m)$, $L(m)$ and $B(m)$ by

$$\tilde{f}(u_k^{(m)}) = u_{k+1}^{(m)}, \quad \tilde{f}(u_k^{(m)} + qL(m)) = u_{k+1}^{(m)} + qL(m)$$

$$\tilde{e}(u_k^{(m)}) = u_{k-1}^{(m)}, \quad \tilde{e}(u_k^{(m)} + qL(m)) = u_{k-1}^{(m)} + qL(m)$$

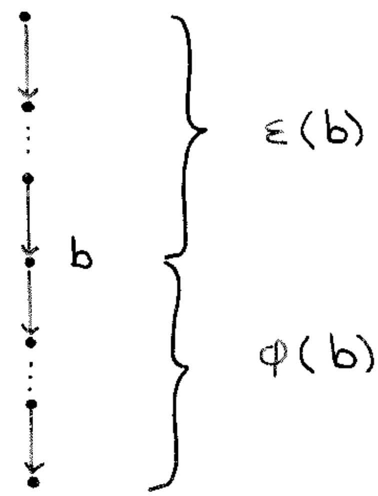
Define operators $\tilde{e}$ and $\tilde{f}$ on $V \overset{\phi}{\cong} \bigoplus_{l_i \in S \subseteq \mathbb{Z}} V(l_i)$, L and B, if (L, B) is a crystal base of V, by $\tilde{f}(v) = \phi^{-1}(\tilde{f}\,\phi(v))$ for $v \in V$ (or L) and $\tilde{f}(v + qL) = \tilde{f}v + qL$.

The crystal graph of V is the graph whose set of vertices is B and directed edges given by: $\underset{b \in B}{b \longrightarrow \tilde{f}b}$ if $\tilde{f}b \neq 0$.

Define φ and ε on B by

$\varphi(b) := \max\{k \mid \tilde{f}^k b \neq 0\}$ and

$\varepsilon(b) := \max\{k \mid \tilde{e}^k b \neq 0\}$



Let V and V' be finite dimensional integrable $\mathcal{U}_q(\mathfrak{sl}_2)$-modules with crystal bases (L, B) and (L', B') respectively, then $(L \otimes L', B \otimes B')$ is a crystal bases of $V \otimes V'$ and for $b \in B$ and $b' \in B'$,

$$\tilde{f}(b \otimes b') = \begin{cases} \tilde{f}(b) \otimes b' & \text{if} \quad \varphi(b) > \varepsilon(b') \\ b \otimes \tilde{f}(b') & \text{if} \quad \varphi(b) \leq \varepsilon(b') \end{cases}$$

and

$$\tilde{e}(b \otimes b') = \begin{cases} \tilde{e}(b) \otimes b' & \text{if} \quad \varphi(b) \geq \varepsilon(b') \\ b \otimes \tilde{e}(b') & \text{if} \quad \varphi(b) < \varepsilon(b') \end{cases}$$

A decomposition of $V(\ell) \otimes V(1)$:

Define, for $0 \le k \le \ell+1$,

$$w_k := u_k^{(\ell)} \otimes u_0^{(1)} + q^{\ell-k+1} u_{k-1}^{(\ell)} \otimes u_1^{(1)}$$

and, for $0 \le k \le \ell-1$,

$$v_k := q^{-k}\frac{[\ell-k]}{[\ell]} u_k^{(\ell)} \otimes u_1^{(1)} - \frac{q[k+1]}{[\ell]} u_{k+1}^{(\ell)} \otimes u_0^{(1)} .$$

Then $V(\ell) \otimes V(1) = \left(\bigoplus_{k=0}^{\ell+1} \mathbb{Q}(q) w_k\right) \oplus \left(\bigoplus_{k=0}^{\ell-1} \mathbb{Q}(q) v_k\right)$.

Let $L := \left(\bigoplus_{k=0}^{\ell+1} A_0 w_k\right) \oplus \left(\bigoplus_{k=0}^{\ell-1} A_0 v_k\right)$ and

$$B := \{\bar{w}_0, \dots, \bar{w}_{\ell+1}, \bar{v}_0, \dots, \bar{v}_{\ell-1}\} \quad (\bar{v} := v \bmod q)$$

We have $\check{f} w_k = [k] w_{k+1}$, $\check{e} w_k = [\ell+2-k] w_{k-1}$, $t w_k = q^{\ell+1-2k} w_k$

and $\check{f} v_k = [k] v_{k+1}$, $\check{e} v_k = [\ell-k] v_{k-1}$, $t v_k = q^{\ell-1-2k} v_k$

Hence we have an isomorphism

$\phi : V(\ell) \otimes V(1) \longrightarrow V(\ell+1) \oplus V(\ell-1)$ such

that $L \cong L(\ell+1) \oplus L(\ell-1)$ and

$$B \cong B(\ell+1) \sqcup B(\ell-1) .$$

For $0 \le k \le \ell$, $\quad w_k = u_k^{(\ell)} \otimes u_0^{(1)} \mod qL$

and $\quad w_{\ell+1} = u_\ell^{(\ell)} \otimes u_1^{(1)} \mod qL$.

For $0 \le k \le \ell-1$, $\quad v_k = u_k^{(\ell)} \otimes u_1^{(1)} \mod qL$.

For $0 \le k \le \ell-1$,

$$\tilde{f}(\bar{u}_k^{(\ell)} \otimes \bar{u}_0^{(1)}) = \tilde{f}(\overline{w_k}) = \overline{w}_{k+1} = \bar{u}_{k+1}^{(\ell)} \otimes \bar{u}_0^{(1)}$$

$$= \tilde{f}(\bar{u}_k^{(\ell)}) \otimes \bar{u}_0^{(1)}$$

(Note that in this case $\varphi(\bar{u}_k^{(\ell)}) > 0 = \varepsilon(\bar{u}_0^{(1)})$)

$$\tilde{f}(\bar{u}_\ell^{(\ell)} \otimes \bar{u}_0^{(1)}) = \tilde{f}(\overline{w}_\ell) = \overline{w}_{\ell+1} = \bar{u}_\ell^{(\ell)} \otimes \bar{u}_1^{(1)}$$

$$= \bar{u}_\ell^{(\ell)} \otimes \tilde{f}(\bar{u}_0^{(1)})$$

(Note that $\varphi(\bar{u}_\ell^{(\ell)}) = 0 \le 0 = \varepsilon(\bar{u}_0^{(1)})$)

$$\tilde{f}(\bar{u}_\ell^{(\ell)} \otimes \bar{u}_1^{(1)}) = \tilde{f}(\overline{w_{\ell+1}}) = 0 = \bar{u}_\ell^{(\ell)} \otimes \tilde{f}(u_1^{(1)}).$$

(Note that $\varphi(\bar{u}_\ell^{(\ell)}) = 0 \le 1 = \varepsilon(\bar{u}_1^{(1)})$.)

For $0 \le k \le \ell - 2$,

$$\tilde{f}(\bar{u}_k^{(\ell)} \otimes \bar{u}_1^{(1)}) = \tilde{f}(\bar{v}_k) = \bar{v}_{k+1} = \bar{u}_{k+1}^{(\ell)} \otimes \bar{u}_1^{(1)}$$
$$= \tilde{f}(\bar{u}_k^{(\ell)}) \otimes \bar{u}_1^{(1)}$$

(Note that in this case $\varphi(\bar{u}_k^{(\ell)}) > 1 = \varepsilon(\bar{u}_1^{(1)})$)

$$\tilde{f}(\bar{u}_{\ell-1}^{(\ell)} \otimes \bar{u}_1^{(1)}) = \tilde{f}(\bar{v}_{\ell-1}) = 0 = \bar{u}_{\ell-1}^{(\ell)} \otimes \tilde{f}(\bar{u}_1^{(1)})$$

(Note that $\varphi(\bar{u}_{\ell-1}^{(\ell)} = 1 \le 1 = \varepsilon(\bar{u}_1^{(1)})$).

Similarly for $\tilde{e}$.