For $0 \le k \le \ell - 2$,

$$\tilde{f}\left(\bar{u}_k^{(\ell)} \otimes \bar{u}_1^{(1)}\right) = \tilde{f}\left(\bar{v}_k\right) = \bar{v}_{k+1} = \bar{u}_{k+1}^{(\ell)} \otimes \bar{u}_1^{(1)}$$

$$= \tilde{f}\left(\bar{u}_k^{(\ell)}\right) \otimes \bar{u}_0^{(1)}$$

(Note that in this case $\varphi\left(\bar{u}_k^{(\ell)}\right) > 1 = \varepsilon\left(\bar{u}_1^{(1)}\right)$)

$$\tilde{f}\left(\bar{u}_{\ell-1}^{(\ell)} \otimes \bar{u}_1^{(1)}\right) = \tilde{f}\left(\bar{v}_{\ell-1}\right) = 0 = \bar{u}_{\ell-1}^{(\ell)} \otimes \tilde{f}\left(\bar{u}_1^{(1)}\right)$$

(Note that $\varphi\left(\bar{u}_{\ell-1}^{(\ell)}\right) = 1 \le 1 = \varepsilon\left(\bar{u}_1^{(1)}\right)$).

$B(\ell)$:

$B(2)$:

$$\begin{tikzcd}
 & \bar{u}_0^{(\ell)} \arrow[r] & \bar{u}_1^{(\ell)} \arrow[r] & \cdots \arrow[r] & \bar{u}_{\ell-1}^{(\ell)} \arrow[r] & \bar{u}_\ell^{(\ell)} \\
\bar{u}_0^{(1)} \arrow[dd] & \bar{u}_0^{(\ell)} \otimes \bar{u}_0^{(1)} \arrow[r] \arrow[d, equal] & \bar{u}_1^{(\ell)} \otimes \bar{u}_0^{(1)} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \bar{u}_{\ell-1}^{(\ell)} \otimes \bar{u}_0^{(1)} \arrow[r] \arrow[d, equal] & \bar{u}_\ell^{(\ell)} \otimes \bar{u}_0^{(1)} \arrow[d, equal] \\
 & \bar{w}_1 & \bar{w}_2 & & \bar{w}_{\ell-1} & \bar{w}_\ell \arrow[d] \\
\bar{u}_1^{(1)} & \bar{u}_0^{(\ell)} \otimes \bar{u}_1^{(1)} \arrow[r] \arrow[d, equal] & \bar{u}_1^{(\ell)} \otimes \bar{u}_1^{(1)} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \bar{u}_{\ell-1}^{(\ell)} \otimes \bar{u}_1^{(1)} \arrow[d, equal] & \bar{u}_\ell^{(\ell)} \otimes \bar{u}_1^{(1)} \arrow[d, equal] \\
 & \bar{v}_1 & \bar{v}_2 & & \bar{v}_{\ell-1} & \bar{w}_{\ell+1}
\end{tikzcd}$$

Let V be a finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module such that $V \cong \bigoplus_{\substack{\text{some} \\ \ell_i\text{'s}}} V(\ell_i)$

and (L, B) a crystal base of V.

Then

(1) L is a free A_0- submodule of L such that

$V \cong \mathbb{Q}(q) \otimes_{A_0} L$

(2) B is a basis of the $\mathbb{Q}$ vector space L/qL

(3) $L = \oplus L_k$, where $L_k = L \cap M_k$, and

$B = \sqcup B_k$, where $B_k = B \cap (L_k / qL_k)$

(4) $\tilde{e} L \subseteq L$ and $\tilde{f} L \subseteq L$

(5) $\tilde{e} B \subseteq B \cup \{0\}$ and $\tilde{f} B \cup \{0\}$

(6) If $b, b' \in B$, $b = \tilde{f} b' \iff \tilde{e} b = b'$

Conversely, if (L, B) satisfies (1)–(6), then it is a crystal basis

Proof: (By induction on the # of distinct l_i's)

First assume that $V \cong \underbrace{V(m) \oplus \cdots \oplus V(m)}_{k\text{-times}}$.

Let $B_m = \{b_1, \dots, b_k\}$ and let $v_1, \dots, v_k \in L_m$ such that $\overline{v_i} = b_i$, $1 \le i \le k$.

$\{v_1, v_2, \dots, v_k\}$ is an A_0-basis of L_m.

For $1 \le i \le k$ and $0 \le j \le m$, let

$$v_{ij} := \frac{f^j}{[j]!} v_i$$

Then $v_{ij} = \tilde{f}^j v_i \in L$ and $\overline{v_{ij}} = \tilde{f}^j \overline{v_i} \in B \cup \{0\}$, and $V = \bigoplus_{i=1}^{k} \mathcal{U}_q(\mathfrak{sl}_2) v_i$, $L = \bigoplus_{i,j} A_0 \tilde{f}^j v_i$ and

$$B = \{ \tilde{f}^j \overline{v}_i \}_{i,j}.$$

Now let m be a maximal weight of V.

Then $V \cong V(m)^{\oplus k} \oplus \left(\bigoplus_{\ell_i < m} V(\ell_i) \right)$.

Let $\{v_1, \ldots, v_k\}$ be an A_0-basis of L_m.

Then $\mathcal{U}_q(\mathfrak{sl}_2) v_i \cong V(m)$.

Let $W := \bigoplus_{i=1}^{k} \mathcal{U}_q(\mathfrak{sl}_2) v_i$ and let X be the complementary submodule of W in V. So $V = W \oplus X$ and $X \cong \bigoplus_{\ell_i < m} V(\ell_i)$.

Let $L_W = L \cap W$, $B_W = B \cap L_W / qL_W$,

$$L_X = L \cap X, \quad B_X = B \cap L_X / qL_X$$

We show that $L = L_W \oplus L_X$. This will imply that $W \cong \mathbb{Q}(q) \otimes_{A_0} L_W$ and

$$X \cong \mathbb{Q}(q) \otimes_{A_0} L_X.$$

$L_m = (L_W)_m$ and $(L_X)_m = 0$.

So $L_m = (L_W)_m \oplus (L_X)_m$.

Let $r < m$ and $v \in L_r$.

Then $v = w + x$ for some $w \in W_r$ & $x \in X_r$.

$\tilde{e}v = \tilde{e}w + \tilde{e}x \in L_{r+2} = (L_W)_{r+2} \oplus (L_X)_{r+2}$

Hence $\tilde{e}w \in (L_W)_{r+2}$ and $\tilde{e}x \in (L_X)_{r+2}$

Then $w \in (L_W)_r$.

Therefore $x = v - w \in L \cap X = L_X$

We now show that $B = B_W \sqcup B_X$.

This implies B_W is a basis of L_W/qL_W

and B_X is a basis of L_X/qL_X

$B_m = (B_w)_m$ and $(B_x)_m = 0$.

So $B_m = (B_w)_m \oplus (B_x)_m$

Let $r < m$ and $b \in B_r$.

If $\tilde{e}\, b = b' \neq 0$, $b' \in B_{m+2} = (B_w)_{m+2} \sqcup (B_x)_{m+2}$

So $b' \in B_w$ or $b' \in B_x$

Hence $b = \tilde{f}\, b' \in B_w \sqcup B_x$

If $\tilde{e}\, b = 0$. Let $b = \underset{L_w/qL_w}{\underset{\Cap}{b_1}} + \underset{L_x/qL_x}{\underset{\Cap}{b_2}}$.

Then $\tilde{e}\, b_1 = 0$ and $\tilde{e}\, b_2 = 0$.

$\tilde{e}\, b_1 = 0 \Rightarrow b_1 = 0$ since $wt(b_1) < m$.

Therefore $b = b_2 \in B_x$.

We can check that (3)-(6) are satisfied by (L_x, B_x). So by induction we are done.

Theorem:

Let V and V' be finite dimensional integrable $\mathcal{U}_q(\mathfrak{sl}_2)$-modules with crystal bases (L, B) and (L', B') respectively, then $(L \otimes L', B \otimes B')$ is a crystal bases of $V \otimes V'$ and for $b \in B$ and $b' \in B'$,

$$\tilde{f}(b \otimes b') = \begin{cases} \tilde{f}(b) \otimes b' & \text{if } \varphi(b) > \varepsilon(b') \\ b \otimes \tilde{f}(b') & \text{if } \varphi(b) \leq \varepsilon(b') \end{cases}$$

and

$$\tilde{e}(b \otimes b') = \begin{cases} \tilde{e}(b) \otimes b' & \text{if } \varphi(b) \geq \varepsilon(b') \\ b \otimes \tilde{e}(b') & \text{if } \varphi(b) < \varepsilon(b') \end{cases}$$

We saw that the theorem is true for $V(\ell) \otimes V(1)$.

If V be an integrable $\mathcal{U}_q(\mathfrak{sl}_2)$-module, then the theorem is also true for $V \otimes V(1)$.

It suffices to show the theorem for $V(\ell) \otimes V(m)$, $m \geq 2$.

Assume the theorem is true for $V(\ell) \otimes V(m-1)$.

Then $(L(\ell) \otimes L(m-1), B(\ell) \otimes B(m-1))$ is a c.b. of $V(\ell) \otimes V(m-1)$.

By the special case $V \otimes V(1)$, $(L(\ell) \otimes L(m-1) \otimes L(1), B(\ell) \otimes B(m-1) \otimes B(1))$ is a c.b of $V(\ell) \otimes V(m-1) \otimes V(1)$.

We have $\quad V(m-1)\otimes V(1) \cong V(m)\oplus V(m-2)$

$$L(m-1)\otimes L(1) \cong L(m)\oplus L(m-2)$$

$$B(m-1)\otimes B(1) \cong B(m)\sqcup B(m-2)$$

Hence

$$\big(L(\ell)\otimes L(m)\oplus L(\ell)\otimes L(m-2),\ B(\ell)\otimes B(m)\sqcup B(\ell)\otimes B(m-2)\big)$$

is a c.b of $\quad V(\ell)\otimes V(m)\oplus V(\ell)\otimes V(m-2)$.

$\qquad\qquad\qquad B(\ell)\otimes B(m))$

$(L(\ell)\otimes L(m))$ satisfy (1)-(6) for $V(\ell)\otimes V(m)$

and hence it is a crystal base.

For $0 \le h \le m-1$,

$$u_k^{(\ell)} \otimes u_h^{(m)} \longleftrightarrow u_k^{(\ell)} \otimes \underbrace{u_h^{(m-1)} \otimes u_0^{(1)}}_{\substack{\| \\ w_h}}$$

$$u_k^{(\ell)} \otimes u_m^{(m)} \longleftrightarrow u_k^{(\ell)} \otimes \underbrace{u_{m-1}^{(m-1)} \otimes u_1^{(1)}}_{\substack{\| \\ w_m}}$$

For $0 \le h < m-1$,

$$\tilde{f}(u_k^{(\ell)} \otimes u_h^{(m)}) \longleftrightarrow \tilde{f}(u_k^{(\ell)} \otimes u_h^{(m-1)} \otimes u_0^{(1)})$$

$$= \tilde{f}(u_k^{(\ell)} \otimes u_h^{(m-1)}) \otimes u_0^{(1)}$$

$$= \begin{cases} u_{k+1}^{(\ell)} \otimes u_h^{(m-1)} \otimes u_0^{(1)} & \text{if } k+h < \ell \\ u_k^{(\ell)} \otimes u_{h+1}^{(m-1)} \otimes u_0^{(1)} & \text{if } k+h \geq \ell \end{cases}$$

$$\longleftrightarrow \begin{cases} \tilde{f} u_k^{(\ell)} \otimes u_h^{(m)} & \text{if } k+h < \ell \\ u_k^{(\ell)} \otimes \tilde{f} u_h^{(m)} & \text{if } k+h \geq \ell \end{cases}$$

Note that $\phi(u_k^{(\ell)}) = \ell - k$ and $\varepsilon(u_h^{(m)}) = h$.