$\mathfrak{sl}_3$ – trace 0 3×3 matrices over $\mathbb{Q}$.

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$e_1, f_1, h_1, e_2, f_2, h_2$ generate $\mathfrak{sl}_3$ and satisfy

- $[h_1, h_1] = [h_2, h_2] = [h_1, h_2] = 0$
- $[e_1, f_1] = h_1$, $[e_1, f_2] = 0$, $[e_2, f_1] = 0$, $[e_2, f_2] = h_2$
- $[h_1, e_1] = 2\, e_1$, $[h_2, e_1] = -1\, e_1$

  $[h_1, e_2] = -1\, e_2$, $[h_2, e_2] = 2\, e_2$
- $[h_1, f_1] = -2\, f_1$, $[h_2, f_1] = 1\, f_1$

  $[h_1, f_2] = 1\, f_2$, $[h_2, f_2] = -2\, f_2$
- $[e_1, [e_1, e_2]] = [f_1, [f_1, f_2]] = [e_1, [e_1, e_2]] = [f_1, [f_1, f_2]] = 0$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ – the Cartan matrix of $\mathfrak{sl}_3$.

$\langle e_1, f_1, h_1, e_1, f_1, h_1 \mid \text{above relations} \rangle = \mathfrak{sl}_3$

A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called a Cartan matrix if
- $a_{ii} = 2$
- $a_{ij} \le 0$ if $i \ne j$
- $a_{ij} = 0 \iff a_{ji} = 0$
- all principal minors are > 0

The algebra generated by

$$e_1, f_1, h_1, e_2, f_2, h_2, \dots, e_n, f_n, h_n,$$

subject to the relations

- $[h_i, h_j] = 0$ • $[e_i, f_j] = \delta_{ij} h_i$
- $$\begin{array}{llll} [h_1, e_1] = a_{11} e_1, & [h_2, e_1] = a_{12} e_1, & \dots, & [h_n, e_1] = a_{1n} e_1 \\ [h_1, e_2] = a_{21} e_2, & [h_2, e_2] = a_{22} e_2, & \dots, & [h_n, e_2] = a_{2n} e_2 \\ \vdots & \vdots & & \vdots \\ [h_1, e_n] = a_{n1} e_n, & [h_2, e_n] = a_{n2} e_n, & \dots, & [h_n, e_n] = a_{nn} e_n \end{array}$$
- $[h_i, f_j] = -a_{ji} f_j$
- $[\underbrace{e_i, [e_i, \dots, [e_i}_{-a_{ji}+1}, e_j]\dots]] = [\underbrace{f_i, [f_i, \dots, [f_i}_{-a_{ji}+1}, f_j]\dots]] = 0$

is a semisimple finite dimensional Lie algebra and every f.d. s.s. L. algebra is of this form.

$\mathfrak{h} := \operatorname{span}\{h_1, h_2, \dots, h_n\}$ is called the Cartan subalg.

Define $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathfrak{h}^*$ by $\alpha_i(h_j) = a_{ij}$.

$\alpha_1, \alpha_2, \dots, \alpha_n$ are called the simple roots and are linearly independant.

A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called a generalized Cartan matrix if

- $a_{ii} = 2$
- $a_{ij} \le 0$ if $i \ne j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

The algebra $\mathfrak{g}$ generated by

$e_1, f_1, h_1, e_2, f_2, h_2, \ldots, e_n, f_n, h_n, h_{n+1}, \ldots, h_{2n-\mathrm{rank}A}$

subject to the relations

- $[h_i, h_j] = 0$
- $[e_i, f_j] = \delta_{ij} h_i$
- $[h_i, e_j] = a_{ji} e_j$ for $1 \le i \le n$
- $[h_i, f_j] = -a_{ji} f_j$ for $1 \le i \le n$
- $[\underbrace{e_i, [e_i, \ldots, [e_i}_{-a_{ji}+1}, e_j] \ldots]] = [\underbrace{f_i, [f_i, \ldots, [f_i}_{-a_{ji}+1}, f_j] \ldots]] = 0$

is called a Kac-Moody Lie algebra.

$\mathfrak{h} := \mathrm{span}\{h_1, h_2, \ldots, h_n, h_{n+1}, \ldots\}$ - Cartan subalgebra

Choose (the simple roots) $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathfrak{h}^*$ s.t. $\alpha_i(h_j) = a_{ij}$ for $1 \le j \le n$ and s.t. they are lin. ind.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with 1 given by

$$\left\langle e_1, f_1, h_1, e_2, f_2, h_2, \ldots, e_n, f_n, h_n, h\text{'s} \;\middle|\; \begin{array}{c}\text{relations}\\ \text{above}\end{array} \right\rangle$$

and comultiplication

$$\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i$$

$$\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i$$

$$\Delta(h) = h \otimes 1 + 1 \otimes h$$

Let A be a generalized Cartan matrix.

A is called symmetrizable if $\exists$ a diagonal invertible matrix D with integer entries such that DA is symmetric. The corresponding K-M Lie algebra is called symmetrizable

$\mathfrak{g}$ - Kac-Moody Lie algebra

The category $\mathcal{O}_{int}$ consists of $\mathcal{U}(\mathfrak{g})$-modules V such that

- $V = \bigoplus_{\lambda \in P} V_\lambda$, where $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z}, 1 \le i \le n\}$ and $V_\lambda = \{x \in V \mid h \cdot x = \lambda(h) x \ \forall h \in \mathfrak{h}\}$
- $\dim V_\lambda < \infty$
- $\exists\ \lambda_1, \ldots, \lambda_k \in P$ such that if $V_\lambda \neq \{0\}$, $\lambda = \lambda_i - \sum_{j=1}^{n} a_j \alpha_j$ for some i and $a_j \in \mathbb{N}$.
- V is the direct sum of finite dimensional $\mathcal{U}(\mathfrak{sl}_2^i) := \langle e_i, f_i, h_i \rangle$ -modules for each i.

Let $\lambda \in P_+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, h_i \rangle \in \mathbb{N}, 1 \le i \le n\}$.

Let I_λ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by e_i, $h_i - \lambda(h_i) 1$, $f_i^{1 + \langle \lambda, h_i \rangle}$.

If $\mathfrak{g}$ is symmetrizable, $V(\lambda) := \mathcal{U}(\mathfrak{g}) / I_\lambda \in \mathcal{O}_{int}$ and it is irreducible.

$\mathcal{O}_{int}$ is semisimple and every irreducible module in $\mathcal{O}_{int}$ is isomorphic to $V(\lambda)$, $\lambda \in P_+$.

An example - $\mathfrak{sl}_3$

Define $\Lambda_1, \Lambda_2 \in P_+$ by

$$\Lambda_1(h_1) = 1, \ \Lambda_1(h_2) = 0$$

$$\Lambda_2(h_2) = 1, \ \Lambda_2(h_1) = 0.$$

Let $\lambda := \Lambda_1 + \Lambda_2$ and $u_\lambda := 1 + I_\lambda$.

$$V(\lambda) = \operatorname{span}\{ u_\lambda, f_1 u_\lambda, f_2 u_\lambda, f_1 f_2 u_\lambda, f_2 f_1 u_\lambda, f_1^2 f_2 u_\lambda, f_2^2 f_1 u_\lambda, f_1 f_2^2 f_1 u_\lambda \}$$

$wt(u_\lambda) = \lambda$

$wt(f_1 u_\lambda) = \lambda - \alpha_1$

$wt(f_1 f_2 u_\lambda) = \lambda - \alpha_1 - \alpha_2$

$\vdots$

$\mathfrak{sl}_2^1$ - decomposition

- u_λ
- $f_1 u_\lambda$

- $(-\frac{1}{2} f_1 f_2 + f_2 f_1) u_\lambda$

- $f_2 u_\lambda$
- $f_1 f_2 u_\lambda$
- $f_1^2 f_2 u_\lambda$

- $f_2^2 f_1 u_\lambda$
- $f_1 f_2^2 f_1 u_\lambda$

Let $\mathfrak{g}$ be symmetrizable. $D=\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix}$

The quantized universal enveloping algebra of $\mathfrak{g}$ is the Hopf algebra over $\mathbb{Q}(q)$ with 1 given by

$$\langle e_1, f_1, t_1, t_1^{-1}, e_2, f_2, t_2, t_2^{-1}, \dots, e_n, f_n, t_n, t_n^{-1}, t_{n+1}, t_{n+1}^{-1}, \dots \mid \text{relations below} \rangle$$

- $t_i t_i^{-1} = 1$, $t_i t_j = t_j t_i$
- $t_j e_i t_j^{-1} = q^{\alpha_i(h_j)} e_i$, $t_j f_i t_j^{-1} = q^{-\alpha_i(h_j)} f_i$
- $[e_i, f_j] = \delta_{ij} \dfrac{t_i^{d_i} - t_i^{-d_i}}{q^{d_i} - q^{-d_i}}$
- $\displaystyle\sum_{k=0}^{1-a_{ji}} (-1)^k \begin{bmatrix} 1-a_{ji} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ji}-k} e_j e_i^k = 0$ for $i \neq j$
- $\displaystyle\sum_{k=0}^{1-a_{ji}} (-1)^k \begin{bmatrix} 1-a_{ji} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ji}-k} f_j f_i^k = 0$ for $i \neq j$

where $[k]_{q_i} = \dfrac{q^{d_i k} - q^{-d_i k}}{q^{d_i} - q^{-d_i}}$ and $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \dfrac{[m]_{q_i}}{[m-k]_{q_i} [k]_{q_i}}$

$\mathfrak{g}$ - Kac-Moody Lie algebra

The category $\mathcal{O}^q_{int}$ consists of $\mathcal{U}_q(\mathfrak{g})$-modules V^q such that

- $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$, where $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z}, 1 \leq i \leq n\}$

  and $V^q_\lambda = \{x \in V \mid \cancel{h \cdot x = \lambda(h)x \ \forall h \in \mathfrak{h}}\}$ $t_i x = q^{\lambda(h_i)} \ \forall i$

- $\dim V^q_\lambda < \infty$

- $\exists \ \lambda_1, \dots, \lambda_k \in P$ such that if $V^q_\lambda \neq \{0\}$, $\lambda = \lambda_i - \sum_{j=1}^{n} a_j \alpha_j$ for some i and $a_j \in \mathbb{N}$.

- V^q is the direct sum of finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2^i) := \langle e_i, f_i, h_i \rangle$ -modules for each i.

Let $\lambda \in P_+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, h_i \rangle \in \mathbb{N}, 1 \leq i \leq n\}$.

Let I^q_λ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by e_i, $\cancel{h_i - \lambda(h_i)1}$ $t_i - q^{\lambda(h_i)}1$, $f_i^{1+\langle \lambda, h_i \rangle}$.

If $\mathfrak{g}$ is symmetrizable,
$V^q(\lambda) := \mathcal{U}_q(\mathfrak{g}) / I^q_\lambda \in \mathcal{O}^q_{int}$ and it is irreducible.

$\mathcal{O}^q_{int}$ is semisimple and every irreducible module in $\mathcal{O}^q_{int}$ is isomorphic to $V^q(\lambda)$, $\lambda \in P_+$.

Let U_{A_1} be the A_1-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by e_i, f_i, $1 \le i \le n$ and

$$t_i, t_i^{-1}, \frac{t_i - 1}{q - 1} \quad \forall i.$$

Then $U_{A_1} / (q-1)U_{A_1} \cong \mathcal{U}(\mathfrak{g})$

Let $V_{A_1}(\lambda) := U_{A_1} u_\lambda^q$. Then $V_{A_1}(\lambda) / (q-1)V_{A_1}(\lambda) \cong V(\lambda)$

Comultiplication

$\Delta(t_i) = t_i \otimes t_i$

$\Delta(e_i) = e_i \otimes t_i^{-d_i} + 1 \otimes e_i$

$\Delta(f_i) = f_i \otimes 1 + t_i^{d_i} \otimes f_i$

Antipode

$S(t_i) = t_i^{-1}$, $S(e_i) = -e_i t_i^{d_i}$, $S(f_i) = -t_i^{-d_i} f_i$.

An example

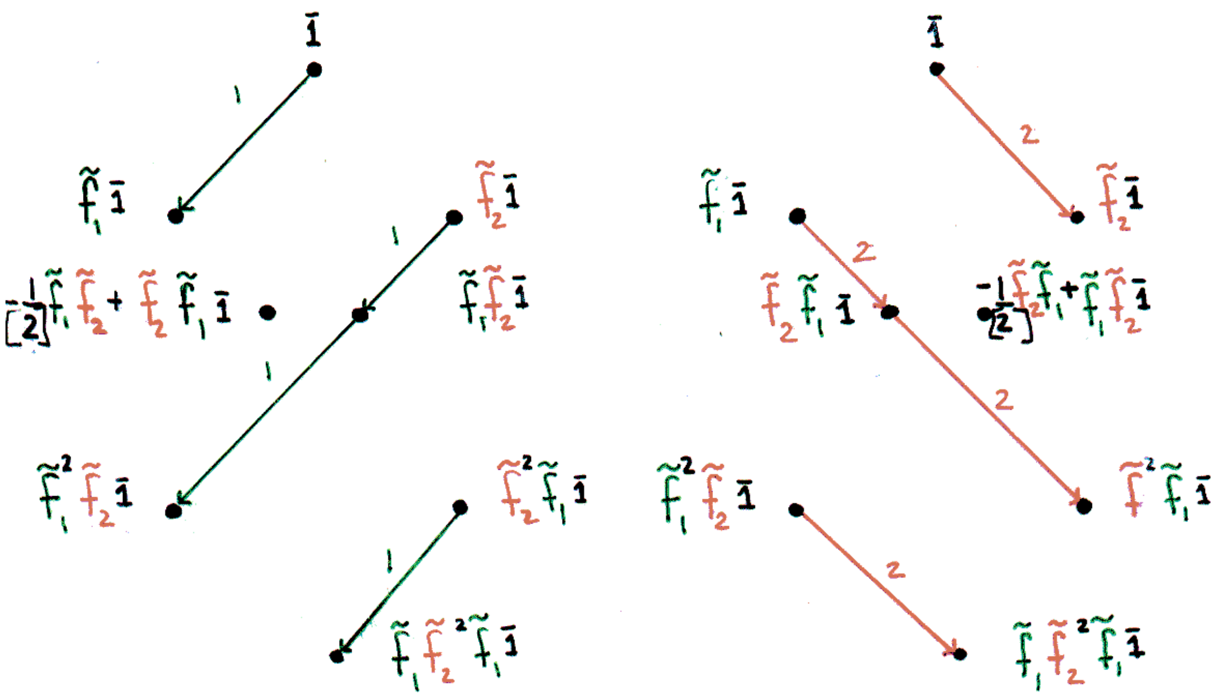
$\mathcal{U}_q(\mathfrak{sl}_3) = \langle e_1, f_1, h_1, e_2, f_2, h_2 \mid \text{some relations} \rangle$

Let Λ_1 and $\Lambda_2 \in \mathfrak{h}^*$ be defined by

$$\Lambda_1(h_1) = 1 \,,\; \Lambda_1(h_2) = 0 \quad \text{and} \quad \Lambda_2(h_2) = 1 \,,\; \Lambda_2(h_1) = 0 \,.$$

Decomposition of $V^q(\Lambda_1 + \Lambda_2)$ as :

$\mathcal{U}_q^1(\mathfrak{sl}_2)$ - module $\qquad\qquad \mathcal{U}_q^2(\mathfrak{sl}_2)$ - module



An example

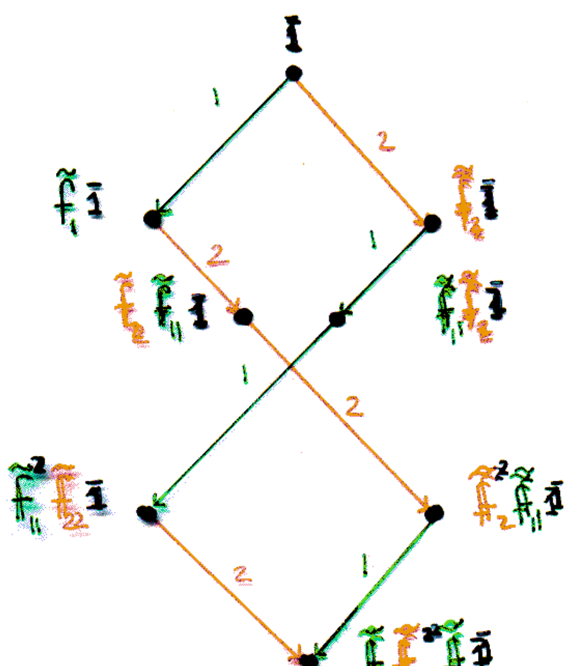
$\mathcal{U}_q(\mathfrak{sl}_3) = \langle e_1, f_1, h_1, e_2, f_2, h_2 \mid \text{some relations} \rangle$

Let Λ_1 and $\Lambda_2 \in \mathfrak{h}^*$ be defined by

$\Lambda_1(h_1) = 1,\ \Lambda_1(h_2) = 0$ and $\Lambda_2(h_2) = 1,\ \Lambda_2(h_1) = 0$.

Decomposition of $V^q(\Lambda_1 + \Lambda_2)$ as:

$\mathcal{U}_q^{2}(\mathfrak{sl}_2)$-module



Let V be an integrable module of $\mathcal{U}_q(\mathfrak{g})$.

For each $1 \le i \le n$, $\langle e_i, f_i, t_i, t_i^{-1} \rangle$

$$\overset{\shortparallel}{\mathcal{U}_q^i(\mathfrak{sl}_2)}$$

V decomposes into the direct sum of irreducible $\mathcal{U}_q^i(\mathfrak{sl}_2)$-modules, so we can define $\tilde{e}_i$ and $\tilde{f}_i$ on V.

A crystal base of V is a pair (L, B) such that $\boxed{(L,B) \text{ is a crystal base of the } \mathcal{U}_q^i(\mathfrak{sl}_2)\text{-module } V \text{ for all } i}$

(1) L is a free A_0-submodule of V s.t. $V \cong \mathbb{Q}(q) \otimes_{A_0} L$

(2) B is a basis of the $\mathbb{Q}$-vector space L/qL

(3) $L = \bigoplus_{\mu \in P} L_\mu$, where $L_\mu = L \cap V_\mu$ and

$B = \sqcup B_\lambda$ where $B_\mu = B \cap (L_\mu / q L_\mu)$

(4) $\tilde{e}_i L \subseteq L$ and $\tilde{f}_i L \subseteq L$

(5) $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B \cup \{0\}$

(6) If $b, b' \in B$, $b = \tilde{f}_i b' \Leftrightarrow \tilde{e}_i b = b'$.

Theorem: Let $\lambda \in P_+$.

Let $L(\lambda) :=$ A_0-submodule of $V(\lambda)$ spanned by
$$\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} u_\lambda \mid 1 \le i_1, \ldots, i_\ell \le n \}$$
and $B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \bar{u}_\lambda \neq 0 \mid 1 \le i_1, \ldots, i_\ell \le n \}$.

Then $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.

Theorem: Let V and W be integrable $\mathcal{U}_q(\mathfrak{g})$-modules with crystal bases (L_1, B_1) and (L_2, B_2), resp.

Then $(L_1 \oplus L_2, B_1 \sqcup B_2)$ is a crystal base of $V \oplus W$.

Theorem (Tensor Product Theorem)

Fact: $\dim V_\mu = \#$ of elements in B_μ

Note:
- $B(\lambda)$ is connected
- $\bar{u}_\lambda$ is the unique element of $B(\lambda)$ such that $\tilde{e}_i \bar{u}_\lambda = 0 \quad \forall i$