CRYSTAL BASES
DEFN & PROPERTIES

Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(g)$-module in $\mathcal{O}_{int}$

Crystal base (L, B) :

1. L free A_0-Submodule of M st $M \simeq Q(q) \otimes_{A_0} L$
2. B basis of L/qL
3. $(L, B) = \bigoplus_{\lambda \in P} (L_\lambda, B_\lambda)$, each (L_λ, B_λ) verifies 1, 2 for M_λ
4. For $\forall i = 1 \dots n$, $\exists$ $U_q(g)_i$ - isomorphism

$$\theta_i : M \xrightarrow{\sim} \bigoplus_j \underbrace{V(l_j)}_{U_q(sl_2)\text{-irred}}$$

such that $(L, B) \xrightarrow{\theta} \dot{\bigcup_j} \{u_k^{(l_j)}\}_{0 \le k \le l_j}$

Define $\tilde{e}_i, \tilde{f}_i$

Fix i and take θ_i. Let $b \in B$ st. $\theta_i(b) = u_k^{(l_j)}$

$$\tilde{f}_i : \begin{cases} \text{- if } 0 \le k < l_j & \tilde{f}_i(b) \text{ is so that } \theta_i(\tilde{f}_i b) = u_{k+1}^{(l_j)} \\ \text{- if } k = l_j & \tilde{f}_i(b) = 0 \end{cases}$$

$$\tilde{e}_i : \begin{cases} \text{- if } 0 < k \le l_j & \tilde{e}_i(b) : \theta_i(\tilde{e}_i b) = u_{k-1}^{(l_j)} \\ \text{- if } k = 0 & \tilde{e}_i(b) = 0 \end{cases}$$

$\tilde{f}_i, \tilde{e}_i$ are called *Kashiwara operators*

Remark 1. This does not depend on the choice of θ_i

$$\underbrace{\bullet \xrightarrow{i} \cdots \xrightarrow{i}}_{\varepsilon_i(b)} \underset{b}{\bullet} \underbrace{\xrightarrow{i} \cdots \xrightarrow{i} \bullet}_{\varphi_i(b)}$$

2. If $\theta_i(b) = u_k^{(\ell_j)}$, $\varepsilon_i(b) = k$ and $\varphi_i(b) = \ell_j - k$

$$\langle h_i, \mathrm{wt}\, b\rangle = \ell_j - 2k$$

ie. $\boxed{\langle h_i, \mathrm{wt}\, b\rangle = \varphi_i(b) - \varepsilon_i(b)}$

PROPERTIES:

① $\{M_j\}$ family of module in $\mathcal{O}_{int}(\mathfrak{g})$, with (L_j, B_j) crystal bases. Then

$\bigoplus (L_j, B_j)$ crystal base for $\bigoplus M_j$

②! (L_1, B_1) cb for M_1, (L_2, B_2) cb for M_2

1. $(L_1, B_1) \otimes (L_2, B_2) = (L_1 \otimes L_2, B_1 \times B_2)$ cb for $M_1 \otimes_Q M_2$

2. $$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

Proof : → Reduce to $U_q(sl_2)$-case.

Remark:

$$wt(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1),\ \varepsilon_i(b_2) - \langle h_i, wt\, b_1\rangle)$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2),\ \varphi_i(b_1) + \langle h_i, wt\, b_2\rangle)$$

Corollary

a) $b_1 \otimes b_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$ is a highest weight vector (i.e. $\tilde{e}_i(b_1 \otimes b_2) = 0,\ \forall i$) iff $\tilde{e}_i b_1 = 0$ and
$\langle h_i, wt\, b_1\rangle \geq \varepsilon_i(b_2)$, $\forall i$

b) $b_1 \otimes \dots \otimes b_N \in \mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_N$ h.w. vector iff $b_1 \otimes \dots \otimes b_k$ is a h.w. vector, $\forall k = 1, \dots, N$.

3 EXISTENCE

i) $\exists!$ $V(\lambda)$ irred. $(L(\lambda), B(\lambda))$ s.t. $B(\lambda)_\lambda = \{\underset{\text{h.w.}}{u_\lambda}\}$

ii) $B(\lambda) = \{ \tilde{f}_{i_1}^{a_1} \tilde{f}_{i_2}^{a_2} \cdots \tilde{f}_{i_k}^{a_k} u_\lambda ,\ \underset{\forall i_1 \dots i_k}{a_i > 0} \} \setminus \{0\}$

CONSEQ.:

ii) $\to$ $B(\lambda)$ connected

Dec. of M into irred $\longleftrightarrow$ Dec of B into connected comp $\longleftrightarrow$ h.w vectors in B

CLASSIFICATION OF GCM

Let $A = (a_{ij})_{1 \le i,j \le n}$ be an indecomposable GCM.

(Finite) $\det A \ne 0$; $\exists u > 0$ s.t. $Au > 0$;
$Av \geq 0 \Rightarrow v > 0$ or $v = 0$ $\Big\}$ $\Leftrightarrow$ all princ minors > 0

(Affine) rank $A = n-1$; $\exists u > 0$ s.t. $Au = 0$
$Av \geq 0 \Rightarrow Av = 0$ $\Big\}$ $\Leftrightarrow$ $\det A = 0$, all proper princ min > 0

(Indefinite) $\exists u > 0$ s.t. $Au < 0$
$Av \geq 0$ and $v \geq 0 \Rightarrow v = 0$

DYNKIN DIAGRAMS = graph assoc to A

vertices $1, 2, \ldots, n$

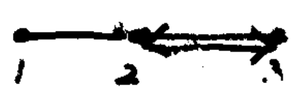
edges and arrows :

- If $a_{ij} a_{ji} \le 4$ and $|a_{ij}| \geq |a_{ji}|$
  i —— j $\to$ $|a_{ij}|$ edges , arrow toward i if $|a_{ij}| > 1$
- $a_{ij} a_{ji} > 4$, i —— j a bold faced edge with $(|a_{ij}|, |a_{ji}|)$

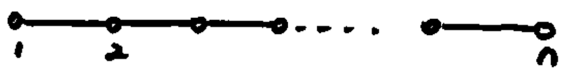
Ex :

1. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

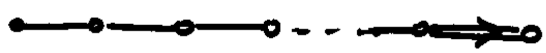
2. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$



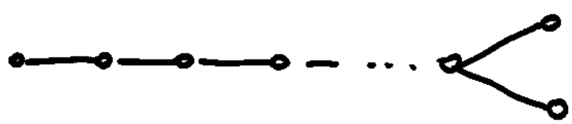
CLASSIFICATION OF SIMPLE FINITE-DIM LIE ALGEBRAS

$A_n, n>0$ 

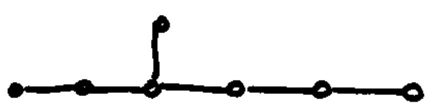
$sl_{n+1}(\mathbb{C})$

$B_n, n>1$  $so_{2n+1}(\mathbb{C})$

$C_n, n>2$  $sp_{2n}(\mathbb{C})$

$D_n, n>3$  $so_{2n}(\mathbb{C})$

E_6 

E_7  exceptional

E_8 

F_4 

G_2 

CRYSTAL BASES FOR A_{n-1}

Realized by $sl_n(\mathbb{C}) = \{T \in M_n(\mathbb{C}) \,/\, tr\,T = 0\}$

Cartan data:

$$E_{ij} = \begin{cases} 1 & \text{in } (i,j) \\ 0 & \text{otherwise} \end{cases} \qquad \begin{aligned} e_i &= E_{i,i+1} \\ f_i &= E_{i+1,i} \\ h_i &= E_{ii} - E_{i+1\,i+1} \end{aligned}$$

$$\mathfrak{g} = sl_n(\mathbb{C}) = \langle e_i, f_i, h_i,\ i = 1 \ldots n-1 \rangle$$

$$\mathfrak{h} = \langle h_i \rangle = \text{diagonal matrices of trace } 0$$

$$\varepsilon_i \in \mathfrak{h}^* \ , \quad \varepsilon_i(D) = d_{ii}$$

simple roots $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$

fundam. weights $\Lambda_i := \varepsilon_1 + \ldots + \varepsilon_i$

The quantum special linear algebra $U_q(sl_n)$

$= \langle e_i, f_i, t_i, t_i^{-1} \,/\, \text{relations} \rangle$

GCM
$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 \\ 0 & \cdots & & & 2 & -1 \\ 0 & \cdots & & & -1 & 2 \end{pmatrix}_{n-1 \times n-1}$$

Dynkin diagram

The weight lattice

$$P = \mathbb{Z}\Lambda_1 \oplus \dots \oplus \mathbb{Z}\Lambda_{n-1}$$

$$P_+ = \mathbb{Z}_{\geq 0}\Lambda_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\Lambda_{n-1}$$ dominant integral weights

Recall The irred in $\mathcal{O}^q_{int}(\mathfrak{sl}_n)$ (or any $\mathfrak{g}$) are $V(\lambda)$, $\lambda \in P_+$.

CRYSTAL BASES FOR $V(\lambda)$, $\lambda \in P_+$

Notation:

If $\lambda = \lambda_1\Lambda_1 + \dots + \lambda_{n-1}\Lambda_{n-1} \in P_+ \iff \lambda_i \geq 0$

$\updownarrow$

$\tilde{\lambda} = (\lambda_1 + \dots + \lambda_{n-1}, \dots, \lambda_{n-1})$ partition notation

① The vector representation

$V(\Lambda_1)$, $\lambda = 1\cdot\Lambda_1 + 0 \dots + 0\Lambda_{n-1}$

$V(\Lambda_1) = \bigoplus_{i=1}^{n} \mathbb{Q}\boxed{i}$ with $U_q(\mathfrak{sl}_n)$-action

$$e_j\boxed{i} = \delta_{j\,i-1}\boxed{i-1}$$

$$f_j\boxed{i} = \delta_{ji}\boxed{i+1}$$

$$t_j^{\pm}\boxed{i} = q^{\pm\langle h_j, \varepsilon_i\rangle}\boxed{i} \qquad wt\boxed{i} = \varepsilon_i$$

$\{\boxed{i}, 1 \leq i \leq n\}$ crystal base

$$B(\Lambda_1): \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \dots \longrightarrow \boxed{n-1} \xrightarrow{n-1} \boxed{n}$$

Let Y be a Young diagram with at most $(n-1)$ rows and $B(Y)$ the set of all cs tableau of shape Y filled with $1, 2, \ldots, n$

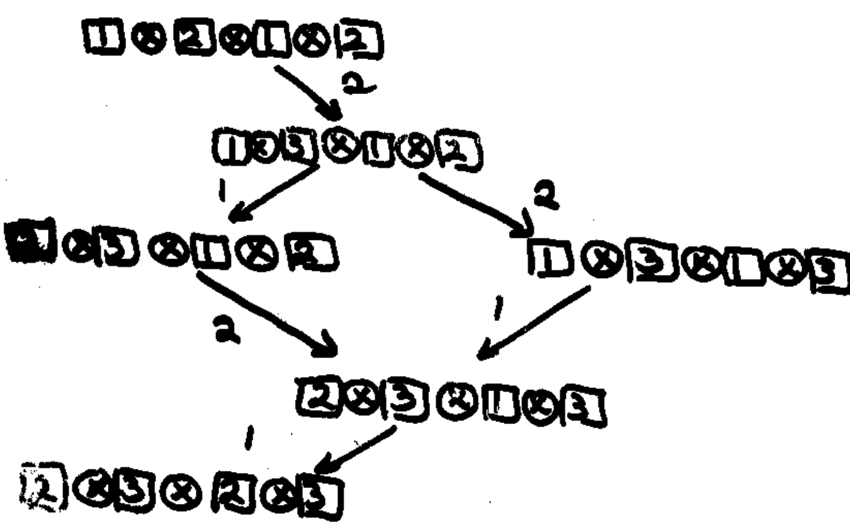
Using (any) reading embed $B(Y) \subset B^{\otimes N}$ where $B = B(\Lambda_1)$ and $N = |Y|$.

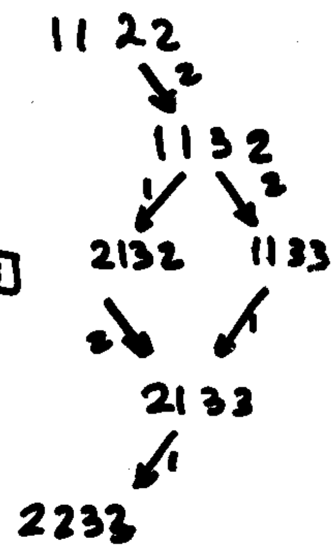
<u>Ex</u>: $n = 3$, $Y = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$

B: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$

$$B(Y) = \left\{ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline\end{array} \right\}$$

1212	1312	2312	1313	2313	2323
1122	1132	2132	1133	2133	2233





General representations $V(\lambda)$

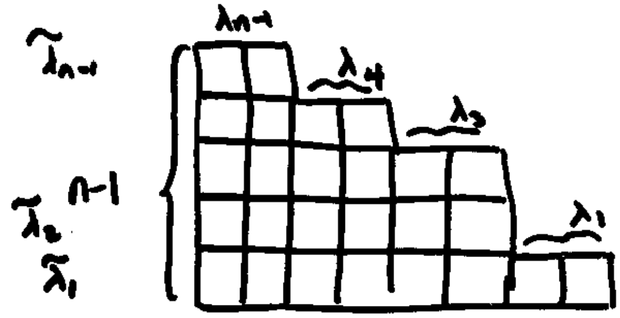
Idea: $\lambda = \lambda_1 \Lambda_1 + \dots + \lambda_{n-1} \Lambda_{n-1}$

Embed $B(\lambda) \hookrightarrow B(\Lambda_1)^{\otimes \lambda_1} \otimes B(\Lambda_2)^{\otimes \lambda_2} \dots$

$$\hookrightarrow B(\Lambda_1)^{\otimes \lambda_1} \otimes \left(B(\Lambda_1) \otimes B(\Lambda_1)\right)^{\otimes \lambda_2} \dots$$

$$= B(\Lambda_1)^{\lambda_1 + 2\lambda_2 + \dots + (n-1)\lambda_{n-1}}$$

For $\lambda \in P_+$ associate $Y(\lambda) = \tilde{\lambda}$ the Young diagram



$\lambda_i = \#$ col. of length i

$N = \tilde{\lambda}_1 + \dots + \tilde{\lambda}_{n-1} = \sum i \lambda_i$

$$\lambda = \lambda_1 \Lambda_1 + \dots + \lambda_{n-1} \Lambda_{n-1} = \tilde{\lambda}_1 \varepsilon_1 + \dots + \tilde{\lambda}_{n-1} \varepsilon_{n-1}$$

Column strict tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 4 \\ \hline 3 & 4 & 5 \\ \hline 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 2 & 2 & 3 \\ \hline \end{array}$$

weight T = content T

$= (2,3,4,3,5) =$

$= 2\varepsilon_1 + 3\varepsilon_2 + 4\varepsilon_3 + 3\varepsilon_4 + 5\varepsilon_5$

$= -1\Lambda_1 - \Lambda_2 + \Lambda_3 + 2\Lambda_4 + 5\Lambda_5$

Reading

1. chinese $\quad 324235\,1341234 = \boxed{3} \otimes \boxed{2} \otimes \dots$

2. middle eastern $\quad 3221143325434$

THEOREM

If $\lambda \in P_+$ then $\mathcal{B}(\lambda)$ is isomorphic to the $U_q(sl_n)$-crystal $\mathcal{B}(Y)$ consisting of c.s. tableaux of shape $Y = \bar{\lambda}$.

The action of the Kashiwara operators on $B(Y)$ can be described as:

$$\tilde{e}_i, \tilde{f}_i$$

$b = \boxed{a_1} \otimes \cdots \otimes \boxed{a_N}$ in some reading $B^{\otimes N}$

- ignore $a_i \neq i, i+1$
- view $\boxed{i}$ as "(" and $\boxed{i+1}$ as ")"

  pair all parenthesis
- $\boxed{\tilde{e}_i}$ changes an $\boxed{i+1}$ into $\boxed{i}$ so that no other pairs of parenth.

  i.e. the rightmost ")" into "("

  $$))) . \boxed{)} ((((\longrightarrow))) . \boxed{(} (((($$

- $\tilde{f}_i$ changes $\boxed{i}$ into $\boxed{i+1}$

  $$))) ..) \boxed{(} ((.. (\longrightarrow))) .. \boxed{)} ((. ($$

Ex:

3 2 3 | 2 2 3 2 2 for e_2, f_2

) () | (() ((

e_2: (() (() ((

[illegible]3|22322

f_2:) ()) () ((

3 2 3 | 3 2 2 3 2 2