$U_q(\mathfrak{g})$-crystals

Set $\mathcal{B}$ with maps $\begin{cases} wt : \mathcal{B} \longrightarrow P \\ \tilde{e}_i, \tilde{f}_i : \mathcal{B} \longrightarrow \mathcal{B} \cup \{0\} \\ \varepsilon_i, \varphi_i : \mathcal{B} \longrightarrow \mathbb{Z} \cup \{-\infty\} \end{cases}$

st.

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt\, b \rangle$
2. $wt(\tilde{e}_i b) = wt\, b + \alpha_i$, $wt(\tilde{f}_i b) = wt\, b - \alpha_i$
3. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i b + 1$ if $\tilde{e}_i b \in \mathcal{B}$
4. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, if $\tilde{f}_i b \in \mathcal{B}$
5. $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}$
6. If $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$ then $\tilde{e}_i b = \tilde{f}_i b = 0$.

<u>Ex</u>: ① $\mathcal{B}(\lambda)$, $\lambda \in P^+$ is a $U_q(\mathfrak{g})$-crystal

② Let $\lambda \in P$, $T_\lambda = \{t_\lambda\}$ and define

$$wt\, t_\lambda = \lambda$$

$$\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$$

$$\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$$

T_λ is a $U_q(\mathfrak{g})$-crystal.

Crystal morphism

B_1, B_2 two $U_q(g)$ - crystals

$\psi : B_1 \to B_2$ is a crystal morphism if

$$\psi : B_1 \cup \{0\} \longrightarrow B_2 \cup \{0\} \text{ and}$$

(1) $\psi(0) = 0$

(2) If $b \in B_1$, $\psi(b) \in B_2$ then

$$wt(\psi(b)) = wt\, b$$
$$\varepsilon_i(\psi(b)) = \varepsilon_i(b)$$
$$\varphi_i(\psi(b)) = \varphi_i(b) \quad \forall i$$

(3) If $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$
then $\tilde{f}_i \psi(b) = \psi(b')$ and $\psi(b) = \tilde{e}_i \psi(b')$.

If $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ bijection

ψ is called isomorphism

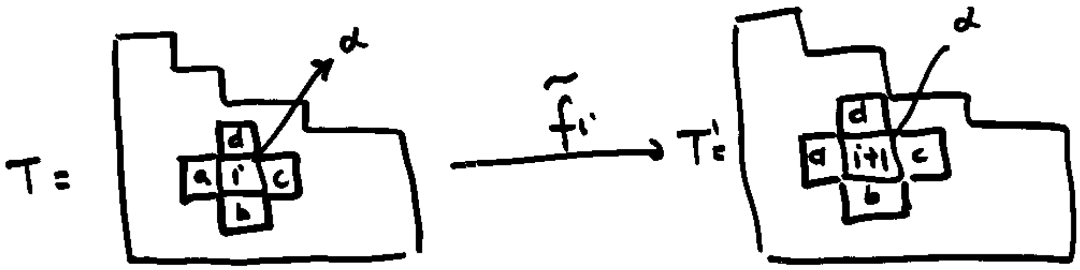
THEOREM

Let $\psi : B(Y) \longrightarrow B^{\otimes N}$ be a reading and $\tilde{e}_i, \tilde{f}_i$ defined on $B^{\otimes N}$ (by the parentheses rule).

Then $\tilde{e}_i$ and $\tilde{f}_i$ are well defined on $B(Y)$ (i.e. $B(Y)$ has a $U_q(sl_n)$ - crystal structure).

Pf:

Let $T \in B(Y) \hookrightarrow B^{\otimes N}$. Applying $\tilde{f}_i$ we change an $\boxed{i}$ into $\boxed{i+1}$



Show T' is C.S.

$a \leq i \Rightarrow a < i+1$ OK

$b < i \Rightarrow b < i+1$

$i < d$. What if $\underline{d = i+1}$

①  We claim that in this case the cell α is paired in the reading, hence it cannot be changed ✗

$i \leq c$. If $c = i$

②  in this case in the reading we claim that i in position c is not paired and precedes α in reading, thus α is not changing ✗

Assume the parenthesis rule in $B^{\otimes N}$.

A standard tableau decreasing in rows and increasing in col $\leftrightarrow$ Reading

$$\begin{array}{|c|c|c|}\hline 6 & & \\ \hline 5 & 4 & \\ \hline 3 & 2 & 1 \\ \hline\end{array}$$

THEOREM

The induced $U_q(\mathfrak{gl}_n)$-structure on $B(Y)$ does not depend on the choice of the reading.

$T \in B(Y)$
and let R_1, R_2 two readings
and $\omega_T^{R_1}, \omega_T^{R_2}$ the readings of T

Show ① $\omega_T^{R_1} \overset{K}{\underset{d}{\simeq}} \omega_T^{R_2}$

② $\tilde{f}_i \omega_T^{R} \overset{K}{\underset{d}{\simeq}} \tilde{f}_i \omega_T^{R}$ $(\text{or } \tilde{e}_i)$

Here $\overset{K}{\underset{d}{\simeq}}$ dual Knuth relation obtained via column insertion

<u>Knuth</u>

$xzy \simeq zxy$, if $x \le y < z$
$yxz \simeq yzx$, if $x < y \le z$

<u>Dual Knuth</u>

$$\begin{cases} xzy \equiv zxy & x < y \le z \\ yxz \equiv yzx, & x \le y < z \end{cases}$$

Robinson - Schensted correspondence

Let w be a word in $\{1, 2, \ldots, n\}$. Then

$$w \xleftrightarrow[1-1]{R\text{-}S} (P, Q)$$

P = insertion tableau $\leftrightarrow$ column strict

Q = recording tableau $\leftrightarrow$ standard

1. **Thm** $w_1 \overset{K}{\simeq} w_2$ iff $P(w_1) = P(w_2)$

insertion: row $\leftrightarrow$ Knuth; column $\leftrightarrow$ dual Knuth

2. **Prop** Let P be a CS tableau, R a reading and $\overline{R}$ the standard tableau of R.

Then $w_P^R \leftrightarrow (P, \overline{R})$

$$R = \begin{matrix} 6 & & \\ 5 & 4 & \\ 3 & 2 & 1 \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} 6 & & \\ 4 & 5 & \\ 1 & 2 & 3 \end{matrix} = \overline{R}$$

THEOREM

Let $\lambda = \tilde{\lambda}_1 \varepsilon_1 + \dots + \tilde{\lambda}_{n-1} \varepsilon_{n-1} \in P_+$.
Then $B(\lambda)$ the crystal graph of $V(\lambda)$ is isomorphic to $\mathcal{B}(Y) =$ c.s tableaux of shape $Y = \tilde{\lambda}$.

Recall $\mathbb{B} = B(\Lambda_1) = B(1 \cdot \varepsilon_1)$. From the complete reducibility Thm

$$\mathbb{B}^{\otimes N} = \dot{\bigcup} \text{ connected components}$$

each component $\simeq B(\mu)$, $\mu \in P_+$
i.e. contains a h.w. vector of weight μ

Since $\mathcal{B}(Y) \hookrightarrow \mathbb{B}^{\otimes N}$ via a reading

show $\mathcal{B}(Y)$ isom to a connected component that contains a h.w. vector of wt λ.

Let $T_Y =$

$$\begin{array}{|c|c|c|c|c|c|}
\hline
3 & \cdots & & & & \\
\hline
2 & 2 & \cdots & 2 & & \\
\hline
1 & 1 & 1 & 1 & \cdots & 1 \\
\hline
\end{array}$$

<u>Claim</u> T_Y is a hw vector of wt λ

$$\text{wt } T_Y = \tilde{\lambda}_1 \varepsilon_1 + \tilde{\lambda}_2 \varepsilon_2 + \dots + \tilde{\lambda}_{n-1} \varepsilon_{n-1} = \lambda$$

If we read ME

$$\underbrace{1\,1\,1 \dots 1}_{\tilde{\lambda}_1} \quad \underbrace{2\,2 \dots 2}_{\tilde{\lambda}_2} \quad 3 \dots$$

$$((((\quad)))),$$

$$\tilde{e}_i(T_Y) = 0 \quad \forall\, i = 1, \dots, n-1$$

Claim 2 $B(Y)$ is connected. Moreover $\forall T \in B(Y)$

$\exists i_1, \dots, i_r$ s.t. $T = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} T_Y$

(or $\tilde{e}_{i_1} \dots \tilde{e}_{i_r} T = T_Y$).

Lemma

Claim 2 $\Longleftrightarrow$ if $\tilde{e}_i T = 0$, $\forall i = 1, \dots, n-1$ then $T = T_Y$.

Proof lemma

$1 \Rightarrow 2$ if $T = T_Y$ done

otherwise $\exists\, r \geq 1$ s.t. $T = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} T_Y$ and $i_1, \dots, i_r$

apply $\tilde{e}_{i_1}$ so $\underset{0}{\underset{\shortparallel}{\tilde{e}_{i_1} T}} = \underbrace{\tilde{f}_{i_2} \cdots \tilde{f}_{i_r} T_Y}_{\neq 0}$ $\times$

$2 \Rightarrow 1$ Let $T \in B(Y)$. Assume $T \neq T_Y$ then $\exists\, i_1$ s.t. $\tilde{e}_{i_1} T \neq 0$, if $\tilde{e}_{i_1} T = T_Y$ done, otherwise continue.

Note: the sequence $\tilde{e}_{i_1} \dots \tilde{e}_{i_r} T$ if not zero contains distinct elements as

$$wt(\tilde{e}_{i_1} \dots \tilde{e}_{i_r} T) = wt\, T + \alpha_{i_1} + \dots + \alpha_{i_r}.$$

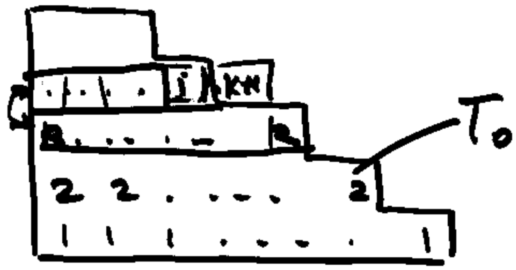
Graph finite $\to$ STOP is.

$\exists\, i_1 \dots i_r$ s.t. $\tilde{e}_{i_1} \cdots \tilde{e}_{i_r} T = T_Y$

$\square$

$e_{i_1} \dots e_{i_r} T \neq T$

Let $T \in B(Y)$ s.t. $\tilde{e}_i T = 0$, $\forall i$. Let k be the largest s.t. the first k rows in T are the same as in T_Y $(k \geq 0)$

$T =$ 

Note $j > k+1$

Read $T \overset{ME}{=} T_0 \otimes j \otimes \ldots$ the rest...

By Lemma (before) we must have $\tilde{e}_i (T_0 \otimes j) = 0$, same lemma (corollary)

$$\boxed{\tilde{e}_i (T_0) = 0 \quad \& \quad \varphi_i (T_0) \geq \varepsilon_i (j)}, \forall i$$

$\varphi_i (T_0) = 0$, $\forall i \geq k+1$

$\varepsilon_i (j) = 0$, $\forall i \geq k+1$

$\varepsilon_i(\boxed{j}) = \begin{cases} 1, & i = j-1 \\ 0, & \text{otherwise} \end{cases}$

Recall $\boxed{1} \xrightarrow{1} \boxed{2} \ldots \xrightarrow{j-1} \boxed{j} \xrightarrow{j} \ldots \boxed{n+1}$

$j \leq k+1$ $\cancel{\times}$. So $T = T_Y$.

Note $\psi : \underset{T_Y}{B(Y)} \xrightarrow{\sim} \underset{u}{B(\lambda)}$ iso

$\quad\; T_Y \longrightarrow u$

$B(Y) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} T_Y\}$ $\quad B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u\}$

ψ commutes with $\tilde{f}_i, \tilde{e}_i$

LITTLEWOOD-RICHARDSON RULE

Decompose $V(\lambda) \otimes V(\mu) = \bigoplus V(\nu)$

$\lambda, \mu, \nu \in P_+$

i.e. decompose $B(\lambda) \otimes B(\mu)$ into connected components by finding <u>the h.w. vectors</u>

<u>Def</u>:

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1})$ be a partition.

We denote by $\lambda[j]$ the partition ~~obtai~~

$(\lambda_1, \ldots \lambda_j + 1, \ldots \lambda_{n-1})$ if possible

and $\emptyset$ if $(\lambda_{j-1} < \lambda_j + 1)$.

Do L-R in two steps:

<u>THM 1</u>

$$B(\lambda) \otimes B = \bigoplus_{\boxed{j} \in B} B(\lambda[j])$$

<u>Pf</u>:

By Corollary the h.w. vectors in $B(\lambda) \otimes B$ are

$T_y \otimes \boxed{j}$ s.t. $\varepsilon_i(\boxed{j}) \leq \varphi_i(T_y) = \lambda_i - \lambda_{i+1}$

$$\varepsilon_i(\boxed{j}) = \begin{cases} 1, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow \lambda_{j-1} - \lambda_j > 0 \Rightarrow \lambda[j]$ is a partition

of weight $\lambda + \varepsilon_j$

Example

$n=3$, $\lambda = (2,1,1) = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

$\lambda[1] =$  , $\lambda[2] =$  , $\lambda[3] =$  $= \phi$.

$\mathcal{B}(\lambda) \otimes \mathbb{B} = \mathcal{B}($  $) \oplus \mathcal{B}($  $)$

THEOREM (Littlewood - Richardson rule)

Let $\lambda, \mu \in P_+$

$$\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) = \bigoplus_{\boxed{j_1} \otimes \cdots \otimes \boxed{j_N} \in \mathcal{B}(\mu)} \mathcal{B}(\lambda[j_1, \ldots, j_N])$$

with $N = |\mu|$

and $\lambda[j_1 \ldots j_N] = \phi$ unless $\lambda[j_1, \ldots j_k]$ partition, $\forall \; k = 1, \ldots, N$.

Pf:

Embed $\mathcal{B}(\mu) \hookrightarrow \mathbb{B}^{\otimes N}$ via a reading

Let $T \otimes T' \in \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ a h.w. vector

$$T \otimes T' = T \otimes \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}$$

$$T' = \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}$$

By Corollary

$T \otimes \boxed{j_1} \otimes \cdots \otimes \boxed{j_k}$ is h.w. vector, $\forall k = 1, \ldots, N$

▼ $\lambda[j_1, \ldots, j_k]$ the partition obtained by adding a box in $j_1, j_2, \cdots j_k$ row s.t. all intermediate steps are partitions (i.e. $\lambda[j_1]$, $\lambda[j_1, j_2]$, $\ldots$)

$T \otimes \boxed{j_1} \otimes \cdots \otimes \boxed{j_N}$ is a h.w vector

iff $\lambda[j_1, \ldots, j_k]$ partition, $\forall k$

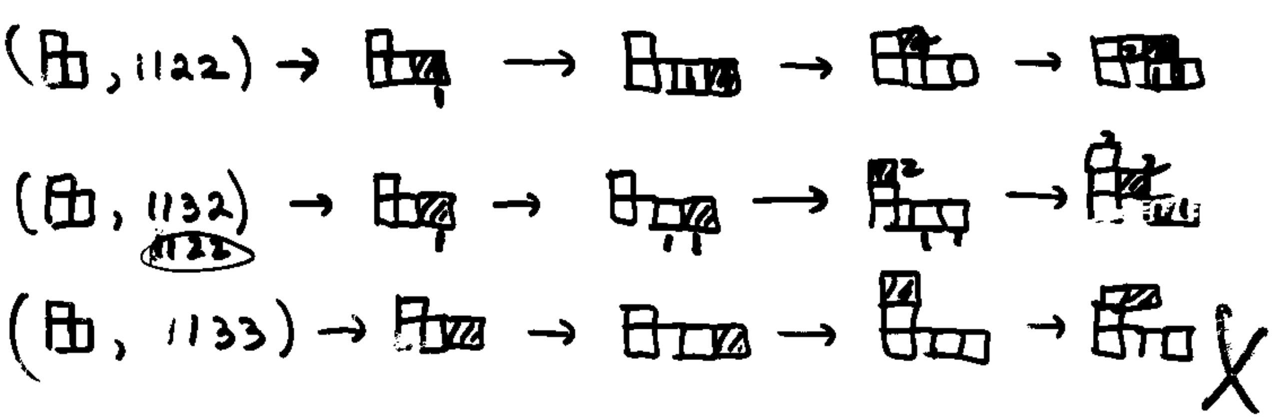
Thus we get . . .

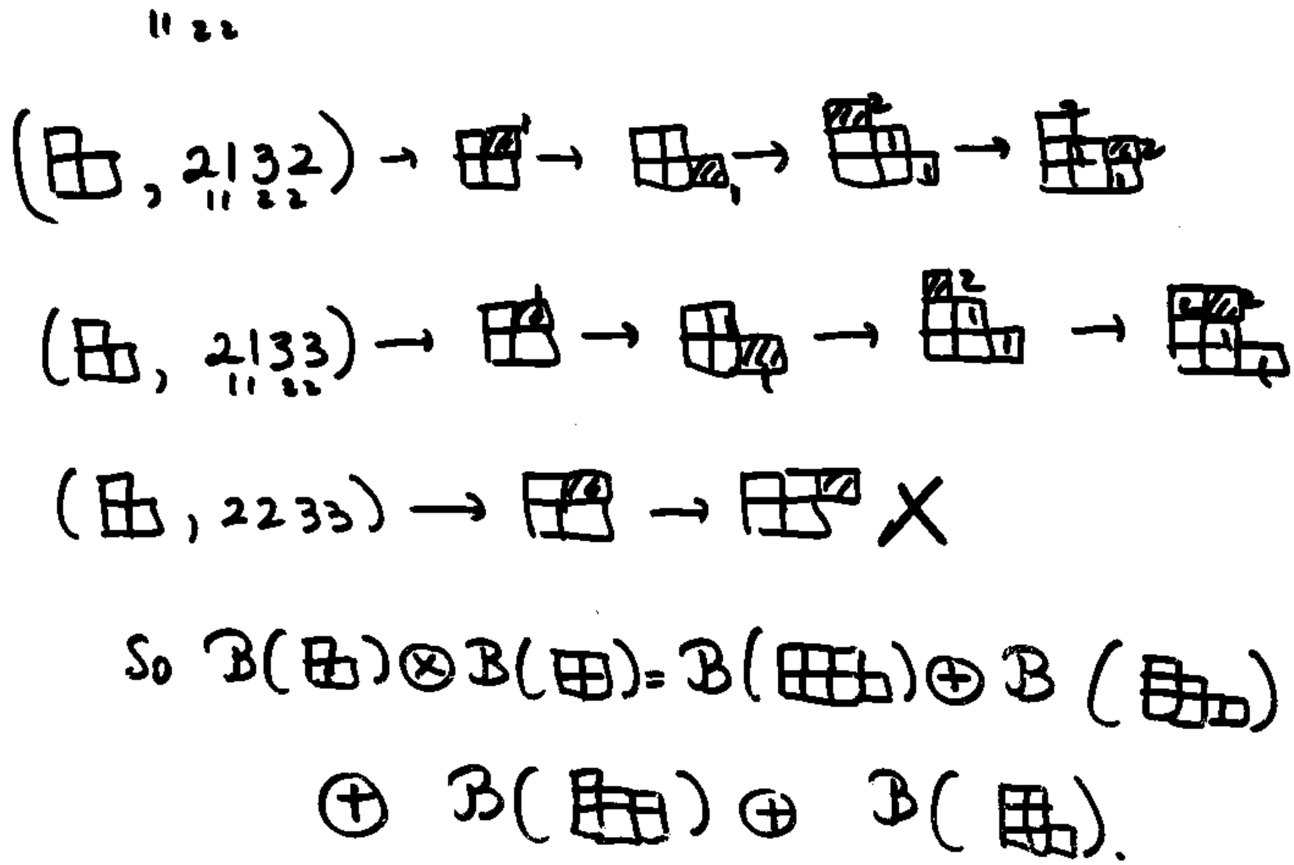
<u>Example</u>

$n = 3$, $\lambda =$  , $\mu =$ 

$$B(Y) = \left\{ \begin{matrix} 2\,2 \\ 1\,1 \end{matrix},\ \begin{matrix} 2\,3 \\ 1\,1 \end{matrix},\ \begin{matrix} 3\,3 \\ 1\,1 \end{matrix},\ \begin{matrix} 2\,3 \\ 1\,2 \end{matrix},\ \begin{matrix} 3\,3 \\ 1\,2 \end{matrix},\ \begin{matrix} 3\,3 \\ 2\,2 \end{matrix} \right\}$$

ME 1122 1132 1133 2132 2133 2233





Remark

Equivalent interpretations ?