Quantum Affine Algebras

Cartan matrix of affine type

$$A = (a_{ij})_{i,j\in I} \qquad I = \{0,1,2,\dots,n\}$$

dual weight lattice

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$$ free abelian group

Cartan subalgebra $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$

simple roots α_i: $\alpha_i(h_j) = a_{ij}$ $\alpha_i(d) = \delta_{i0}$

simple coroots h_i

fundamental weights Λ_i: $\Lambda_i(h_j) = \delta_{ij}$ $\Lambda_i(d) = 0$

simple roots $\Pi = \{\alpha_i \mid i \in I\}$ simple coroots $\Pi^\vee = \{h_i \mid i \in I\}$

affine weight lattice(+)

$$\boxed{\begin{array}{c}\text{affine Cartan datum} \\ (A, \Pi, \Pi^\vee, P, P^\vee)\end{array}}$$

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$$
$$= \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{\delta}{d_0}$$

null root

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n$$

$d_0 = 1$ except for $A^{(2)}_{2n}$

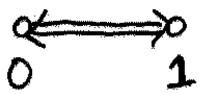
Canonical central element

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n$$

affine dominant integral weights

$$P^+ = \{ \lambda \in P \mid \lambda(h_i) \in \mathbb{N} \ \forall i \in I \}$$

level of λ $\quad \ell = \lambda(c) = \langle \lambda, c \rangle$

type: $A_1^{(1)}$ 

$\alpha_0(d) = 1 \quad \alpha_1(d) = 0$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \longrightarrow \begin{matrix} \alpha_0(h_0) = 2 & \alpha_0(h_1) = -2 \\ \alpha_1(h_0) = -2 & \alpha_1(h_1) = 2 \end{matrix}$$

null root $\delta = \alpha_0 + \alpha_1$

canonical central element $c = h_0 + h_1$

$\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$

$\alpha_1 = -2\Lambda_0 + 2\Lambda_1$

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$$

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}d$$

$$P^+ = \mathbb{N}\Lambda_0 \oplus \mathbb{N}\Lambda_1 \oplus \mathbb{Z}\delta$$

$$\Pi = \{\alpha_0, \alpha_1\} \qquad \Pi^\vee = \{h_0, h_1\}$$

To each Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$

can associate an affine Kac-Moody algebra

$\mathfrak{g}$ center of $\mathfrak{g}$ is generated by c - canonical central element

for $A_n^{(1)}$ $\mathfrak{g}$ is $\widehat{sl}_n$

also can associate a quantum affine algebra $U_q(\mathfrak{g})$

this is the associative algebra over $\mathbb{C}(q)$ generated by $e_i, f_i, t_i^{\pm}, q^h$ $\quad K_i^{\pm} = q^{\pm h_i}$

$i \in I \quad t_i^{\pm} = K_i^{\pm} \quad h \in P^\vee$

with relations given by Alejandra on 10/10/2002

Classical weights $\quad \bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$

$\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$

classical dominant integral weights $\quad \bar{P}^+ = \{\lambda \in \bar{P} \mid \lambda(h_i) \geq 0 \ \forall i \in I\}$

$U_q'(g)$ is the subalgebra of $U_q(g)$ generated by e_i f_i $K_i^{\pm 1}$ $(i \in I)$

$K_i^{\pm 1} = t_i^{\pm d_i}$

(does not contain all elements q^h for $h \in P^\vee$) missing q^d

Example $U_q'(\widehat{sl_2})$ is Hopf algebra

generated by $e_0, e_1, f_0, f_1, K_0, K_1, K_0^{-1}, K_1^{-1}$

$\Delta(K_0) = K_0 \otimes K_0 \quad \Delta(K_1) = K_1 \otimes K_1$

$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$

$\varepsilon(K_i) = 1 \quad \varepsilon(e_i) = \varepsilon(f_i) = 0$

$S(K_i) = K_i^{-1} \quad S(e_i) = -e_i K_i \quad S(f_i) = -K_i^{-1} f_i$

$[K_i, K_j] = 0$

etc..

$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \begin{aligned} \alpha_0 &= 2\Lambda_0 - 2\Lambda_1 \\ \alpha_1 &= 2\Lambda_1 - 2\Lambda_0 \end{aligned}$

$\mathcal{B}$ crystal for $U_q'(\mathfrak{g})$

$$\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i \qquad \phi(b) = \sum_{i \in I} \phi_i(b)\Lambda_i$$

where
$$\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k(b) \neq 0\}$$
$$\phi_i(b) = \max\{k \mid \tilde{f}_i^k(b) \neq 0\}$$

Note $wt(b) = \phi(b) - \varepsilon(b)$

Example $U_q'(\widehat{s\ell_3})$ module $\qquad A_2^{(1)}$

$$V = \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3$$

$$e_i v_j = \begin{cases} v_{j-1} & \text{if } j \equiv i+1 \pmod 3 \\ 0 & \text{else} \end{cases}$$

$$f_i v_j = \begin{cases} v_{j+1} & \text{if } j \equiv i \pmod 3 \\ 0 & \text{else} \end{cases} \qquad c = h_0 + h_1 + h_2$$

$$K_i v_j = \begin{cases} q v_j & \text{if } j \equiv i \pmod 3 \\ q^{-1} v_j & \text{if } j \equiv i+1 \pmod 3 \\ v_j & \text{else} \end{cases}$$

$$\varepsilon(\boxed{1}) = \Lambda_0$$
$$\varepsilon(\boxed{2}) = \Lambda_1$$
$$\varepsilon(\boxed{3}) = \Lambda_2$$
$$\phi(\boxed{1}) = \Lambda_1$$
$$\phi(\boxed{2}) = \Lambda_2$$
$$\phi(\boxed{3}) = \Lambda_0$$

$$\Longleftarrow \qquad \begin{tikzcd} \boxed{1} \arrow[d, "1"] \\ \boxed{2} \arrow[d, "2"] \\ \boxed{3} \arrow[uu, bend right, "0"'] \end{tikzcd}$$

$$\tilde{f}_1(\boxed{1}) = \boxed{2} \qquad \tilde{e}_1(\boxed{2}) = \boxed{1}$$
$$\tilde{f}_2(\boxed{2}) = \boxed{3} \qquad \tilde{e}_2(\boxed{3}) = \boxed{2}$$
$$\tilde{f}_0(\boxed{3}) = \boxed{1} \qquad \tilde{e}_0(\boxed{1}) = \boxed{3}$$
$$\tilde{f}_1(\boxed{2}) = \tilde{f}_1(\boxed{3}) = \tilde{f}_2(\boxed{1}) = \ldots =$$
$$= \tilde{e}_0(\boxed{3}) = 0$$

$\mathcal{B}$ is a perfect crystal of level ℓ

$$\bar{P}^+_\ell = \{\lambda \in \bar{P}^+ \mid \langle c, \lambda \rangle = \ell\}$$

(1) there exists a finite dimensional $U_q'(g)$ with crystal basis whose crystal graph is isomorphic to $\mathcal{B}$

(2) $\mathcal{B} \otimes \mathcal{B}$ is connected

(3) there is a $\lambda_0 \in \bar{P}$ such that for all $b \in \mathcal{B}$ $\quad \varepsilon(b) = \lambda_0 - \sum_{i \neq 0} a_i^b \alpha_i \quad a_i^b \geq 0$

AND exists! b_{λ_0} with $wt(b_{\lambda_0}) = \lambda_0$

(4) for all $b \in \mathcal{B}$ $\quad \langle c, \varepsilon(b) \rangle \geq \ell$

(5) For each $\lambda \in \bar{P}^+_\ell$ there are <u>unique</u> vectors b_λ and b^λ

$$\varepsilon(b^\lambda) = \lambda \qquad \phi(b_\lambda) = \lambda$$

(5) is important. Says ε and ϕ are bijections
$\{b \in \mathcal{B} \mid \langle c, \varepsilon(b) \rangle = \ell\} \cong \bar{P}^+_\ell$

Example:

$$\begin{tikzcd} \boxed{1} \arrow[d, "1"'] \\ \boxed{2} \arrow[d, "2"'] \\ \boxed{3} \arrow[uu, bend right, "0"'] \end{tikzcd}$$

$U_q'(\widehat{sl}_3)$-crystal

$\mathcal{B}\otimes\mathcal{B}$

$$\begin{tikzcd} \boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{2}\otimes\boxed{1} \arrow[r, "2"] \arrow[d, "1"] & \boxed{3}\otimes\boxed{1} \arrow[d, "1"] \\ \boxed{1}\otimes\boxed{2} \arrow[d, "2"] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"] & \boxed{3}\otimes\boxed{2} \arrow[d, "2"] \arrow[ll, bend left, "0"] \\ \boxed{1}\otimes\boxed{3} \arrow[r, "1"] \arrow[uu, bend left, "0"] & \boxed{2}\otimes\boxed{3} \arrow[uu, bend left, "0"] & \boxed{3}\otimes\boxed{3} \arrow[ll, bend left, "0"] \end{tikzcd}$$

b^{Λ_0}
$\shortparallel$

$wt(\boxed{1}) = \Lambda_1 - \Lambda_0 \qquad \varepsilon(\boxed{1}) = \Lambda_0 \qquad \phi(\boxed{1}) = \Lambda_1$

$wt(\boxed{2}) = \Lambda_2 - \Lambda_1 \qquad \varepsilon(\boxed{2}) = \Lambda_1 \qquad \phi(\boxed{2}) = \Lambda_2$

$wt(\boxed{3}) = \Lambda_0 - \Lambda_2 \qquad \varepsilon(\boxed{3}) = \Lambda_2 \qquad \phi(\boxed{3}) = \Lambda_0$

all level 1

$\alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2 \qquad \alpha_2 = -\Lambda_0 - \Lambda_1 + 2\Lambda_2$

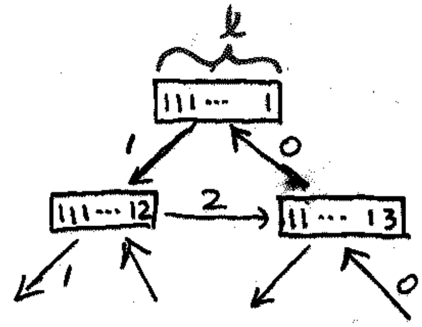
$wt(\boxed{3}) = wt(\boxed{2}) - \alpha_2 \qquad wt(\boxed{2}) = wt(\boxed{1}) - \alpha_1$

$= wt(\boxed{1}) - \alpha_2 - \alpha_1 \qquad \Lambda_2 - \Lambda_1 = \Lambda_1 - \Lambda_0 + \Lambda_0 - 2\Lambda_1 + \Lambda_2$

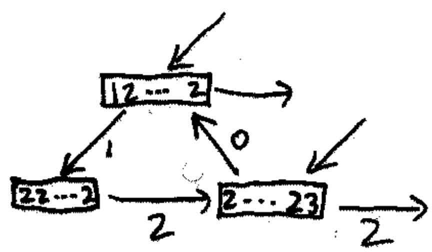
More generally $\quad \varepsilon\left(\overset{a}{\boxed{1\cdots1}}\overset{b}{\boxed{2\cdots2}}\overset{c}{\boxed{3\cdots3}}\right) = a\Lambda_0 + b\Lambda_1 + c\Lambda_2$

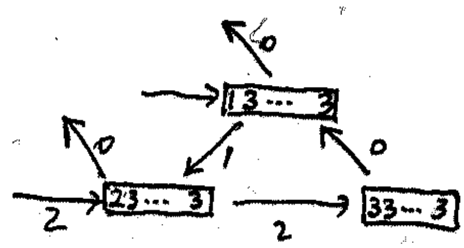
$$\phi\left(\overset{a}{\boxed{1\cdots1}}\overset{b}{\boxed{2\cdots2}}\overset{c}{\boxed{3\cdots3}}\right) = c\Lambda_0 + a\Lambda_1 + b\Lambda_2$$

all elements of level ℓ



$U_q'(\widehat{sl_3})$





let $b^\lambda = \overset{a_0}{\boxed{1\cdots1}}\overset{a_1}{\boxed{2\cdots2}}\overset{a_2}{\boxed{3\cdots3}}$

$$\varepsilon(b^\lambda) = \lambda = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2$$

let $b_\lambda = \overset{a_1}{\boxed{11\cdots1}}\overset{a_2}{\boxed{2\cdots2}}\overset{a_0}{\boxed{3\cdots3}}$

then $\phi(b_\lambda) = \lambda$

$$\bar{P}^+_\ell = \{a\Lambda_0 + b\Lambda_1 + c\Lambda_2 \mid a+b+c=\ell\}$$