A crystal $\mathcal{B}$ is simple if

1. It is the crystal of a $U_q'(\mathfrak{g})$ module
2. There is a weight $\lambda \in \bar{P}^+$ such that there is a unique $u(\mathcal{B})$ element with $wt(u(\mathcal{B})) = \lambda$ and the weight of any extremal vector is in $\overline{W}\lambda$

A crystal $\mathcal{B}$ is perfect of level ℓ

1. $\mathcal{B}$ is simple
2. $\min\{\langle \varepsilon(b), c \rangle : b \in \mathcal{B}\} = \ell$
3. $\forall \lambda \in \bar{P}_\ell^+$ there are elements b_λ and b^λ with $\langle \varepsilon(b_\lambda), c \rangle = \ell$

$$\phi(b_\lambda) = \lambda \quad \text{and} \quad \varepsilon(b^\lambda) = \lambda$$

"J-components" of a crystal are the connected components of the crystal if we erase the 0-arrows.

<u>Thrm</u> 1. $\mathcal{B}$ simple $\Rightarrow \mathcal{B}\otimes\mathcal{B}$ simple

2. $\mathcal{B}$ simple $\Rightarrow \mathcal{B}$ connected
(but not necess. J-connected)

<u>Thrm</u> $\mathcal{B}_1$ and $\mathcal{B}_2$ are simple

1. There is an isomorphism

$$\sigma_{\mathcal{B}_2,\mathcal{B}_1} : \mathcal{B}_2\otimes\mathcal{B}_1 \longrightarrow \mathcal{B}_1\otimes\mathcal{B}_2$$

2. There is a function $H = H_{\mathcal{B}_2,\mathcal{B}_1} : \mathcal{B}_2\otimes\mathcal{B}_1 \to \mathbb{Z}$ constant on J-components

$$H(\tilde{e}_0(b_2\otimes b_1)) = H(b_2\otimes b_1) + \begin{cases} -1 & \text{if } \varepsilon_0(b_2) > \varphi_0(b_1) \text{ and } \varepsilon_0(b_1') > \varphi_0(b_2') \\ 1 & \text{if } \varepsilon_0(b_2) \le \varphi_0(b_1) \text{ and } \varepsilon_0(b_1') \le \varphi_0(b_2') \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma(b_2\otimes b_1) = b_1'\otimes b_2'$

Notes

H is called the local energy function
σ is called the local isomorphism

H is unique up to global additive constant. By convenience:

$$H(u(B_2) \otimes u(B_1)) = 0$$

If φ is an isomorphism of crystals then φ preserves weights and sends extremal vectors to extremal vectors.

In particular $\sigma(u(B_2) \otimes u(B_1)) = u(B_1) \otimes u(B_2)$

Graded crystal $(\mathcal{B}, D)$

$\mathcal{B}$ simple

D intrinsic energy function

$$D : \mathcal{B} \longrightarrow \mathbb{Z} \quad \text{constant on } J \text{ components}$$

(normalize so that $D(u(\mathcal{B})) = 0$)

$(\mathcal{B}_1, D_1)$ $(\mathcal{B}_2, D_2)$ graded crystals

then $$D_{\mathcal{B}_2 \otimes \mathcal{B}_1} = H_{\mathcal{B}_2, \mathcal{B}_1} + D_1 \pi_1 + D_2 \pi_2$$

where $\pi_1(b_2 \otimes b_1) = b_1$ $\quad \pi_2(b_2 \otimes b_1) = b_2$

Example $\mathcal{B}_1 = \mathcal{B}_2 = $
$$\begin{tikzcd} \boxed{1} \arrow[r, "1"'] & \boxed{2} \arrow[r, "2"'] & \boxed{3} \arrow[ll, bend right, "0"'] \end{tikzcd}$$

$$D_1(b) = D_2(b) = 0$$

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{2}\otimes\boxed{1} \arrow[r, "2"] \arrow[d, "1"] & \boxed{3}\otimes\boxed{1} \arrow[d, "1"] \\
\boxed{1}\otimes\boxed{2} \arrow[d, "2"'] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"] & \boxed{3}\otimes\boxed{2} \arrow[d, "2"] \arrow[ll, bend left, "0"] \\
\boxed{1}\otimes\boxed{3} \arrow[r, "1"] \arrow[uu, bend left, "0"] & \boxed{2}\otimes\boxed{3} \arrow[uu, bend left, "0"] & \boxed{3}\otimes\boxed{3} \arrow[ll, bend left, "0"]
\end{tikzcd}$$

$$wt(\boxed{1}\otimes\boxed{1}) = 2\Lambda_1 \qquad wt(\boxed{1}\otimes\boxed{2}) = \Lambda_2$$

$$H(\boxed{1}\otimes\boxed{1}) = 0 \qquad H(\boxed{1}\otimes\boxed{2}) = 1$$

$$X(\mathcal{B}_2\otimes\mathcal{B}_1, \Lambda_2; q) = q$$

$$X(\mathcal{B}_2\otimes\mathcal{B}_1, 2\Lambda_1; q) = 1$$

My apologies but the definition given in the paper does not agree with the definition of the energy function which is consistent with this example. The example came from the book and I will try to obtain a good definition

One dimensional sums

$\lambda \in \bar{P}$ and $B = B_L \otimes \cdots \otimes B_1$
where $B_j = B^{r_j, s_j}$ (the conjectured perfect crystals from last time).
with energy function D_j

$P(B,\lambda) =$ set of J-highest weight vectors in B of weight λ.

$$X(B,\lambda,q) = \sum_{b \in P(B,\lambda)} q^{D_B(b)}$$

"classically restricted 1d sums"

$W_s^{(r)}$ corresponds to crystal $B^{r,s}$

$$\begin{bmatrix} p+m \\ m \end{bmatrix}_q = \frac{(q^{p+1};q)_\infty (q^{m+1};q)_\infty}{(q;q)_\infty (q^{p+m+1};q)_\infty}$$

$$(x;q) = \prod_{j=0}^{\infty} 1 - xq^j$$

index set $H_\ell = \{(a,i) \mid 1 \le a \le n,\ 1 \le i \le t_a \ell\}$

Let $\lambda = \lambda_1 \bar{\Lambda}_1 + \cdots + \lambda_n \bar{\Lambda}_n$

and $W = W_L \otimes \cdots \otimes W_1$ with $W_i = W_{s_i}^{(r_i)}$

Let $\bar{C}(\lambda, W)$ = the set of sequences $\{ m_i^{(a)} \in \mathbb{Z}_{\ge 0} \mid (a,i) \in H_\ell \}$

withe the following conditions

1. $p_i^{(a)} \ge 0$ where

$$p_i^{(a)} = \sum_{j=1}^{t_a \ell} \nu_j^{(a)} \min(i,j) - \frac{1}{t_a^\vee} \sum_{(b,k) \in H_\ell} (\tilde{\alpha}_a | \tilde{\alpha}_b) \min(t_b i, t_a k) \cdot m_k^{(b)}$$

where if type $X_N^{(r)}$ and $r=1$ then
$t_i = t_i^\vee = 1$ and if $X_N^{(r)} = A_{2n}^{(2)}$ then
$(t_a, t_a^\vee) = (1,2)$ for $1 \le a \le n$.

$\nu_j^{(a)}$ = # of occurences of $W_j^{(a)}$ in W
$\tilde{\alpha}_a = \alpha_a$ except for $A_{2n}^{(2)}$ case

2. $$\sum_{(a,i)\in H_\ell} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in H_\ell} i \nu_i^{(a)} \Lambda_a - \lambda$$

Set for $\{m\} \in \bar{C}(\lambda, W)$

$$c(\{m\}) = \frac{1}{2} \sum_{(a,j),(b,k)\in H_\ell} (\alpha_a|\alpha_b) \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)} - \sum_{a=1}^{n} \sum_{1\le j,k\le \ell} \min(j,k)\, \nu_j^{(a)} m_k^{(a)}$$

$$M(W,\lambda;q) = \sum_{\{m\}\in \bar{C}(\lambda,W)} q^{c(\{m\})} \prod_{(a,i)\in H_\ell} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}$$

Conjecture

$$X = M$$

with $q \to q^{-1}$ and by a factor of q^N on one side.

This is a generalization of the type $A_n^{(1)}$ case where much more is known (although not enough)

The highest weight elements of B of the J-components $P(\lambda, B)$ corresponds to LR tableaux in type $A_n^{(1)}$

A bijection is known between LR tableaux and rigged configurations

Kirillov, Shimozono, Schilling

Rigged configurations

λ partition $\quad R=(R_1,\ldots,R_L)$

sequence of rectangles $\quad R_j$ has r_j rows
s_j cols

configurations:

$C(\lambda,R)=$ set of sequences of partitions

$$\nu=(\nu^{(1)},\nu^{(2)},\ldots)$$

with $\displaystyle |\nu^{(k)}|=-\sum_{j=1}^{k}\lambda_j+\sum_{a=1}^{L}s_a\min\{r_a,k\}$

and $\quad P_i^{(k)}(\nu)\geq 0 \quad$ for $k\geq 1$ $i\geq 0$

where $\displaystyle P_i^{(k)}(\nu)=Q_i(\nu^{(k-1)})-2Q_i(\nu^{(k)})+Q_i(\nu^{(k+1)})+Q_i(\xi^{(k)}(R))$

and $\quad Q_i(\rho)$ is the size of the first i columns of the partition ρ

and $\quad \xi^{(k)}(R)$ is the partion whose parts are the widths of the rectangles in R of height k.

Rigged configurations:

$RC(\lambda, R) =$ the set of pairs (ν, J) where $\nu \in C(\lambda, R$

and $J = \{J^{(k,i)}\}_{i,k \geq 1}$

such that $J^{(k,i)}$ is a partition that fits inside a box of width $P_i^{(k)}(\nu)$ and height $m_i(\nu^{(k)}) = \#$ of part of $\nu^{(k)}$ of size i.

Cocharge:

$$cc(\nu, J) = cc(\nu) + \sum_{i,k \geq 0} |J^{(k,i)}|$$

$$cc(\nu) = \sum_{k,i \geq 0} \alpha_i^{(k)} (\alpha_i^{(k)} - \alpha_i^{(k+1)})$$

where $\alpha_i^{(k)} =$ size of i^{th} column in $\nu^{(k}$

$$M_{A_n^{(1)}}(W, \lambda; q) = \sum_{(\nu, J) \in RC(\lambda, R)} q^{cc(\nu, J)}$$

Conjecture for types $B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$

there is a symmetric function expression

$$H_{X_n^{(1)},R}[X;q] = \Delta^{\perp}_{X_n^{(1)}} H_R[X;q]$$

$$K^{X_n^{(1)}}_{\lambda R}(q) = H_{X_n^{(1)},R}[X;q]\Big|_{s_\lambda[X]}$$

where $H_R[X;q]$ is the type $A_n^{(1)}$ case

Conjecture

$$K = X = M$$