INTERLACINGS, REPRESENTATION THEORY, AND THE INTERCHANGE PROCESS ON WEIGHTED GRAPHS

LET α BE A NONNEGATIVE FUNCTION ON S_n which is SUPPORTED ON THE SET OF TRANSPOSITIONS.

DEFINE THE FOURIER TRANSFORM THROUGH

$$\hat{\alpha}(\rho) = \sum_{\sigma \in S_n} \alpha(\sigma)\, \rho(\sigma). \quad , \forall \rho \text{ REPRESENTATION}$$

$$\tilde{\alpha}(\rho) = \sum_{\sigma \in S_n} \alpha(\sigma) \left(\rho(id) - \rho(\sigma) \right)$$

RECALL THAT THE IRREDUCIBLE REPRESENTATIONS OF S_n ARE INDEXED BY PARTITIONS $\lambda \vdash n$.
WE WRITE $\{\rho^\lambda : \lambda \vdash n\}$ FOR THE IRR. REPR. OF n.

__THM__ FOR ANY $\lambda \vdash n$, $\lambda \neq (n)$

$$\sup_{\substack{\|x\|=1 \\ x \in \mathbb{R}^{d_\lambda}}} x' \hat{\alpha}(\rho^\lambda) x \leq \sup_{\substack{\|y\|=1 \\ y \in \mathbb{R}^{n-1}}} y' \hat{\alpha}(\rho^{(n-1,1)})\, y$$

THE PROOF WILL RELY ON AN INDUCTION ARGUMENT, THE BASIS BEING $n=2$.
FOR THE PROOF, WE NEED TWO LEMMAS.

__LEMMA__ (D. '10) SET $\beta_{ij} = \alpha_{ij} + \dfrac{\alpha_{in}\alpha_{jn}}{\alpha_{1n} + \dots + \alpha_{n-1,n}}$, $1 \leq i < j \leq n-1$

$\beta_{i} = 0$

$$\exists\ v = v(\alpha): \qquad \tilde{\alpha}\left(\rho^{(n-1,1)}\right) = \tilde{\beta}\left(\rho^{(n-1,1)}\right) + vv'$$

THIS IMMEDIATELY LEADS TO THE FOLLOWING INTERLACING PROPERTY:

$$\mu_1^{(n-1,1)}(\beta) \le \mu_1^{(n-1,1)}(\alpha) \le \mu_2^{(n-1,1)}(\beta) \le \mu_2^{(n-1,1)}(\alpha) \le$$
$$\dots \le \mu_{n-1}^{(n-1,1)}(\beta) \le \mu_{n-1}^{(n-1,1)}(\alpha),$$

WHERE $\mu_k^{(n-1,1)}(\gamma)$ IS THE k-TH SMALLEST EIGENVALUE OF $\tilde{\gamma}\left(\rho^{(n-1,1)}\right)$, $\forall \gamma$.

NOTE: $\mu_1^{(n-1,1)}(\beta) = 0$. BY THE BRANCHING RULE, AND THAT ALSO

$$\mu_2^{(n-1,1)}(\beta) = \mu_1^{(n-2,1)}(\beta)$$ [ABUSE OF NOTATION]

LEMMA "OCTOPUS INEQUALITY" (CLR '10)

FOR ANY $k \ge 2$, $\gamma_1, \dots, \gamma_k \ge 0$, AND ANY $g: S_n \to \mathbb{R}$, WE HAVE

$$\sum_{1 \le i < j \le k-1} \sum_{\sigma \in S_k} \frac{\gamma_i \gamma_j}{\gamma_1 + \dots + \gamma_k} \left(g(\sigma) - g((ij)\sigma)\right)^2$$
$$\le \sum_{i=1}^{k} \sum_{\sigma \in S_k} \gamma_i \left(g(\sigma) - g((ik)\sigma)\right)^2.$$

IMPLIES: $\mu_1^{\lambda}(\beta) \le \mu_1^{\lambda}(\alpha) \quad \forall \alpha$.

PROOF OF THE THEOREM: FOR $\lambda \ne (n), (n-1,1)$:

$$\mu_1^{(n-1,1)}(\alpha) \underset{\substack{\uparrow \\ \text{INTERLACING}}}{\leq} \mu_1^{(n-2,1)}(\beta) \underset{\substack{\uparrow \\ \text{INDUCTION} \\ \text{HYP.}}}{=} \inf_{\substack{\lambda' \vdash n-1 \\ \lambda' \neq (n-1)}} \mu_1^{\lambda'}(\beta)$$

$$= \inf_{\substack{\lambda \vdash n \\ \lambda \neq (n), (n-1,1)}} \mu_1^{\lambda}(\beta) \leq \inf_{\substack{\lambda \vdash n \\ \lambda \neq (n), (n-1,1)}} \mu_1^{\lambda}(\alpha)$$