THE RESTRICTION OF $Gl_n(\mathbb{C})$ MODULES TO THE SUBGROUP OF S_n

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The character of a $Gl_n(\mathbb{C})$ module is given by the formula

$$ch_{Gl_n(\mathbb{C})}(M) = \sum_{b \in M} \left[b \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & x_n \end{pmatrix} \right]_{coef\ b}$$

where the sum runs over a basis of the module M. This expression is a symmetric function in the indeterminates x_i.

The character determines the decomposition of the module into irreducibles. If the module has a character given by a Schur function in the variables $x_1, x_2, \dots, x_n$ then the module is irreducible. The character is a function of the eigenvalues of the matrix to $\mathbb{C}$.

Since the symmetric group (represented as permutation matrices) is a natural subgroup of $Gl_n(\mathbb{C})$ we ask the following question:

Question: Given an irreducible $Gl_n(\mathbb{C})$ module, how does it decompose into irreducible S_n modules?

This turns out to be easy to compute for specific examples. I don't know the answer to this question in general and it would be very useful to have a clear expression for this decomposition. The character of a $Gl_n(\mathbb{C})$ module M at a permutation matrix A_σ will be the evaluation of the symmetric function $ch_{Gl_n(\mathbb{C})}(M)(x_1, x_2, \dots, x_n)$ at the eigenvalues of the matrix A_σ.

Now the eigenvalues of the matrix A_σ are determined by the cycle structure of the permutation σ and they will be $\Xi_{\lambda_1}, \Xi_{\lambda_2}, \dots, \Xi_{\lambda_r}$ where $\Xi_m = 1, e^{2\pi i/m}, e^{4\pi i/m}, \dots, e^{2(m-1)\pi i/m}$ (the m roots of unity).

Example: So for instance, the irreducible $Gl_3(\mathbb{C})$ module with character $s_{(5)}(x_1, x_2, x_3)$ considered as an S_3 module has an S_3 character that when evaluated at the identity has character equal to $s_{(5)}(1, 1, 1) = 21$. The character evaluated at the permutation (12)(3) will be $s_{(5)}(1, -1, 1) = 3$. The character evaluated at the permutation (123) will be $s_{(5)}(1, e^{2\pi i/3}, e^{4\pi i/3}) = 0$.

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As an example, we consider M as the $Gl_n(\mathbb{C})$ module consisting of the polynomials in n variables of degree k. That is, $M = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} : \alpha_1 + \cdots + \alpha_n = k\}$. Note that M has a character given by

$$ch_{Gl_n(\mathbb{C})}(M) = \sum_{y^\alpha \in M} \left[y^\alpha \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & x_n \end{pmatrix} \right]_{coef\ y^\alpha} = \sum_{y^\alpha \in M} x^\alpha = s_{(k)}(x_1, \dots, x_n).$$

We define the Frobenius characteristic of an S_n character as

$$\mathcal{F}_{S_n}(\chi) = \sum_{\lambda \vdash n} \chi(\sigma(\lambda)) p_\lambda / z_\lambda$$

where $\sigma(\lambda)$ is a permutation of cycle type λ. For a $Gl_n(\mathbb{C})$ character $f(x_1, x_2, \dots, x_n)$ we have that

$$\mathcal{F}_{S_n}(f(x_1, x_2, \dots, x_n)) = \sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda / z_\lambda.$$

Example: The example of the module with character $s_{(5)}(x_1, x_2, x_3)$, we have already calculated the Frobenius image as $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3)) = 21p_{(111)}/6 + 3p_{(21)}/2$. But we have also determined that the module is isomorphic to the polynomials of degree 5 in three variables. Since we know that the Frobenius image of the polynomial ring in 3 variables has a graded Frobenius image of $h_3\left[\frac{X}{1-q}\right]$ hence we know that the coefficient of q^5 in the expression will be equal to the Frobenius image $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3))$. Since we know that

$$h_3\left[\frac{X}{1-q}\right] = s_{(3)}[X] s_{(3)}\left[\frac{1}{1-q}\right] + s_{(21)}[X] s_{(21)}\left[\frac{1}{1-q}\right] + s_{(111)}[X] s_{(111)}\left[\frac{1}{1-q}\right]$$

then we know that the coefficient of $q^5 s_\lambda[X]$ will be the number of column strict tableaux of shape λ with entries in $0, 1, 2, 3, \dots$ whose entries sum to 5.

For $\lambda = (3)$ we know that the tableaux are given by $\boxed{0|0|5}$, $\boxed{0|1|4}$, $\boxed{0|2|3}$, $\boxed{1|1|3}$, $\boxed{1|2|2}$.

For $\lambda = (2, 1)$ the tableaux are given by $\begin{array}{|c|c}\hline 5 & \\ \hline 0 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 4 & \\ \hline 0 & 1 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 1 & \\ \hline 0 & 4 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 3 & \\ \hline 0 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 2 & \\ \hline 0 & 3 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array}$.

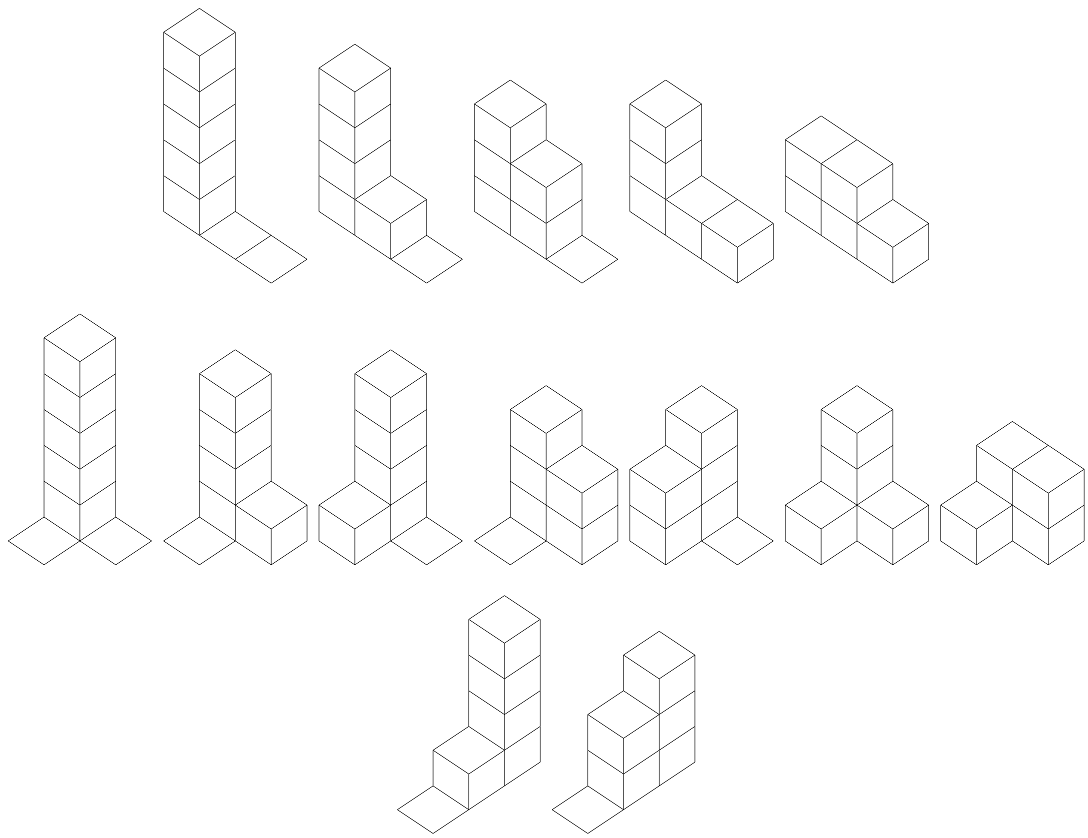
For $\lambda = (1, 1, 1)$ the tableaux are given by $\begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 0 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 0 \\ \hline\end{array}$.
We conclude that $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3)) = 5s_{(3)} + 7s_{(21)} + 2s_{(111)}$.

Notice that in general that we can express the special case

$$\mathcal{F}_{S_n}(s_{(k)}(x_1, \dots, x_n)) = \sum_{\lambda \vdash n} c_\lambda^{(k)} s_\lambda$$

where $c_\lambda^{(k)}$ is the number of column strict tableaux of shape λ whose entries sum to k.

Macdonald also gives another interpretation of this coefficient (p. 81 example 14). $c_\lambda^{(k)}$ is also the number of column strict plane partitions of shape λ. A plane partition is a stack of blocks with adjacent stacks which are weakly decreasing in height. A plane partition is column strict if the stacks are strictly decreasing in the columns.



Note that this is similar to the last interpretation except that the order of the alphabet is reversed since these objects are equivalent to column strict tableaux of shape λ with entries that are weakly decreasing rows and strictly decreasing in columns.

We remark that given two $Gl_n(\mathbb{C})$ modules, their (inner) tensor product will have character as the product of the product of the modules. That is, for modules M, N,

$$ch_{Gl_n(\mathbb{C})}(M \otimes N) = ch_{Gl_n(\mathbb{C})}(M) ch_{Gl_n(\mathbb{C})}(N).$$

This follows directly from the definition of the character.

We also have that the Frobenius image satisfies $\mathcal{F}_{S_n}(fg) = \mathcal{F}_{S_n}(f) * \mathcal{F}_{S_n}(g)$ where $*$ is the inner tensor product (Kronecker product). The Kronecker product on symmetric functions is defined as $p_\lambda/z_\lambda * p_\mu/z_\mu = \delta_{\lambda\mu} p_\lambda/z_\lambda$. It then follows that

$$\begin{aligned}
\mathcal{F}_{S_n}(f) * \mathcal{F}_{S_n}(g) &= \left(\sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda / z_\lambda \right) * \left(\sum_{\lambda \vdash n} g[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda / z_\lambda \right) \\
&= \sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] g[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda / z_\lambda \\
&= \mathcal{F}_{S_n}(fg)
\end{aligned}$$

Macdonald (p. 50 example 17) talks about the evaluation of a Schur function at the sum of roots of unity. $s_\lambda[\Xi_m] = \pm 1$ if λ has an empty m-core (i.e. it can be tiled with ribbons of length m) and $s_\lambda[\Xi_m] = 0$ otherwise. The sign of $s_\lambda[\Xi_m]$ will be the sign of the unique permutation σ such that $\lambda + \delta_m = \sigma(\delta_m) \pmod m$ where $\delta_m = (m-1, m-2, \ldots, 1, 0)$.

It is quite simple to write a short program which accepts a symmetric function and a value of n (determining which copy of $Gl_n(\mathbb{C})$ on is working in) and which returns a symmetric function which returns the Frobenius image of this function as an S_n character.

```
> with(SF)
> psum:=proc(lst,k) local i;
add(lst[i],i=1..k);
end:

> toSnFrob:=proc(expr,n) local i,lambda,j;
add( mul( cat(p,lambda[i]), i=1..nops(lambda))/zee(lambda)*
simplify(subs(seq(seq(x[psum(lambda,i-1)+j+1]=exp(2*Pi*I*j/lambda[i]),
j=0..lambda[i]-1), i=1..nops(lambda)),
evalsf(expr, add(x[i],i=1..convert(lambda,`+`))))),
lambda=Par(n));
end:
```

The program above substitutes the roots of unity in for the variables of the symmetric function after evaluating the symmetric function at n variables.

We compute an example using this program:

```
> for i from 1 to 6 do
> tos(toSnFrob(s[1],i));
> od;
```

$$s_{(1)}$$
$$s_{(2)} + s_{(11)}$$
$$s_{(3)} + s_{(21)}$$
$$s_{(4)} + s_{(31)}$$

$$s_{(5)} + s_{(41)}$$
$$s_{(6)} + s_{(51)}$$

Observing the data below, one simple conjecture to make (that should not be that hard to prove) is

Conjecture 1.

$$\mathcal{F}_{S_n}(s_{(1^k)}(x_1, x_2, \ldots, x_n)) = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$$

Here is data for $\mathcal{F}_{S_n}(s_\lambda(x_1, x_2, \ldots, x_n))$ for $n \leq 6$ and $|\lambda| \leq 5$. Note that if $n \leq \ell(\lambda)$ then $s_\lambda(x_1, x_2, \ldots, x_n) = 0$.

$$F_{S_1}(s_{(1)}) = s_{(1)}$$
$$F_{S_1}(s_{(2)}) = s_{(1)}$$
$$F_{S_1}(s_{(3)}) = s_{(1)}$$
$$F_{S_1}(s_{(4)}) = s_{(1)}$$
$$F_{S_1}(s_{(5)}) = s_{(1)}$$
$$F_{S_2}(s_{(1)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(2)}) = 2s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(11)}) = s_{(11)}$$
$$F_{S_2}(s_{(3)}) = 2s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(21)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(4)}) = 3s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(31)}) = s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(22)}) = s_{(2)}$$
$$F_{S_2}(s_{(5)}) = 3s_{(2)} \ + \ 3s_{(11)}$$
$$F_{S_2}(s_{(41)}) = 2s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(32)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_3}(s_{(1)}) = s_{(21)} \ + \ s_{(3)}$$
$$F_{S_3}(s_{(2)}) = 2s_{(3)} \ + \ 2s_{(21)}$$
$$F_{S_3}(s_{(11)}) = s_{(111)} \ + \ s_{(21)}$$
$$F_{S_3}(s_{(3)}) = s_{(111)} \ + \ 3s_{(3)} \ + \ 3s_{(21)}$$
$$F_{S_3}(s_{(21)}) = s_{(111)} \ + \ s_{(3)} \ + \ 3s_{(21)}$$
$$F_{S_3}(s_{(111)}) = s_{(111)}$$
$$F_{S_3}(s_{(4)}) = s_{(111)} \ + \ 4s_{(3)} \ + \ 5s_{(21)}$$
$$F_{S_3}(s_{(31)}) = 3s_{(111)} \ + \ 2s_{(3)} \ + \ 5s_{(21)}$$
$$F_{S_3}(s_{(22)}) = 2s_{(3)} \ + \ 2s_{(21)}$$
$$F_{S_3}(s_{(211)}) = s_{(111)} \ + \ s_{(21)}$$

$$F_{S_3}(s_{(5)}) = 2s_{(111)} + 5s_{(3)} + 7s_{(21)}$$
$$F_{S_3}(s_{(41)}) = 4s_{(111)} + 4s_{(3)} + 8s_{(21)}$$
$$F_{S_3}(s_{(32)}) = 2s_{(111)} + 3s_{(3)} + 5s_{(21)}$$
$$F_{S_3}(s_{(311)}) = 2s_{(111)} + 2s_{(21)}$$
$$F_{S_3}(s_{(221)}) = s_{(3)} + s_{(21)}$$
$$F_{S_4}(s_{(1)}) = s_{(4)} + s_{(31)}$$
$$F_{S_4}(s_{(2)}) = 2s_{(4)} + 2s_{(31)} + s_{(22)}$$
$$F_{S_4}(s_{(11)}) = s_{(211)} + s_{(31)}$$
$$F_{S_4}(s_{(3)}) = s_{(211)} + 3s_{(4)} + 4s_{(31)} + s_{(22)}$$
$$F_{S_4}(s_{(21)}) = 2s_{(211)} + s_{(4)} + 3s_{(31)} + 2s_{(22)}$$
$$F_{S_4}(s_{(111)}) = s_{(1111)} + s_{(211)}$$
$$F_{S_4}(s_{(4)}) = 2s_{(211)} + 5s_{(4)} + 6s_{(31)} + 3s_{(22)}$$
$$F_{S_4}(s_{(31)}) = 2s_{(4)} + 3s_{(22)} + 5s_{(211)} + s_{(1111)} + 7s_{(31)}$$
$$F_{S_4}(s_{(22)}) = 2s_{(4)} + 3s_{(22)} + s_{(211)} + 3s_{(31)}$$
$$F_{S_4}(s_{(211)}) = s_{(22)} + 3s_{(211)} + s_{(1111)} + s_{(31)}$$
$$F_{S_4}(s_{(1111)}) = s_{(1111)}$$
$$F_{S_4}(s_{(5)}) = 6s_{(4)} + 4s_{(22)} + 4s_{(211)} + 10s_{(31)}$$
$$F_{S_4}(s_{(41)}) = 5s_{(4)} + 7s_{(22)} + 9s_{(211)} + 2s_{(1111)} + 12s_{(31)}$$
$$F_{S_4}(s_{(32)}) = 4s_{(4)} + 5s_{(22)} + 6s_{(211)} + s_{(1111)} + 9s_{(31)}$$
$$F_{S_4}(s_{(311)}) = 3s_{(22)} + 6s_{(211)} + 3s_{(1111)} + 3s_{(31)}$$
$$F_{S_4}(s_{(221)}) = s_{(4)} + 2s_{(22)} + 2s_{(211)} + 3s_{(31)}$$
$$F_{S_4}(s_{(2111)}) = s_{(211)} + s_{(1111)}$$
$$F_{S_5}(s_{(1)}) = s_{(5)} + s_{(41)}$$
$$F_{S_5}(s_{(2)}) = 2s_{(5)} + 2s_{(41)} + s_{(32)}$$
$$F_{S_5}(s_{(11)}) = s_{(311)} + s_{(41)}$$
$$F_{S_5}(s_{(3)}) = s_{(311)} + 3s_{(5)} + 2s_{(32)} + 4s_{(41)}$$
$$F_{S_5}(s_{(21)}) = 2s_{(311)} + s_{(221)} + s_{(5)} + 2s_{(32)} + 3s_{(41)}$$
$$F_{S_5}(s_{(111)}) = s_{(311)} + s_{(2111)}$$
$$F_{S_5}(s_{(4)}) = 2s_{(311)} + s_{(221)} + 5s_{(5)} + 4s_{(32)} + 7s_{(41)}$$
$$F_{S_5}(s_{(31)}) = 2s_{(5)} + 7s_{(41)} + 5s_{(32)} + 6s_{(311)} + s_{(2111)} + 2s_{(221)}$$
$$F_{S_5}(s_{(22)}) = 2s_{(5)} + 3s_{(41)} + 4s_{(32)} + s_{(311)} + 2s_{(221)}$$
$$F_{S_5}(s_{(211)}) = s_{(41)} + s_{(32)} + 3s_{(311)} + 2s_{(2111)} + 2s_{(221)}$$
$$F_{S_5}(s_{(1111)}) = s_{(2111)} + s_{(11111)}$$
$$F_{S_5}(s_{(5)}) = 11s_{(41)} + 7s_{(32)} + 5s_{(311)} + 2s_{(221)} + 7s_{(5)}$$
$$F_{S_5}(s_{(41)}) = 14s_{(41)} + 3s_{(2111)} + 6s_{(221)} + 5s_{(5)} + 11s_{(32)} + 11s_{(311)}$$

$$F_{S_5}(s_{(32)}) = 10s_{(41)} + 2s_{(2111)} + 5s_{(221)} + 4s_{(5)} + 10s_{(32)} + 8s_{(311)}$$
$$F_{S_5}(s_{(311)}) = 3s_{(41)} + 5s_{(2111)} + s_{(11111)} + 5s_{(221)} + 4s_{(32)} + 8s_{(311)}$$
$$F_{S_5}(s_{(221)}) = 3s_{(41)} + s_{(2111)} + 4s_{(221)} + s_{(5)} + 4s_{(32)} + 3s_{(311)}$$
$$F_{S_5}(s_{(2111)}) = 3s_{(2111)} + s_{(11111)} + s_{(221)} + s_{(311)}$$
$$F_{S_5}(s_{(11111)}) = s_{(11111)}$$
$$F_{S_6}(s_{(1)}) = s_{(51)} + s_{(6)}$$
$$F_{S_6}(s_{(2)}) = 2s_{(51)} + s_{(42)} + 2s_{(6)}$$
$$F_{S_6}(s_{(11)}) = s_{(51)} + s_{(411)}$$
$$F_{S_6}(s_{(3)}) = 4s_{(51)} + 2s_{(42)} + s_{(33)} + 3s_{(6)} + s_{(411)}$$
$$F_{S_6}(s_{(21)}) = 3s_{(51)} + 2s_{(42)} + s_{(6)} + 2s_{(411)} + s_{(321)}$$
$$F_{S_6}(s_{(111)}) = s_{(3111)} + s_{(411)}$$
$$F_{S_6}(s_{(4)}) = 5s_{(6)} + 7s_{(51)} + 5s_{(42)} + s_{(33)} + s_{(321)} + 2s_{(411)}$$
$$F_{S_6}(s_{(31)}) = s_{(3111)} + 2s_{(6)} + 7s_{(51)} + 5s_{(42)} + 2s_{(33)} + 3s_{(321)} + 6s_{(411)}$$
$$F_{S_6}(s_{(22)}) = 2s_{(6)} + 3s_{(51)} + 4s_{(42)} + s_{(411)} + s_{(33)} + 2s_{(321)} + s_{(222)}$$
$$F_{S_6}(s_{(211)}) = s_{(51)} + s_{(42)} + 3s_{(411)} + 2s_{(321)} + 2s_{(3111)} + s_{(2211)}$$
$$F_{S_6}(s_{(1111)}) = s_{(21111)} + s_{(3111)}$$
$$F_{S_6}(s_{(5)}) = 12s_{(51)} + 8s_{(42)} + 5s_{(411)} + 3s_{(321)} + 7s_{(6)} + 3s_{(33)}$$
$$F_{S_6}(s_{(41)}) = 5s_{(6)} + 14s_{(51)} + 13s_{(42)} + 12s_{(411)} + 4s_{(33)} + 8s_{(321)} + 3s_{(3111)} + s_{(222)} + s_{(2211)}$$
$$F_{S_6}(s_{(32)}) = 4s_{(6)} + 10s_{(51)} + 11s_{(42)} + 8s_{(411)} + 5s_{(33)} + 8s_{(321)} + 2s_{(3111)} + s_{(222)} + s_{(2211)}$$
$$F_{S_6}(s_{(311)}) = 3s_{(51)} + 4s_{(42)} + 8s_{(411)} + s_{(33)} + 7s_{(321)} + 6s_{(3111)} + s_{(222)} + 2s_{(2211)} + s_{(21111)}$$
$$F_{S_6}(s_{(221)}) = s_{(6)} + 3s_{(51)} + 4s_{(42)} + 3s_{(411)} + 2s_{(33)} + 5s_{(321)} + s_{(3111)} + 2s_{(222)} + 2s_{(2211)}$$
$$F_{S_6}(s_{(2111)}) = s_{(411)} + s_{(321)} + 3s_{(3111)} + 2s_{(2211)} + 2s_{(21111)}$$
$$F_{S_6}(s_{(11111)}) = s_{(21111)} + s_{(111111)}$$

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES II: GUIDING FORMULAS

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First we list some properties of the Frobenius map which sends a an S_n character $\chi : S_n \to \mathbb{C}$ into a symmetric function by the map

$$\mathcal{F}(\chi) = \sum_{\lambda \vdash n} \chi(\sigma(\lambda)) p_\lambda / z_\lambda.$$

For an S_n module M, denote $Frob(M) = \mathcal{F}(char_{S_n}(M))$.

Let M^λ be a $Gl_n(\mathbb{C})$ module with character equal to the symmetric function $s_\lambda(x_1, x_2, \dots, x_n)$.

I mentioned in the last writeup that $Frob(M^{(1^k)}) = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$. I received an elementary proof of this proposition from Adriano Garsia:

$$M^{(1^k)} \simeq \mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\} \simeq Ind_{S_k \times S_{n-k}}^{S_n} \mathcal{L}\{x_1 \wedge x_2 \wedge \cdots \wedge x_k\}$$

where the action of S_{n-k} is trivial on the module and S_k has the sign action on $x_1 \wedge x_2 \wedge \cdots \wedge x_k$.

Therefore $Frob(M^{(1^k)}) = s_{(1^k)} s_{(n-k)} = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$.

That two line proof should be broken down into lemmas

Lemma 1.

$$M^{(1^k)} = \mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

Proof. We compute the $Gl_n(\mathbb{C})$ character of $\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$. Let $diag(y_1, y_2, \cdots, y_k)$ represent a diagonal matrix of $Gl_n(\mathbb{C})$ which acts on the variables by $diag(y_1, y_2, \cdots, y_k) x_i = y_i x_i$.

$$\sum_{i_1 < i_2 < \cdots < i_k} diag(y_1, y_2, \cdots, y_k) x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \Big|_{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}} = \sum_{i_1 < i_2 < \cdots < i_k} y_{i_1} y_{i_2} \cdots y_{i_k} = s_{(1^k)}(y_1, y_2, \dots, y_n)$$

Therefore $\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ is the irreducible module with character $s_{(1^k)}(y_1, y_2, \dots, y_n)$. □

Lemma 2.

$$\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\} \simeq Ind_{S_k \times S_{n-k}}^{S_n} \mathcal{L}\{x_1 \wedge x_2 \wedge \cdots \wedge x_k\}$$

Date: December 6, 2006.

Proof. Recall that $Ind_{S_k\times S_{n-k}}^{S_n}\mathcal{L}\{b_i\} = \mathcal{L}\{\sigma \otimes_{S_k\times S_{n-k}} b_i : \sigma \in S_n, b_i\}$ where we have the tensor *over* a group satisfies the relations

$$\sigma h \otimes_H v = \sigma \otimes_H hv.$$

The isomorphism is given by

$$x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \mapsto \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{pmatrix} \otimes_{S_k\times S_{n-k}} x_1 \wedge x_2 \wedge \cdots \wedge x_k$$

where $\{j_1, j_2, \ldots, j_{n-k}\} = [n]\backslash\{i_1, i_2, \ldots, i_k\}$.

The proof is to show that this map is equivariant with respect to the action of the symmetric group S_n and it suffices to show that the action of the simple transpositions $(j, j+1)$ are equal on both basis elements. There will be four cases to consider, namely both j and $j+1$ are in $\{i_1, i_2, \ldots, i_k\}$, j is in $\{i_1, i_2, \ldots, i_k\}$ and $j+1$ is not, $j+1$ is in $\{i_1, i_2, \ldots, i_k\}$ and j is not, and $j, j+1 \notin \{i_1, i_2, \ldots, i_k\}$.

We leave the remainder of the proof as an exercise to the reader. □

The third step of Adriano's proof is that we need to know some properties of the Frobenius map. We list below some of the images of common S_n modules.

trivial S_n module	$\rightarrow$	$s_{(n)}$
sign S_n module	$\rightarrow$	$s_{(1^n)}$
permutation representation $\{1, 2, \ldots, n\}$	$\rightarrow$	$s_{(n-1)}s_{(1)} = s_{(n)} + s_{(n-1,1)}$
regular representation	$\rightarrow$	$s_{(1)}^n$
$Ind_{S_k\times S_{n-k}}^{S_n} M \otimes N$	$\rightarrow$	$Frob_{S_k}(M)Frob_{S_{n-k}}(N)$
$\bigoplus_{k=0}^n Res_{S_k\times S_{n-k}}^{S_n} M$	$\rightarrow$	$\Delta(Frob_{S_n}(M))$
$Res_{S_k\times S_{n-k}}^{S_n} M$	$\rightarrow$	$\sum_{\lambda\vdash k} s_\lambda^\perp Frob_{S_n}(M) \otimes s_\lambda$
$M \otimes N$ (with S_n acting diagonally)	$\rightarrow$	$Frob_{S_n}(M) \odot Frob_{S_n}(N)$

Adriano also provided me with a construction of the irreducible $Gl_n(\mathbb{C})$ modules. Let T be a standard tableaux of shape λ a partition of k. Let

$$N(T) = \sum_{\sigma\in col(T)} sgn(\sigma)\sigma$$

$$P(T) = \sum_{\sigma\in row(T)} \sigma$$

$$h_\lambda = \text{product of the hooks of } \lambda$$

$col(T)$ and $row(T)$ are the column and row group of the tableau T and are subgroups of S_k. S_k will act on positions of letters in words, that is, a *right* action. $Gl_n(\mathbb{C})$ will have a *left* action on the variables.

Next set

$$E_T = N(T)P(T)/h_\lambda.$$

Now set
$$M^\lambda \simeq \mathcal{L}\{wE_T : w \in [n]^k\}.$$
Note that the elements wE_T form a spanning set and not a basis so one will have to linearly reduce these elements to a basis.

What is interesting to note from this construction is that it is possible to decompose M^λ into submodules of a fixed content. We define the content of a word to be the tuple representing the number of 1s, the number 2s, ..., the number of ns in the word. This tuple is then sorted so that order of the elements does not matter. 0 entries are allowed in this tuple so that the content of a word will be $content(w) = 0^{n_0}1^{n_1}\cdots r^{n_r}$ where $n_0 + n_1 + \cdots + n_r = n$ and $0n_0 + 1n_1 + \cdots rn_r = k$ (e.g. if $n = 4$ then $content(114414) = content(222111) = (0^2 3^2)$). Next we set
$$M^\lambda_\alpha = \{wE_T : w \in [n]^r, content(w) = \alpha\}.$$

Now we have reduced the problem of finding a decomposition of the module M^λ as an S_n module to finding a decomposition of the module M^λ_α for each α. For many special cases of α this is not a difficult problem.

Proposition 3.
$$Frob_{S_{|\lambda|}}(M^\lambda_{1^{|\lambda|}}) = s_\lambda.$$

Proof. This is Schur-Weyl duality.
$$M^\lambda_{1^{|\lambda|}} = \{wE_T : w \in S_n\}$$
□

Proposition 4.
$$Frob_{S_n}(M^\lambda_{0^{n_0}1^{n_1}\cdots r^{n_r}}) = s_{(n_0)} Frob_{S_{n-n_0}}(M^\lambda_{1^{n_1}\cdots r^{n_r}}).$$

Proof. (idea) Show that
$$M^\lambda_{0^{n_0}1^{n_1}\cdots r^{n_r}} \simeq Ind^{S_n}_{S_{n_0}\times S_{n-n_0}} M^\lambda_{1^{n_1}\cdots r^{n_r}}$$
where the action of S_{n_0} is trivial. □

Proposition 5.
$$Frob_{S_n}(M^{(k)}_{0^{n_0}1^{n_1}\cdots r^{n_r}}) = s_{(n_0)} s_{(n_1)} \cdots s_{(n_r)}.$$

Proof. I showed what the decomposition of $M^{(k)}$ was last time using only symmetric function theory. This is (somewhat) a refinement of the statement that
$$Frob_{S_n}(M^{(k)}) = \sum_T s_{\lambda(T)}[X]$$
where the sum is over all column strict tableaux T (non-neg entries less than or equal to n) with content that sums to k.

What I am saying here is that
$$Frob_{S_n}(M^{(k)}_\alpha) = \sum_T s_{\lambda(T)}[X]$$

where the sum is over all column strict tableaux T of size n whose content is α.

By definition we have that

$$M^{(k)}_\alpha = \mathcal{L}\{wE_T : content(w) = \alpha\} = \mathcal{L}\{\tau w E_T : \tau \in S_n\} \tag{1}$$

(where in the second equality w is any single word such that $content(w) = \alpha$).

Since $E_T = \sum_{\sigma \in S_k} \sigma$ (remember that it acts on positions of the word), we can easily show that

$$wE_T = c_w \sum_u u$$

where the sum is over the all words such that the number of is in u is equal to the number of is in w. So fix w to be the smallest word in lex order such that $content(w) = \alpha$ (really I am just choosing any such word but that is OK). Now, for $\alpha = (0^{n_0}1^{n_1}...k^{n_k})$. Check that $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k} \times S_{n_0} \subseteq S_n$ has the trivial action on the element wE_T. By equation (1) we see that

$$M^{(k)}_\alpha = Ind^{S_n}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k} \times S_{n_0}} \mathcal{L}\{wE_T\}.$$

Therefore the Frobenius image is equal to $s_{(n_0)} s_{(n_1)} \cdots s_{(n_r)}$ as stated. $\square$

Conjecture 6.

$$Frob_{S_n}(M^\lambda_{(1^{|\lambda|-\ell}\ell)}) = s_{(1)} s^\perp_{(\ell)} s_\lambda$$

Conjecture 7.

$$Frob_{S_n}(M^\lambda_{(1^{n_1}2^{n_2}\ldots\ell^{n_\ell})}) = \sum_{\mu \vdash n_1} \sum_{\gamma \vdash k-n_1} c^\lambda_{\mu\gamma} Frob_{S_{n-n_1}}(M^\gamma_{(2^{n_2}\ldots\ell^{n_\ell})}) s_\mu$$

Conjecture 8.

$$Frob_{S_n}(M^\lambda_{(d^{n_d}\ldots\ell^{n_\ell})}) = \sum_{\mu \vdash dn_d} \sum_{\gamma \vdash k-dn_d} c^\lambda_{\mu\gamma} Frob(M^\gamma_{((d+1)^{n_{d+1}}\ldots\ell^{n_\ell})}) Frob(M^\mu_{d^{n_d}})$$

This last conjecture is a 'master' conjecture since it implies all the others. Using it we have reduced the calculation from the determination of the decomposition of M^λ_α to the decomposition of $M^\lambda_{(a^b)}$.

I still don't know how to compute $Frob_{S_b}(M^\lambda_{(a^b)})$, but I do have the following clues:

Conjecture 9.

$$Frob_{S_2}(M^\lambda_{(a^2)}) = \begin{cases} s_{(11)} & \textit{if } \lambda = (2a-b, b) \textit{ with } b \textit{ odd} \\ s_{(2)} & \textit{if } \lambda = (2a-b, b) \textit{ with } b \textit{ even} \\ 0 & \textit{else} \end{cases}$$

This conjecture is implied by the more general theorem:

Conjecture 10.

$$Frob_{S_b}(M^{(1^b)+\lambda}_{(a^b)}) = s_{(1^b)} \odot Frob_{S_b}(M^\lambda_{((a-1)^b)})$$

I have a good idea on how to prove most of the conjectures above since their very formulas suggest that there is some module isomorphism that can be used to demonstrate them.

The first cases where one of the conjectures above does not apply is $\alpha = (2^3)$. I was able to compute by process of elimination (since I can compute $Frob(M^\lambda)$ and $Frob(M^\lambda_\beta)$ for $\beta \neq \alpha$, then we can deduce $Frob(M^\lambda_\alpha)$) that

$$Frob_{S_3}(M^{(33)}_{(222)}) = s_{(1^3)}$$
$$Frob_{S_3}(M^{(42)}_{(222)}) = s_{(2)}s_{(1)}$$
$$Frob_{S_3}(M^{(51)}_{(222)}) = s_{(21)}$$

Any clues about why?

Example: We have by Conjecture 8

$$Frob_{S_4}(M^{(42)}_{(2211)}) = Frob_{S_2}(M^{(4)}_{(22)})s_{(2)} + Frob_{S_2}(M^{(31)}_{(22)})s^2_{(1)} + Frob_{S_2}(M^{(22)}_{(22)})s_{(2)}$$

since $Frob_{S_2}(M^\lambda_{(11)}) = s_\lambda$.

In addition we know $Frob_{S_2}(M^{(4)}_{(22)}) = 2$ by Proposition 5 and $Frob_{S_2}(M^{(31)}_{(22)}) = s_{(11)}$ and $Frob_{S_2}(M^{(22)}_{(22)}) = s_{(2)}$ by Conjecture 9. Therefore,

$$Frob_{S_4}(M^{(42)}_{(2211)}) = 2s_{(2)}s_{(2)} + s_{(11)}s^2_{(1)}$$

I have placed data for λ a partition of 6 for my conjectures on the decomposition of homogeneous components on the web page for this seminar.

I have used the conjectures above to compute some Frobenius images for $\alpha = (2,2)$, $(2,2,2)$, $(2,2,2,2)$, $(2,2,2,2,2)$, $(2,2,2,2,2,2)$, $(3,3)$, $(3,3,3)$, $(3,3,3,3)$, $(4,4)$, $(4,4,4)$, $(4,4,4,4)$, $(5,5,5)$, $(6,6,6)$. Note that the n that I use is determined by $\ell(\alpha)$.

From this data I conjecture the following:

Conjecture 11. *Let $\lambda = (m, \tilde{\lambda})$ where $\tilde{\lambda} \vdash k$. Then for $a \geq \tilde{\lambda}_1$,*

$$\mathcal{F}(M^{\lambda}_{(a^b)}) = \mathcal{F}(M^{(b\tilde{\lambda}_1 - k, \tilde{\lambda})}_{(\tilde{\lambda}_1^b)}) \tag{2}$$

I am pretty sure that this is easy to prove and that it is a matter of showing

$$M^{\lambda}_{(a^b)} \simeq M^{(b\tilde{\lambda}_1 - k, \tilde{\lambda})}_{(\tilde{\lambda}_1^b)}.$$

I am calling it a conjecture until I check the details.

This is useful because I can't calculate using my programs (presently) $\mathcal{F}(M^{(14,2,2)}_{(333333)})$, but I know that this is equal to $\mathcal{F}(M^{(8,2,2)}_{(222222)})$ which I can calculate.

Since we know by Schur-Weyl duality that $\mathcal{F}(M^{\lambda}_{(1^{|\lambda|})}) = s_\lambda$, we can conclude

Corollary 12. *For $k < b$,*

$$\mathcal{F}(M^{(ab-k,1^k)}_{(a^b)}) = s_{(b-k,1^k)}.$$

$$\mathcal{F}(M^{(4)}_{(22)}) = s_{(2)}$$
$$\mathcal{F}(M^{(31)}_{(22)}) = s_{(11)}$$
$$\mathcal{F}(M^{(22)}_{(22)}) = s_{(2)}$$

$$\mathcal{F}(M^{(6)}_{(33)}) = s_{(2)}$$
$$\mathcal{F}(M^{(51)}_{(33)}) = s_{(11)}$$
$$\mathcal{F}(M^{(42)}_{(33)}) = s_{(2)}$$
$$\mathcal{F}(M^{(33)}_{(33)}) = s_{(11)}$$

$$\mathcal{F}(M^{(8)}_{(44)}) = s_{(2)}$$
$$\mathcal{F}(M^{(71)}_{(44)}) = s_{(11)}$$
$$\mathcal{F}(M^{(62)}_{(44)}) = s_{(2)}$$
$$\mathcal{F}(M^{(53)}_{(44)}) = s_{(11)}$$
$$\mathcal{F}(M^{(44)}_{(44)}) = s_{(2)}$$

$$\mathcal{F}(M^{(6)}_{(222)}) = \mathcal{F}(M^{(9)}_{(333)}) = \mathcal{F}(M^{(12)}_{(444)}) = \mathcal{F}(M^{(15)}_{(555)}) = s_{(3)}$$
$$\mathcal{F}(M^{(51)}_{(222)}) = \mathcal{F}(M^{(81)}_{(333)}) = \mathcal{F}(M^{(11,1)}_{(444)}) = \mathcal{F}(M^{(14,1)}_{(555)}) = s_{(21)}$$
$$\mathcal{F}(M^{(42)}_{(222)}) = \mathcal{F}(M^{(72)}_{(333)}) = \mathcal{F}(M^{(10,2)}_{(444)}) = \mathcal{F}(M^{(13,2)}_{(555)}) = s_{(21)} + s_{(3)}$$
$$\mathcal{F}(M^{(33)}_{(222)}) = s_{(111)}$$
$$\mathcal{F}(M^{(63)}_{(333)}) = \mathcal{F}(M^{(93)}_{(444)}) = \mathcal{F}(M^{(12,3)}_{(555)}) = s_{(3)} + s_{(21)} + s_{(111)}$$
$$\mathcal{F}(M^{(84)}_{(444)}) = \mathcal{F}(M^{(11,4)}_{(555)}) = s_3 + 2\, s_{2,1}$$

$$\mathcal{F}(M^{(411)}_{(222)}) = \mathcal{F}(M^{(711)}_{(333)}) = \mathcal{F}(M^{(10,1,1)}_{(444)}) = \mathcal{F}(M^{(13,1,1)}_{(555)}) = s_{(111)}$$
$$\mathcal{F}(M^{(321)}_{(222)}) = \mathcal{F}(M^{(621)}_{(333)}) = \mathcal{F}(M^{(921)}_{(444)}) = \mathcal{F}(M^{(12,2,1)}_{(555)}) = s_{(21)}$$
$$\mathcal{F}(M^{(531)}_{(333)}) = \mathcal{F}(M^{(831)}_{(444)}) = \mathcal{F}(M^{(11,3,1)}_{(555)}) = s_{(21)} + s_{(111)}$$
$$\mathcal{F}(M^{(741)}_{(444)}) = \mathcal{F}(M^{(10,4,1)}_{(555)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(222)}_{(222)}) = \mathcal{F}(M^{(522)}_{(333)}) = \mathcal{F}(M^{(822)}_{(444)}) = \mathcal{F}(M^{(11,2,2)}_{(555)}) = s_{(3)}$$
$$\mathcal{F}(M^{(432)}_{(333)}) = \mathcal{F}(M^{(732)}_{(444)}) = \mathcal{F}(M^{(10,3,2)}_{(555)}) = s_{(21)}$$

$$\mathcal{F}(M^{(642)}_{(444)}) = \mathcal{F}(M^{(942)}_{(555)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(333)}_{(333)}) = \mathcal{F}(M^{(633)}_{(444)}) = \mathcal{F}(M^{(933)}_{(555)}) = s_{(111)}$$

$$\mathcal{F}(M^{(6)}_{(222)}) = s_{(3)}$$
$$\mathcal{F}(M^{(51)}_{(222)}) = s_{(21)}$$
$$\mathcal{F}(M^{(42)}_{(222)}) = s_{(21)} + s_{(3)}$$
$$\mathcal{F}(M^{(33)}_{(222)}) = s_{(111)}$$
$$\mathcal{F}(M^{(411)}_{(222)}) = s_{(111)}$$
$$\mathcal{F}(M^{(321)}_{(222)}) = s_{(21)}$$
$$\mathcal{F}(M^{(222)}_{(222)}) = s_{(3)}$$

$$\mathcal{F}(M^{(9)}_{(333)}) = s_{(3)}$$
$$\mathcal{F}(M^{(81)}_{(333)}) = s_{(21)}$$
$$\mathcal{F}(M^{(72)}_{(333)}) = s_{(3)} + s_{(21)}$$
$$\mathcal{F}(M^{(63)}_{(333)}) = s_{(3)} + s_{(21)} + s_{(111)}$$
$$\mathcal{F}(M^{(54)}_{(333)}) = s_{(21)}$$
$$\mathcal{F}(M^{(711)}_{(333)}) = s_{(111)}$$
$$\mathcal{F}(M^{(621)}_{(333)}) = s_{(21)}$$
$$\mathcal{F}(M^{(531)}_{(333)}) = s_{(21)} + s_{(111)}$$
$$\mathcal{F}(M^{(522)}_{(333)}) = s_{(3)}$$
$$\mathcal{F}(M^{(441)}_{(333)}) = s_{(3)}$$
$$\mathcal{F}(M^{(432)}_{(333)}) = s_{(21)}$$
$$\mathcal{F}(M^{(333)}_{(333)}) = s_{(111)}$$

$$\mathcal{F}(M^{(12)}_{(444)}) = s_3$$
$$\mathcal{F}(M^{(11,1)}_{(444)}) = s_{2,1}$$
$$\mathcal{F}(M^{(10,2)}_{(444)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(10,1,1)}_{(444)}) = s_{1,1,1}$$

$$\mathcal{F}(M^{(93)}_{(444)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(921)}_{(444)}) = s_{2,1}$$
$$\mathcal{F}(M^{(84)}_{(444)}) = s_3 + 2\,s_{2,1}$$
$$\mathcal{F}(M^{(831)}_{(444)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(822)}_{(444)}) = s_3$$
$$\mathcal{F}(M^{(75)}_{(444)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(741)}_{(444)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(732)}_{(444)}) = s_{2,1}$$
$$\mathcal{F}(M^{(66)}_{(444)}) = s_3$$
$$\mathcal{F}(M^{(651)}_{(444)}) = s_{2,1}$$
$$\mathcal{F}(M^{(642)}_{(444)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(633)}_{(444)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(552)}_{(444)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(543)}_{(444)}) = s_{2,1}$$
$$\mathcal{F}(M^{(444)}_{(444)}) = s_3$$

$$\mathcal{F}(M^{(15)}_{(555)}) = s_3$$
$$\mathcal{F}(M^{(14,1)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(13,2)}_{(555)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(13,1,1)}_{(555)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(12,3)}_{(555)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(12,2,1)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(11,4)}_{(555)}) = s_3 + 2\,s_{2,1}$$
$$\mathcal{F}(M^{(11,3,1)}_{(555)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(11,2,2)}_{(555)}) = s_3$$
$$\mathcal{F}(M^{(10,5)}_{(555)}) = s_3 + 2\,s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(10,4,1)}_{(555)}) = s_3 + s_{2,1} + s_{1,1,1}$$

$$\mathcal{F}(M^{(10,3,2)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(96)}_{(555)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(951)}_{(555)}) = 2\, s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(942)}_{(555)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(933)}_{(555)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(87)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(861)}_{(555)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(852)}_{(555)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(843)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(771)}_{(555)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(762)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(753)}_{(555)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(663)}_{(555)}) = s_3$$
$$\mathcal{F}(M^{(654)}_{(555)}) = s_{2,1}$$
$$\mathcal{F}(M^{(555)}_{(555)}) = s_{1,1,1}$$

$$\mathcal{F}(M^{(18)}_{(666)}) = s_3$$
$$\mathcal{F}(M^{(17,1)}_{(666)}) = s_{2,1}$$
$$\mathcal{F}(M^{(16,2)}_{(666)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M^{(16,1,1)}_{(666)}) = s_{1,1,1}$$
$$\mathcal{F}(M^{(15,3)}_{(666)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(15,2,1)}_{(666)}) = s_{2,1}$$
$$\mathcal{F}(M^{(14,4)}_{(666)}) = s_3 + 2\, s_{2,1}$$
$$\mathcal{F}(M^{(14,3,1)}_{(666)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(14,2,2)}_{(666)}) = s_3$$
$$\mathcal{F}(M^{(13,5)}_{(666)}) = s_3 + 2\, s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M^{(13,4,1)}_{(666)}) = s_3 + s_{2,1} + s_{1,1,1}$$

$$\mathcal{F}(M_{(666)}^{(13,3,2)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(12,6)}) = 2\,s_3 + 2\,s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(12,5,1)}) = 2\,s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(12,4,2)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(12,3,3)}) = s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(11,7)}) = 2\,s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(11,6,1)}) = s_3 + 2\,s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(11,5,2)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(11,4,3)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(10,8)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(10,7,1)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(10,6,2)}) = s_3 + 2\,s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(10,5,3)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(10,4,4)}) = s_3$$
$$\mathcal{F}(M_{(666)}^{(9,9)}) = s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(9,8,1)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(9,7,2)}) = s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(9,6,3)}) = s_3 + s_{2,1} + s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(9,5,4)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(8,8,2)}) = s_3$$
$$\mathcal{F}(M_{(666)}^{(8,7,3)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(8,6,4)}) = s_3 + s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(8,5,5)}) = s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(7,7,4)}) = s_{1,1,1}$$
$$\mathcal{F}(M_{(666)}^{(7,6,5)}) = s_{2,1}$$
$$\mathcal{F}(M_{(666)}^{(6,6,6)}) = s_3$$

$$\mathcal{F}(M^{(8)}_{(2222)}) = s_{(4)}$$
$$\mathcal{F}(M^{(71)}_{(2222)}) = s_{(31)}$$
$$\mathcal{F}(M^{(62)}_{(2222)}) = s_{(4)} + s_{(31)} + s_{(22)}$$
$$\mathcal{F}(M^{(53)}_{(2222)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(M^{(44)}_{(2222)}) = s_{(4)} + s_{(22)}$$
$$\mathcal{F}(M^{(611)}_{(2222)}) = s_{(211)}$$
$$\mathcal{F}(M^{(521)}_{(2222)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(M^{(5111)}_{(2222)}) = s_{(1111)}$$
$$\mathcal{F}(M^{(431)}_{(2222)}) = s_{(31)} + s_{(211)} + s_{(1111)}$$
$$\mathcal{F}(M^{(422)}_{(2222)}) = s_{(4)} + s_{(31)} + s_{(22)}$$
$$\mathcal{F}(M^{(4211)}_{(2222)}) = s_{(211)}$$
$$\mathcal{F}(M^{(332)}_{(2222)}) = s_{(211)}$$
$$\mathcal{F}(M^{(3311)}_{(2222)}) = s_{(22)}$$
$$\mathcal{F}(M^{(3221)}_{(2222)}) = s_{(31)}$$
$$\mathcal{F}(M^{(2222)}_{(2222)}) = s_{(4)}$$

$$\mathcal{F}(M^{(12)}_{(3333)}) = s_4$$
$$\mathcal{F}(M^{(11,1)}_{(3333)}) = s_{3,1}$$
$$\mathcal{F}(M^{(10,2)}_{(3333)}) = s_4 + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(10,1,1)}_{(3333)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(93)}_{(3333)}) = s_4 + 2\, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(921)}_{(3333)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(9111)}_{(3333)}) = s_{1,1,1,1}$$
$$\mathcal{F}(M^{(84)}_{(3333)}) = s_4 + 2\, s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(831)}_{(3333)}) = s_{2,2} + 2\, s_{3,1} + 2\, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(822)}_{(3333)}) = s_4 + s_{2,2} + s_{3,1}$$

$$\mathcal{F}(M^{(8211)}_{(3333)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(75)}_{(3333)}) = 2\, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(741)}_{(3333)}) = s_4 + s_{2,2} + 2\, s_{3,1} + 2\, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(732)}_{(3333)}) = s_4 + s_{2,2} + 2\, s_{3,1} + 2\, s_{2,1,1}$$
$$\mathcal{F}(M^{(7311)}_{(3333)}) = s_{2,2} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(7221)}_{(3333)}) = s_{3,1}$$
$$\mathcal{F}(M^{(66)}_{(3333)}) = s_4 + s_{2,2} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(651)}_{(3333)}) = s_{2,2} + s_{3,1} + 2\, s_{2,1,1}$$
$$\mathcal{F}(M^{(642)}_{(3333)}) = s_4 + 2\, s_{2,2} + 2\, s_{3,1} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(6411)}_{(3333)}) = s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(633)}_{(3333)}) = s_{3,1} + 2\, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(6321)}_{(3333)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(6222)}_{(3333)}) = s_4$$
$$\mathcal{F}(M^{(552)}_{(3333)}) = s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(5511)}_{(3333)}) = s_{2,2} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(543)}_{(3333)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(5421)}_{(3333)}) = s_4 + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(5331)}_{(3333)}) = s_{2,2} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(5322)}_{(3333)}) = s_{3,1}$$
$$\mathcal{F}(M^{(444)}_{(3333)}) = s_4$$
$$\mathcal{F}(M^{(4431)}_{(3333)}) = s_{3,1}$$
$$\mathcal{F}(M^{(4422)}_{(3333)}) = s_{2,2}$$
$$\mathcal{F}(M^{(4332)}_{(3333)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(3333)}_{(3333)}) = s_{1,1,1,1}$$

$$\mathcal{F}(M^{(16)}_{(4444)}) = s_4$$
$$\mathcal{F}(M^{(15,1)}_{(4444)}) = s_{3,1}$$

$$\mathcal{F}(M^{(14,2)}_{(4444)}) = s_4 + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(14,1,1)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(13,3)}_{(4444)}) = s_4 + 2, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(13,2,1)}_{(4444)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(13,1,1,1)}_{(4444)}) = s_{1,1,1,1}$$
$$\mathcal{F}(M^{(12,4)}_{(4444)}) = 2, s_4 + 2, s_{2,2} + 2, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(12,3,1)}_{(4444)}) = s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(12,2,2)}_{(4444)}) = s_4 + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(12,2,1,1)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(11,5)}_{(4444)}) = s_{2,2} + 3, s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(11,4,1)}_{(4444)}) = s_4 + 2, s_{2,2} + 3, s_{3,1} + 3, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(11,3,2)}_{(4444)}) = s_4 + s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(11,3,1,1)}_{(4444)}) = s_{2,2} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(11,2,2,1)}_{(4444)}) = s_{3,1}$$
$$\mathcal{F}(M^{(10,6)}_{(4444)}) = 2, s_4 + 2, s_{2,2} + 2, s_{3,1} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(10,5,1)}_{(4444)}) = s_4 + 2, s_{2,2} + 3, s_{3,1} + 4, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(10,4,2)}_{(4444)}) = 2, s_4 + 3, s_{2,2} + 4, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(10,4,1,1)}_{(4444)}) = s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(10,3,3)}_{(4444)}) = s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(10,3,2,1)}_{(4444)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(10,2,2,2)}_{(4444)}) = s_4$$
$$\mathcal{F}(M^{(9,7)}_{(4444)}) = 2, s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(9,6,1)}_{(4444)}) = s_4 + 2, s_{2,2} + 3, s_{3,1} + 3, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(9,5,2)}_{(4444)}) = s_4 + 2, s_{2,2} + 4, s_{3,1} + 4, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(9,5,1,1)}_{(4444)}) = 2, s_{2,2} + s_{3,1} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(9,4,3)}_{(4444)}) = s_4 + 2, s_{2,2} + 3, s_{3,1} + 3, s_{2,1,1} + s_{1,1,1,1}$$

$$\mathcal{F}(M^{(9,4,2,1)}_{(4444)}) = s_4 + s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(9,3,3,1)}_{(4444)}) = s_{2,2} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(9,3,2,2)}_{(4444)}) = s_{3,1}$$
$$\mathcal{F}(M^{(8,8)}_{(4444)}) = s_4 + 2, s_{2,2}$$
$$\mathcal{F}(M^{(8,7,1)}_{(4444)}) = s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(8,6,2)}_{(4444)}) = 2, s_4 + 3, s_{2,2} + 3, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(8,6,1,1)}_{(4444)}) = s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(8,5,3)}_{(4444)}) = 2, s_{2,2} + 3, s_{3,1} + 4, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(8,5,2,1)}_{(4444)}) = s_4 + s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(8,4,4)}_{(4444)}) = 2, s_4 + 2, s_{2,2} + 2, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(8,4,3,1)}_{(4444)}) = s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(8,4,2,2)}_{(4444)}) = s_4 + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(8,3,3,2)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(7,7,2)}_{(4444)}) = s_{3,1} + 2, s_{2,1,1}$$
$$\mathcal{F}(M^{(7,7,1,1)}_{(4444)}) = s_4 + s_{2,2} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(7,6,3)}_{(4444)}) = s_4 + s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(7,6,2,1)}_{(4444)}) = s_{2,2} + 2, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(7,5,4)}_{(4444)}) = s_{2,2} + 2, s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(7,5,3,1)}_{(4444)}) = s_4 + 2, s_{2,2} + s_{3,1} + 2, s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(7,5,2,2)}_{(4444)}) = s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(7,4,4,1)}_{(4444)}) = s_4 + 2, s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(7,4,3,2)}_{(4444)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(7,3,3,3)}_{(4444)}) = s_{1,1,1,1}$$
$$\mathcal{F}(M^{(6,6,4)}_{(4444)}) = s_4 + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(6,6,3,1)}_{(4444)}) = s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(6,6,2,2)}_{(4444)}) = s_4 + s_{2,2}$$

$$\mathcal{F}(M^{(6,5,5)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(6,5,4,1)}_{(4444)}) = s_{2,2} + s_{3,1} + s_{2,1,1}$$
$$\mathcal{F}(M^{(6,5,3,2)}_{(4444)}) = s_{3,1} + s_{2,1,1} + s_{1,1,1,1}$$
$$\mathcal{F}(M^{(6,4,4,2)}_{(4444)}) = s_{4} + s_{2,2} + s_{3,1}$$
$$\mathcal{F}(M^{(6,4,3,3)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(5,5,5,1)}_{(4444)}) = s_{1,1,1,1}$$
$$\mathcal{F}(M^{(5,5,4,2)}_{(4444)}) = s_{2,1,1}$$
$$\mathcal{F}(M^{(5,5,3,3)}_{(4444)}) = s_{2,2}$$
$$\mathcal{F}(M^{(5,4,4,3)}_{(4444)}) = s_{3,1}$$
$$\mathcal{F}(M^{(4,4,4,4)}_{(4444)}) = s_{4}$$

$$\mathcal{F}(M^{(10)}_{(22222)}) = s_{5}$$
$$\mathcal{F}(M^{(91)}_{(22222)}) = s_{4,1}$$
$$\mathcal{F}(M^{(82)}_{(22222)}) = s_{5} + s_{3,2} + s_{4,1}$$
$$\mathcal{F}(M^{(811)}_{(22222)}) = s_{3,1,1}$$
$$\mathcal{F}(M^{(73)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{3,1,1}$$
$$\mathcal{F}(M^{(721)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + s_{3,1,1}$$
$$\mathcal{F}(M^{(7111)}_{(22222)}) = s_{2,1,1,1}$$
$$\mathcal{F}(M^{(64)}_{(22222)}) = s_{5} + s_{3,2} + s_{4,1} + s_{2,2,1}$$
$$\mathcal{F}(M^{(631)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + 2\, s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(622)}_{(22222)}) = s_{5} + 2\, s_{3,2} + s_{4,1} + s_{2,2,1}$$
$$\mathcal{F}(M^{(6211)}_{(22222)}) = s_{2,2,1} + s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(61111)}_{(22222)}) = s_{1,1,1,1,1}$$
$$\mathcal{F}(M^{(55)}_{(22222)}) = s_{3,1,1}$$
$$\mathcal{F}(M^{(541)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(532)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + 2\, s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(5311)}_{(22222)}) = s_{3,2} + s_{2,2,1} + s_{3,1,1} + s_{2,1,1,1} + s_{1,1,1,1,1}$$

$$\mathcal{F}(M^{(5221)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + s_{3,1,1}$$
$$\mathcal{F}(M^{(52111)}_{(22222)}) = s_{2,1,1,1}$$
$$\mathcal{F}(M^{(442)}_{(22222)}) = s_5 + s_{3,2} + s_{4,1} + s_{2,2,1} + s_{1,1,1,1,1}$$
$$\mathcal{F}(M^{(4411)}_{(22222)}) = s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(433)}_{(22222)}) = s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(4321)}_{(22222)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + s_{3,1,1} + s_{2,1,1,1}$$
$$\mathcal{F}(M^{(43111)}_{(22222)}) = s_{2,2,1}$$
$$\mathcal{F}(M^{(4222)}_{(22222)}) = s_5 + s_{3,2} + s_{4,1}$$
$$\mathcal{F}(M^{(42211)}_{(22222)}) = s_{3,1,1}$$
$$\mathcal{F}(M^{(3331)}_{(22222)}) = s_{2,2,1}$$
$$\mathcal{F}(M^{(3322)}_{(22222)}) = s_{3,1,1}$$
$$\mathcal{F}(M^{(33211)}_{(22222)}) = s_{3,2}$$
$$\mathcal{F}(M^{(32221)}_{(22222)}) = s_{4,1}$$
$$\mathcal{F}(M^{(22222)}_{(22222)}) = s_5$$

$$\mathcal{F}(M^{(15)}_{(33333)}) = s_5$$
$$\mathcal{F}(M^{(14,1)}_{(33333)}) = s_{4,1}$$
$$\mathcal{F}(M^{(13,2)}_{(33333)}) = s_5 + s_{3,2} + s_{4,1}$$
$$\mathcal{F}(M^{(13,1,1)}_{(33333)}) = s_{3,1,1}$$
$$\mathcal{F}(M^{(12,3)}_{(33333)}) = s_5 + s_{3,2} + 2\, s_{4,1} + s_{3,1,1}$$
$$\mathcal{F}(M^{(12,2,1)}_{(33333)}) = s_{3,2} + s_{4,1} + s_{2,2,1} + s_{3,1,1}$$
$$\mathcal{F}(M^{(12,1,1,1)}_{(33333)}) = s_{2,1,1,1}$$
$$\mathcal{F}(M^{(11,4)}_{(33333)}) = s_5 + 2\, s_{3,2} + 2\, s_{4,1} + s_{2,2,1} + s_{3,1,1}$$

$$\mathcal{F}(M^{(12)}_{(222222)}) = s_6$$
$$\mathcal{F}(M^{(11,1)}_{(222222)}) = s_{5,1}$$
$$\mathcal{F}(M^{(10,2)}_{(222222)}) = s_6 + s_{4,2} + s_{5,1}$$
$$\mathcal{F}(M^{(10,1,1)}_{(222222)}) = s_{4,1,1}$$

$$\mathcal{F}(M^{(93)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{5,1} + s_{4,1,1}$$
$$\mathcal{F}(M^{(921)}_{(222222)}) = s_{4,2} + s_{5,1} + s_{3,2,1} + s_{4,1,1}$$
$$\mathcal{F}(M^{(9111)}_{(222222)}) = s_{3,1,1,1}$$
$$\mathcal{F}(M^{(84)}_{(222222)}) = s_{6} + 2\, s_{4,2} + s_{5,1} + s_{3,2,1}$$
$$\mathcal{F}(M^{(831)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{5,1} + 2\, s_{3,2,1} + 2\, s_{4,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(822)}_{(222222)}) = s_{6} + s_{3,3} + 2\, s_{4,2} + s_{5,1} + s_{2,2,2} + s_{3,2,1}$$
$$\mathcal{F}(M^{(8211)}_{(222222)}) = s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(81111)}_{(222222)}) = s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(75)}_{(222222)}) = s_{3,3} + s_{5,1} + s_{3,2,1} + s_{4,1,1}$$
$$\mathcal{F}(M^{(741)}_{(222222)}) = s_{3,3} + 2\, s_{4,2} + s_{5,1} + s_{2,2,2} + 2\, s_{3,2,1} + 2\, s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(732)}_{(222222)}) = s_{3,3} + 2\, s_{4,2} + s_{5,1} + 3\, s_{3,2,1} + 2\, s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(7311)}_{(222222)}) = s_{4,2} + s_{2,2,2} + 2\, s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + 2\, s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(7221)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{5,1} + s_{2,2,2} + 2\, s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1}$$
$$\mathcal{F}(M^{(72111)}_{(222222)}) = s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(711111)}_{(222222)}) = s_{1,1,1,1,1,1}$$
$$\mathcal{F}(M^{(66)}_{(222222)}) = s_{6} + s_{4,2} + s_{2,2,2}$$
$$\mathcal{F}(M^{(651)}_{(222222)}) = s_{4,2} + s_{5,1} + 2\, s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(642)}_{(222222)}) = s_{6} + s_{3,3} + 3\, s_{4,2} + 2\, s_{5,1} + 2\, s_{2,2,2} + 3\, s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(6411)}_{(222222)}) = s_{3,3} + 2\, s_{3,2,1} + 2\, s_{4,1,1} + 2\, s_{2,2,1,1} + 2\, s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(633)}_{(222222)}) = s_{3,3} + s_{3,2,1} + 2\, s_{4,1,1} + s_{2,2,1,1} + 2\, s_{3,1,1,1}$$
$$\mathcal{F}(M^{(6321)}_{(222222)}) = s_{3,3} + 2\, s_{4,2} + s_{5,1} + s_{2,2,2} + 4\, s_{3,2,1} + 2\, s_{4,1,1} + 2\, s_{2,2,1,1} + 2\, s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(63111)}_{(222222)}) = s_{2,2,2} + s_{3,2,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1} + s_{1,1,1,1,1,1}$$
$$\mathcal{F}(M^{(6222)}_{(222222)}) = s_{6} + s_{3,3} + 2\, s_{4,2} + s_{5,1} + s_{2,2,2} + s_{3,2,1}$$
$$\mathcal{F}(M^{(62211)}_{(222222)}) = s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(621111)}_{(222222)}) = s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(552)}_{(222222)}) = s_{3,3} + s_{3,2,1} + 2\, s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$

$$\mathcal{F}(M^{(5511)}_{(222222)}) = s_{4,2} + s_{2,2,2} + s_{3,2,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(543)}_{(222222)}) = s_{4,2} + s_{5,1} + 2\,s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(5421)}_{(222222)}) = s_{3,3}+2\,s_{4,2}+s_{5,1}+s_{2,2,2}+3\,s_{3,2,1}+2\,s_{4,1,1}+2\,s_{2,2,1,1}+2\,s_{3,1,1,1}+s_{2,1,1,1,1}+s_{1,1,1,1,1,1}$$
$$\mathcal{F}(M^{(54111)}_{(222222)}) = s_{3,2,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(5331)}_{(222222)}) = s_{4,2} + s_{2,2,2} + 2\,s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + 2\,s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(5322)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{5,1} + 2\,s_{3,2,1} + 2\,s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(53211)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{2,2,2} + 2\,s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(531111)}_{(222222)}) = s_{2,2,1,1}$$
$$\mathcal{F}(M^{(52221)}_{(222222)}) = s_{4,2} + s_{5,1} + s_{3,2,1} + s_{4,1,1}$$
$$\mathcal{F}(M^{(522111)}_{(222222)}) = s_{3,1,1,1}$$
$$\mathcal{F}(M^{(444)}_{(222222)}) = s_{6} + s_{4,2} + s_{2,2,2} + s_{1,1,1,1,1,1}$$
$$\mathcal{F}(M^{(4431)}_{(222222)}) = s_{3,3} + s_{5,1} + s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(4422)}_{(222222)}) = s_{6} + 2\,s_{4,2} + s_{5,1} + s_{2,2,2} + s_{3,2,1} + s_{2,1,1,1,1}$$
$$\mathcal{F}(M^{(44211)}_{(222222)}) = s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(441111)}_{(222222)}) = s_{2,2,2}$$
$$\mathcal{F}(M^{(4332)}_{(222222)}) = s_{3,2,1} + s_{4,1,1} + s_{2,2,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(43311)}_{(222222)}) = s_{4,2} + s_{2,2,2} + s_{3,2,1} + s_{2,2,1,1}$$
$$\mathcal{F}(M^{(43221)}_{(222222)}) = s_{3,3} + s_{4,2} + s_{5,1} + s_{3,2,1} + s_{4,1,1} + s_{3,1,1,1}$$
$$\mathcal{F}(M^{(432111)}_{(222222)}) = s_{3,2,1}$$
$$\mathcal{F}(M^{(42222)}_{(222222)}) = s_{6} + s_{4,2} + s_{5,1}$$
$$\mathcal{F}(M^{(422211)}_{(222222)}) = s_{4,1,1}$$
$$\mathcal{F}(M^{(3333)}_{(222222)}) = s_{2,2,2}$$
$$\mathcal{F}(M^{(33321)}_{(222222)}) = s_{3,2,1}$$
$$\mathcal{F}(M^{(333111)}_{(222222)}) = s_{3,3}$$
$$\mathcal{F}(M^{(33222)}_{(222222)}) = s_{4,1,1}$$
$$\mathcal{F}(M^{(332211)}_{(222222)}) = s_{4,2}$$

$$\mathcal{F}(M^{(322221)}_{(222222)}) = s_{5,1}$$
$$\mathcal{F}(M^{(222222)}_{(222222)}) = s_6$$

CONJECTURES AND DATA ON THE S_n-IRREP DECOMPOSITION OF A $Gl_n(\mathbb{C})$ MODULE

MIKE ZABROCKI

I have been computing some data and there are some interesting conjectures to be made about families of modules N_α^λ.

I think that these are special cases of the modules that Rota studied with his work on bitableaux bases. It will still take some work to show that these modules are related to the decomposition of $Gl_n(\mathbb{C})$ irreducible modules.

Modules of content (1^k) These are the S_n irreducible and hence,

$$\mathcal{F}(N^\lambda_{(1^{|\lambda|})}) = s_\lambda$$

Modules of content (a^2) We have that $\ell(\lambda) \leq 2$, hence $\lambda = (2a-k, k)$. In this case it is easy to calculate that

$$\mathcal{F}\left(N^{(2a-k,k)}_{(a^2)}\right) = \begin{cases} s_{(2)} & if\ k\ even \\ s_{(11)} & if\ k\ odd \end{cases}$$

Modules of content (2^3)

$$\mathcal{F}(N^{(222)}_{(2^3)}) = s_{(3)}$$
$$\mathcal{F}(N^{(321)}_{(2^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(411)}_{(2^3)}) = s_{(111)}$$
$$\mathcal{F}(N^{(33)}_{(2^3)}) = s_{(111)}$$
$$\mathcal{F}(N^{(42)}_{(2^3)}) = s_{(3)} + s_{(21)}$$
$$\mathcal{F}(N^{(51)}_{(2^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(6)}_{(2^3)}) = s_{(3)}$$

Modules of content (2^4)

$$\mathcal{F}(N^{(2^4)}_{(2^4)}) = s_{(4)}$$

Date: December 29, 2006 .

$$\mathcal{F}(N^{(3221)}_{(2^4)}) = s_{(31)}$$
$$\mathcal{F}(N^{(3311)}_{(2^4)}) = s_{(22)}$$
$$\mathcal{F}(N^{(4211)}_{(2^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(5111)}_{(2^4)}) = s_{(1111)}$$
$$\mathcal{F}(N^{(332)}_{(2^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(431)}_{(2^4)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(422)}_{(2^4)}) = s_{(4)} + s_{(31)} + s_{(22)}$$
$$\mathcal{F}(N^{(521)}_{(2^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(611)}_{(2^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(44)}_{(2^4)}) = s_{(4)} + s_{(22)}$$
$$\mathcal{F}(N^{(53)}_{(2^4)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(62)}_{(2^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(71)}_{(2^4)}) = s_{(31)}$$
$$\mathcal{F}(N^{(8)}_{(2^4)}) = s_{(4)}$$

Modules of content (3^3)

$$\mathcal{F}(N^{(3^3)}_{(3^3)}) = s_{(3)}$$
$$\mathcal{F}(N^{(432)}_{(3^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(441)}_{(3^3)}) = s_{(3)}$$
$$\mathcal{F}(N^{(522)}_{(3^3)}) = s_{(3)}$$
$$\mathcal{F}(N^{(531)}_{(3^3)}) = s_{(21)} + s_{(111)}$$
$$\mathcal{F}(N^{(621)}_{(3^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(54)}_{(3^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(63)}_{(3^3)}) = s_{(3)} + s_{(21)} + s_{(111)}$$
$$\mathcal{F}(N^{(72)}_{(3^3)}) = s_{(3)} + s_{(21)}$$
$$\mathcal{F}(N^{(81)}_{(3^3)}) = s_{(21)}$$
$$\mathcal{F}(N^{(9)}_{(3^3)}) = s_{(21)}$$

Modules of content (3^4)

$$\mathcal{F}(N^{(3^4)}_{(3^4)}) = s_{(1111)}$$
$$\mathcal{F}(N^{(4332)}_{(3^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(4422)}_{(3^4)}) = s_{(22)}$$
$$\mathcal{F}(N^{(5322)}_{(3^4)}) = s_{(31)}$$
$$\mathcal{F}(N^{(6222)}_{(3^4)}) = s_{(4)}$$
$$\mathcal{F}(N^{(5331)}_{(3^4)}) = s_{(22)} + s_{(211)} + s_{(1111)}$$
$$\mathcal{F}(N^{(5421)}_{(3^4)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(6321)}_{(3^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(5511)}_{(3^4)}) = s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(6411)}_{(3^4)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(7311)}_{(3^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(8211)}_{(3^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(9111)}_{(3^4)}) = s_{(1111)}$$
$$\mathcal{F}(N^{(4^3)}_{(3^4)}) = s_{(4)}$$
$$\mathcal{F}(N^{(543)}_{(3^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(552)}_{(3^4)}) = s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(642)}_{(3^4)}) = s_{(4)} + s_{(31)} + 2s_{(22)} + s_{(211)} + s_{(1111)}$$
$$\mathcal{F}(N^{(732)}_{(3^4)}) = s_{(4)} + 2s_{(31)} + s_{(22)} + 2s_{(211)}$$
$$\mathcal{F}(N^{(822)}_{(3^4)}) = s_{(4)} + s_{(31)} + s_{(22)}$$
$$\mathcal{F}(N^{(651)}_{(3^4)}) = s_{(31)} + 2s_{(211)}$$
$$\mathcal{F}(N^{(741)}_{(3^4)}) = s_{(4)} + 2s_{(31)} + s_{(22)} + s_{(211)} + s_{(1111)}$$
$$\mathcal{F}(N^{(831)}_{(3^4)}) = 2s_{(31)} + s_{(22)} + 2s_{(211)}$$
$$\mathcal{F}(N^{(921)}_{(3^4)}) = s_{(31)} + s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(10,1,1)}_{(3^4)}) = s_{(211)}$$
$$\mathcal{F}(N^{(66)}_{(3^4)}) = s_{(4)} + s_{(22)} + s_{(1111)}$$

$$\mathcal{F}(N^{(75)}_{(3^4)}) = 2s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(84)}_{(3^4)}) = s_{(4)} + s_{(31)} + 2s_{(22)} + s_{(211)}$$
$$\mathcal{F}(N^{(93)}_{(3^4)}) = s_{(4)} + 2s_{(31)} + s_{(211)}$$
$$\mathcal{F}(N^{(10,2)}_{(3^4)}) = s_{(4)} + s_{(31)} + s_{(22)}$$
$$\mathcal{F}(N^{(11,1)}_{(3^4)}) = s_{(31)}$$
$$\mathcal{F}(N^{(12)}_{(3^4)}) = s_{(4)}$$

Conjecture, Modules of two row shape For $1 \leq k \leq a$,
$$\mathcal{F}(N^{(ab-k,k)}_{(a^b)}) = S^k(\mathcal{L}\{x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_b\})$$
where $\mathcal{L}\{x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_b\}$ is the S_b module with Frobenius image equal to $s_{(b-1,1)}$.

Proposition For $a \geq 1$
$$\mathcal{F}(N^{((a+1)^a)}_{(a^{(a+1)})}) = \mathcal{F}(N^{((a+1)^{a+1})}_{((a+1)^{a+1})}) = \omega^{a+1}(s_{(a+1)})$$

Conjecture For $a > 1$
$$\mathcal{F}(N^{((a+2)^a)}_{(a^{(a+2)})}) = \sum_{\ell(\lambda)=3} s_\lambda$$
where the sum is over all partitions λ with 3 even parts or 3 odd parts. The first 4 of these I calculated. The last 4 are just conjectured. I observed (conjectured again) that these modules are related to Motzkin numbers. We have that
$$dim\ N^{((a+2)^a)}_{(a^{(a+2)})} + dim\ N^{((a+1)^{a-1})}_{((a-1)^{a+1})} = the\ number\ of\ Motzkin\ paths\ of\ length\ a+1$$
$$\mathcal{F}(N^{(3)}_{(1^3)}) = s_{(3)}$$
$$\mathcal{F}(N^{(44)}_{(2^4)}) = s_{(4)} + s_{(22)}$$
$$\mathcal{F}(N^{(5^3)}_{(3^5)}) = s_{(311)}$$
$$\mathcal{F}(N^{(6^4)}_{(4^6)}) = s_{(6)} + s_{(42)} + s_{(222)}$$
$$\mathcal{F}(N^{(7^5)}_{(5^7)}) = s_{(511)} + s_{(331)}$$
$$\mathcal{F}(N^{(8^6)}_{(6^8)}) = s_{(8)} + s_{(62)} + s_{(44)} + s_{(422)}$$
$$\mathcal{F}(N^{(9^7)}_{(7^9)}) = s_{(711)} + s_{(531)} + s_{(333)}$$
$$\mathcal{F}(N^{(10^8)}_{(8^{10})}) = s_{(10)} + s_{(64)} + s_{(82)} + s_{(622)} + s_{(442)}$$

Calculated:

$$\mathcal{F}(N^{(22)}_{(1^4)}) = s_{(22)}$$
$$\mathcal{F}(N^{(44)}_{(2^4)}) = s_{(4)} + s_{(22)}$$
$$\mathcal{F}(N^{(66)}_{(3^4)}) = s_{(4)} + s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(88)}_{(4^4)}) = s_{(4)} + 2s_{(22)}$$
$$\mathcal{F}(N^{(10,10)}_{(5^4)}) = s_{(4)} + 2s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(12,12)}_{(6^4)}) = 2s_{(4)} + 2s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(14,14)}_{(7^4)}) = s_{(4)} + 3s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(16,16)}_{(8^4)}) = 2s_{(4)} + 3s_{(22)} + s_{(1111)}$$
$$\mathcal{F}(N^{(18,18)}_{(9^4)}) = 2s_{(4)} + 3s_{(22)} + 2s_{(1111)}$$
$$\mathcal{F}(N^{(20,20)}_{(10^4)}) = 2s_{(4)} + 4s_{(22)} + s_{(1111)}$$

conjectured

$$\mathcal{F}(N^{(22,22)}_{(11^4)}) = 2s_{(4)} + 4s_{(22)} + 2s_{(1111)}$$
$$\mathcal{F}(N^{(24,24)}_{(12^4)}) = 3s_{(4)} + 4s_{(22)} + 2s_{(1111)}$$
$$\mathcal{F}(N^{(26,26)}_{(13^4)}) = 2s_{(4)} + 5s_{(22)} + 2s_{(1111)}$$
$$\mathcal{F}(N^{(28,28)}_{(14^4)}) = 3s_{(4)} + 5s_{(22)} + 2s_{(1111)}$$
$$\mathcal{F}(N^{(30,30)}_{(15^4)}) = 3s_{(4)} + 5s_{(22)} + 3s_{(1111)}$$
$$\mathcal{F}(N^{(32,32)}_{(16^4)}) = 3s_{(4)} + 6s_{(22)} + 2s_{(1111)}$$

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES III: GARNIR REPRESENTATIONS

1. The Garnir representation of column strict tableaux

For a column strict tableau T of shape λ define the Garnir polynomial as

$$G(T) = \Delta(x_{T_{11}}, x_{T_{12}}, \ldots, x_{T_{1\lambda'_1}})\Delta(x_{T_{21}}, x_{T_{22}}, \ldots, x_{T_{2\lambda'_2}}) \cdots \Delta(x_{T_{\ell 1}}, x_{T_{\ell 2}}, \ldots, x_{T_{\ell\lambda'_\ell}})$$

where $\Delta(x_1, x_2, \ldots, x_r) = \prod_{1 \le i < j \le r}(x_i - x_j)$ is the Vandermonde determinant.

The content of a column strict tableau will be partition of fixed length n (padded with 0's if necessary). Fix n (we will be creating an S_n module), define

$$N_\alpha^\lambda = \mathcal{L}\{G(T) : T \textit{ column strict tableaux}, \textit{content}(T) = \alpha, \textit{shape}(T) = \lambda\}.$$

The N_α^λ will be the Garnir representation.

In the last pdf file, I defined $E_S = N(S)P(S)$ for a standard tableau S of shape λ with the right action (on positions) and

$$M_\alpha^\lambda = \mathcal{L}\{wE_S : \textit{content}(w) = \alpha\}.$$

In that write-up I conjectured that M_α^λ is a representation of dimension the number of column strict tableaux of shape λ and content α. I believe that I have a proof of this fact now and I will present this in a future write-up.

My original intuition says the following:
It should be that

$$N_\alpha^\lambda \simeq M_\alpha^\lambda.$$

Conjecture 1. *N_α^λ is an S_n module and the Garnir polynomials which defines the space is a basis.* Note

It turns out that both of these statements are FALSE. These were in the first version of this write-up and I am now commenting them out. Keep reading to see some examples.

I want to know how N_α^λ decomposes into irreducibles and understand the relationship between N_α^λ and M_α^λ. I know examples of where N_α^λ has dimension which is smaller than the number of column strict tableaux of shape λ and content α. Studying the module N_α^λ

Date: December 21, 2006.

raises more questions but it might help give us some understanding of the M_α^λ that were not obvious from other perspectives.

Conjecture 2. *If* $\dim N_\alpha^\lambda =$ *the number of column strict tableaux of shape* λ *and content* α*, then* $N_\alpha^\lambda \simeq M_\alpha^\lambda$.

2. Some examples

Example: If the Garnir polynomials are standard (i.e. $\alpha = (1^n)$) then $N_{(1^n)}^\lambda$ is an irreducible module of shape λ and isomorphic to $M_{(1^n)}^\lambda$.

Example: Consider $N_{(21)}^{(21)}$. The two spanning elements are

$$\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} = x_1 - x_2 = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}$$

and therefore $\mathcal{F}(N_{(21)}^{(21)}) = s_{(1^2)}$.

Example: Consider $N_{(32)}^{(32)}$. The two spanning elements are

$$\begin{array}{|c|c|c|}\cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 \\ \hline\end{array} = (x_1 - x_2)^2 = \begin{array}{|c|c|c|}\cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 \\ \hline\end{array}$$

and therefore $\mathcal{F}(N_{(32)}^{(32)}) = s_{(2)}$.

Note that in the last two examples above, $\mathcal{F}(M_{(21)}^{(21)}) = \mathcal{F}(M_{(32)}^{(32)}) = s_{(1)}^2$.

Example: A larger and more complex example where the the Garnir module is not the same as the classical module representation is $N_{(2222)}^{(431)}$. Consider the two basis elements

$$\begin{array}{|c|c|c|c|}\cline{1-1} 4 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 & 4 \\ \hline\end{array} = (x_1 - x_2)(x_1 - x_4)(x_2 - x_4)(x_1 - x_3)(x_2 - x_3)$$

$$\begin{array}{|c|c|c|c|}\cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 4 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 & 3 \\ \hline\end{array} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 - x_4)(x_2 - x_4)$$

These two elements are clearly equal but the dimension of M^λ cannot be less than the number of column strict tableaux of shape λ so we clearly do not have $M^\lambda \simeq \bigoplus_\alpha N_\alpha^\lambda$.

Example:
$\lambda = (4, 4)$ and $\alpha = (2, 2, 2, 2)$. There are 3 column strict tableaux listed below and they correspond to the following Garnir polynomials.

$$\begin{array}{|c|c|c|c|}\hline 2&2&4&4\\\hline 1&1&3&3\\\hline\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 2&3&4&4\\\hline 1&1&2&3\\\hline\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 3&3&4&4\\\hline 1&1&2&2\\\hline\end{array}$$

$$(x_1 - x_2)(x_1 - x_2)(x_3 - x_4)(x_3 - x_4)$$
$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_4)(x_3 - x_4)$$
$$(x_1 - x_3)(x_1 - x_3)(x_2 - x_4)(x_2 - x_4)$$

This is a 3 dimensional module of S_4 and the Frobenius image works out to be $s_{(2,2)}+s_{(4)}$.

3. Garnir representations indexed by rectangles

There are a few special cases that I know for sure. $\mathcal{F}(N^{\lambda}_{(1^{|\lambda|})}) = s_\lambda$. From this interpretation it is clear that each element of $N^{\lambda+(1^n)}_{\alpha+(1^n)}$ is an element of N^{λ}_{α} multiplied by a Vandermonde, hence we have

Proposition 3.

$$\mathcal{F}(N^{\lambda+(1^n)}_{\alpha+(1^n)}) = \omega(\mathcal{F}(N^{\lambda}_{\alpha}))$$

if $\ell(\alpha) = n$ (padded with 0 parts if necessary).

Proposition 4.

$$\mathcal{F}(N^{((a+1)^a)}_{(a^{a+1})}) = \begin{cases} s_{(a+1)} & if\ a\ odd \\ s_{(1^{a+1})} & if\ a\ even \end{cases}$$

Proof. We show by induction that there is exactly one tableau of content (a^{a+1}) and shape $((a+1)^a)$. The statement by induction says that the i^{th} row must be filled with $a+1-i$ i's and i $i+1$'s.

The first row must be filled this way because the 1's all go in the first row and the rest of the row must be filled with one 2 because the last column has the numbers 2 through $a+1$. Assume that the first $i-1$ rows are filled this way, then the i^{th} row must contain the rest of the i's (of which there are $a-(i-1)$) and the remaining entries in the row are filled with numbers greater than i. But since this is the i^{th} row, the entries in these columns must be the entries $i+1$ through $a+1$ because there are a rows in the tableau.

From this description we conclude that the j^{th} column contains the number 1 up to $a+1-j$ and then the numbers $a+1-j+2$ through $a+1$ (in other words, the j^{th} column all of the numbers except $a+2-j$).

The Garnir polynomial corresponding to this tableau is

$$\prod_{j=1}^{a+1} \Delta(x_1, x_2, \ldots, \widehat{x_j}, \ldots, x_{a+1}) = \prod_{1\le i<j\le a+1} (x_i - x_j)^{a-1}.$$

And clearly the action of $\sigma \in S_{a+1}$ on this polynomial is the alternating action if a even and trivial if a odd. $\square$

It is not true in general, that N^λ_α is an irreducible representation (even for $\alpha = (a^b)$). Try for example $N^{(4,2)}_{(2,2,2)}$. There are three tableaux:

$$\begin{array}{|c|c|}\hline 3&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}$$
$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline\end{array}$$

The corresponding polynomials are

$$(x_1 - x_3)^2, (x_1 - x_2)(x_1 - x_3), (x_1 - x_2)^2$$

The Frobenius image of this module (by computing the trace of elements of cycle type):

$$3p_1^3/6 + p_{21}/2 = s_{(2,1)} + s_{(3)}$$

4. A homomorphism from M^λ_α to N^λ_α

For a word w that is the same length as the size of the partitions λ and α, the letter in the i^{th} position of the word corresponds to a cell in the partition λ, (r, s). If $w_i = a$, then set

$$w_i \to x_a^{s-1}$$

where x_a is a commutative variable. I believe that this map sends an element from an element in M^λ_α to (a multiple of) an element in N^λ_α.

Example: Let $\lambda = (3, 2)$. Then $E_S = ((1)-(14))((1)-(25))P(S)$. $wE_S = (w_1w_2w_3w_4w_5 - w_4w_2w_3w_1w_5 - w_1w_5w_3w_4w_2 + w_4w_5w_3w_1w_2)P(S)$. The order of the row group of S is 12. The image of this element under the morphism will be

$$12(x_{w_4}x_{w_5} - x_{w_1}x_{w_5} - x_{w_4}x_{w_2} + x_{w_1}x_{w_2}) = 12(x_{w_4} - x_{w_1})(x_{w_5} - x_{w_2}) = G\left(\begin{array}{|c|c|c}\hline w_4&w_5&\multicolumn{1}{c}{}\\\hline w_1&w_2&\multicolumn{1}{c|}{w_3}\\\hline\end{array}\right)$$

because elements of the row group will preserve the image of these words.

It is not as clear for a general shape, but I claim that this will always happen that the image of wE_S will be the Garnir polynomial filled with the entries of w.

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES IV: SOME CORRESPONDING CHARACTERS AND FROBENIUS IMAGES

Module	$Gl_n(\mathbb{C})$ character	graded S_n-Frobenius image
$\mathbb{Q}\langle X_n\rangle \simeq T(V)$	$\frac{1}{1-(a_1+a_2+\cdots+a_n)} = \sum_{k\geq 0} h_{(1^k)}[A_n]$	$\sum_{d=0}^{n} \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X]$
$\mathbb{Q}[X_n] \simeq S(V)$	$\prod_{i=1}^{n} \frac{1}{1-a_i} = \sum_{k\geq 0} h_k[A_n]$	$h_n\left[\frac{X}{1-q}\right]$
$\bigwedge(V)$	$\prod_{i=1}^{n}(1+a_i) = \sum_{k=0}^{n} e_k[A_n]$	$\sum_{i=0}^{n} q^i h_{n-i}[X]e_i[X]$ $= h_n[(1-q)X]\Big\|_{q\to -q}$
$\mathcal{A}'_n$	$\frac{\prod_{i=1}^{n}(1-a_i)}{1-(a_1+a_2+\cdots+a_n)}$ $= \sum_{k\geq 0}\sum_{i=0}^{k}(-1)^i e_{(i,1^{k-i})}[A_n]$	$\sum_{d=0}^{n} \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X(1-q)]$

Although I think that it is clear what $T(V), S(V)$ and $\bigwedge(V)$ are (because their definitions can be found in a typical algebra text), I should mention that $\mathcal{A}'_n$ is the subspace of $\mathbb{Q}\langle X_n\rangle \simeq T(V)$ killed by all partial derivative operators ∂_{x_i}. Alternatively, $\mathcal{A}'_n$ is the ring of polynomials of all possible brackets of variables (note that $T(V) \simeq \mathcal{A}'_n \otimes S(V)$).

In particular we have

$$\mathcal{F}_{S_n}(e_k[A_n]) = e_k[X]h_{n-k}[X] \tag{1}$$

$$\mathcal{F}_{S_n}(h_k[A_n]) = h_n\left[\frac{X}{1-q}\right]\Big|_{q^k} \tag{2}$$

$$\mathcal{F}_{S_n}(h_{(1^k)}[A_n]) = \sum_{i=1}^{k} S_{k,i} h_{(n-i,1^i)}[X] \tag{3}$$

Date: January 31, 2007.

where $S_{k,i}$ = the number of set partitions of size k with i parts (the Stirling number of the second kind). These

Let me provide an alternate proof of equation (3) using some proof techniques that might be useful later on.

Proof. Because $\mathcal{F}(f[A_n]g[A_n]) = \mathcal{F}(f[A_n]) * \mathcal{F}(g[A_n])$ and $\mathcal{F}(h_1[A_n]) = h_{(n-1,1)}[X]$ from either equation (1) or (2). Therefore we have that

$$\mathcal{F}_{S_n}(h_{(1^k)}[A_n]) = h_{(n-1,1)}[X]^{*k}.$$

Now using techniques of symmetric functions it is not difficult to show that $h_{(n-1,1)} * f = h_1 h_1^{\perp} f$. This follows from the computation

$$\left\langle h_{(n-1,1)} * f, g \right\rangle = \left\langle h_{(n-1,1)}, f * g \right\rangle = \left\langle h_{n-1}, (h_1^{\perp} f) * (h_1^{\perp} g) \right\rangle = \left\langle h_1^{\perp} f, h_1^{\perp} g \right\rangle = \left\langle h_1 h_1^{\perp} f, g \right\rangle$$

Now we proceed by induction since $h_1 h_1^{\perp}(h_{(n-i,1^i)}) = i h_{(n-i,1^i)} + h_{(n-i-1,1^{i+1})}$. Thus if we assume (3) holds for all values smaller than k we then have

$$\begin{aligned}
h_{(n-1,1)}^{*k} &= h_{(n-1,1)} * h_{(n-1,1)}^{*k-1} \\
&= h_1 h_1^{\perp} \left(\sum_{i=1}^{k-1} S_{k-1,i} h_{(n-i,1^i)} \right) \\
&= \sum_{i=1}^{k-1} i S_{k-1,i} h_{(n-i,1^i)} + \sum_{i=1}^{k-1} S_{k-1,i} h_{(n-i-1,1^{i+1})} \\
&= \sum_{i=1}^{k-1} i S_{k-1,i} h_{(n-i,1^i)} + \sum_{i=2}^{k} S_{k-1,i-1} h_{(n-i,1^i)} \\
&= \sum_{i=1}^{k} (i S_{k-1,i} + S_{k-1,i-1}) h_{(n-i,1^i)} = \sum_{i=1}^{k} S_{k,i} h_{(n-i,1^i)}
\end{aligned}$$

□

It is interesting that we can compute very easily $\mathcal{F}(s_\lambda)$ which is equal to the Frobenius image of the S_n module $M^\lambda \simeq \bigoplus_\alpha M_\alpha^\lambda$, but I have no idea what symmetric function corresponds to M_α^λ. We should have that $s_\lambda = \sum_\alpha f_\alpha^\lambda$ such that $\mathcal{F}(f_\alpha^\lambda)$ =the Frobenius image of M_α^λ.

Lets do a special case which can be worked out from what we know here. Recall that the coefficient of $s_\lambda[X] q^k$ in $h_n[X/(1-q)]$ is the number of column strict tableaux of shape λ in the entries $\{0, 1, 2, 3, \ldots\}$ which sum to k. Denote this coefficient by A_λ^k. In this case we have

$$\mathcal{F}_{S_n}(h_k[A_n]) = \sum_{\lambda \vdash n} A_\lambda^k s_\lambda[X].$$

Proposition 1.

$$\mathcal{F}_{S_n}(h_{(k-1,1)}[A_n]) = \sum_{\lambda \vdash n} corners(\lambda) A_\lambda^{k-1} s_\lambda + \sum_{\lambda \vdash n} (\sum_{|\mu \Delta \lambda|=2} A_\mu^{k-1}) s_\lambda$$

where $\mu\Delta\lambda$ are the cells which are not in both μ and λ and $corners(\lambda)$ is the number of corners of λ.

Proof.

$$\begin{aligned}
\mathcal{F}_{S_n}(h_{(k-1,1)}[A_n]) &= \mathcal{F}_{S_n}(h_1[A_n]) * \mathcal{F}_{S_n}(h_{k-1}[A_n]) \\
&= h_1 h_1^{\perp} \sum_{\mu \vdash n} A_\mu^{k-1} s_\mu[X] \\
&= \sum_{\mu \vdash n} A_\mu^{k-1} \left(corners(\mu) s_\mu + \sum_{|\mu\Delta\lambda|=2} s_\lambda \right) \\
&= \sum_{\mu \vdash n} corners(\mu) A_\mu^{k-1} s_\mu + \sum_{\mu \vdash n} A_\mu^{k-1} \sum_{|\mu\Delta\lambda|=2} s_\lambda \\
&= \sum_{\mu \vdash n} corners(\mu) A_\mu^{k-1} s_\mu + \sum_{\lambda \vdash n} \left(\sum_{|\mu\Delta\lambda|=2} A_\mu^{k-1} \right) s_\lambda
\end{aligned}$$

□

However this combinatorial interpretation isn't too helpful when we wish to calculate $\mathcal{F}_{S_n}(s_{(k-1,1)}[A_n])$ since we don't know the difference between the coefficients A_λ^{k-1} and A_λ^k.

I think that my next goal should be to identify some sort of formula for the Frobenius images of the M_α^λ (in particular for the case of $\alpha = (a^b)$).

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES V: ANOTHER PRESENTATION OF THE $Gl_n(\mathbb{C})$ MODULE

Recall that we have defined for a partition $\lambda \vdash k$

$$M^\lambda = \mathcal{L}\{wE_T : w \in [n]^k\}$$

where E_T is an idempotent $N(T)P(T)$ acting on the right (permutation of positions in the word) and T is the super standard tableau of shape λ.

In particular we set

$$M^\lambda_\alpha = \mathcal{L}\{wE_T : content(w) = \alpha\}$$

and we clearly have that $M^\lambda = \bigoplus_\alpha M^\lambda_\alpha$.

I have also defined a Garnir polynomial representation N^λ and N^λ_α and although I originally thought that $N^\lambda_\alpha \simeq M^\lambda_\alpha$, it turns out that these are not isomorphic.

Conjecture 1. *There exists an onto homomorphism from M^λ_α to N^λ_α.*

Problem 2. *Compute the kernel of this homomorphism.*

One thing that might make this easier is a second presentation of the irreducible $Gl_n(\mathbb{C})$ module. I was not aware until recently of the classical definition of the irreducible modules. I learned of the definition of M^λ_α from Adriano and at Aaron Lauve showed me a presentation in terms of matrix minors at a conference in Montreal in January.

Let λ be a partition of k and consider the matrix

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

Let $(i_1\ i_2\ \cdots\ i_r)$, be the determinant of the matrix minor consisting of the first r rows and the columns specified by the sequence $i_1, i_2, \ldots, i_r$. That is

$$(i_1\ i_2\ \cdots\ i_r) = \begin{vmatrix} x_{1i_1} & x_{1i_2} & \cdots & x_{1i_r} \\ x_{2i_1} & x_{2i_2} & \cdots & x_{2i_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{ri_1} & x_{ri_2} & \cdots & x_{ri_r} \end{vmatrix}.$$

Date: February 1, 2007.

For a column strict tableau S of shape λ (with entries S_{ij}), we set

$$X(S) = (S_{11}\ S_{12}\ \cdots\ S_{1\lambda'_1})(S_{21}\ S_{22}\ \cdots\ S_{2\lambda'_2})\cdots(S_{\ell 1}\ S_{\ell 2}\ \cdots\ S_{\ell\lambda'_\ell}).$$

We define

$$P^\lambda = \{X(S) : S \textit{ column strict tableaux shape } \lambda\}.$$

Note that the minors $(i_1\ i_2\ \cdots\ i_r)$ are multilinear since the determinants are. For example we have

$$X\left(\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array}\right) = (1\ 2)(1) = (x_{11}x_{22} - x_{12}x_{21})x_{11}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline\end{array}\right) = (1\ 2)(2) = (x_{11}x_{22} - x_{12}x_{21})x_{12}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array}\right) = (1\ 3)(1) = (x_{11}x_{23} - x_{13}x_{21})x_{11}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 3 \\ \hline\end{array}\right) = (1\ 3)(3) = (x_{11}x_{23} - x_{13}x_{21})x_{13}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline\end{array}\right) = (2\ 3)(2) = (x_{12}x_{23} - x_{13}x_{22})x_{12}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 \\ \hline\end{array}\right) = (2\ 3)(3) = (x_{12}x_{23} - x_{13}x_{22})x_{13}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}\right) = (1\ 3)(2) = (x_{11}x_{23} - x_{13}x_{21})x_{12}$$

$$X\left(\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array}\right) = (1\ 2)(3) = (x_{11}x_{22} - x_{12}x_{21})x_{13}$$

The linear span of theses polynomials is a $Gl_3(\mathbb{C})$ module where for $A \in Gl_n(\mathbb{C})$ and

$$A(x_{ij}) = a_{1j}x_{i1} + a_{2j}x_{i2} + \cdots + a_{nj}x_{in} = \left(\begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{in}\end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\right)_j.$$

This presentation of the $Gl_n(\mathbb{C})$ module has a simple homomorphism from P^λ to N^λ (easier than the one from M^λ to N^λ) which sends x_{ij} to x_j^{i-1}.

The classical definition is nicer in some ways because we are in a commutative polynomial ring. The first coordinate is telling us the ‘row’ of an entry in a tableau. The commutative part erases the column position of the entry (in some ways this is similar to the multiplication by $P(T)$ on the right).

We also define P^λ_α to be the linear span $X(T)$ where T is a column strict tableau of shape λ and content α.

$$P^\lambda_\alpha = \mathcal{L}\{X(T) : T\ CST^\lambda_\alpha\}$$

Conjecture 3.

$$P^{\lambda}_{\alpha} \simeq M^{\lambda}_{\alpha}$$

I will have to think carefully about how to prove this isomorphism exists. We know that because both M^{λ} and P^{λ} are the irreducible $Gl_n(\mathbb{C})$ modules indexed by a partition λ that they are isomorphic.

One of my conjectures from before is clear.

Proposition 4.

$$P^{\lambda+(1^n)}_{\alpha+(1^n)} \simeq P^{(1^n)}_{(1^n)} \otimes P^{\lambda}_{\alpha}$$

Proof. For every $T \in CST^{\lambda}_{\alpha}$ we have that $T \in CST^{\lambda+(1^n)}_{\alpha+(1^n)}$

$$(1\ 2\ \cdots\ n)X(T) = X\left(\begin{array}{|c|}\hline n \\ \hline \vdots \\ \hline \\ \hline 2 \\ \hline 1 \\ \hline \end{array} + T\right)$$

where on the right we mean to attach a full column on the tableaux T on the left. Since $P^{(1^n)}_{(1^n)} = \mathcal{L}\{(1\ 2\ \cdots\ n)\}$ Therefore we have the isomorphism which sends a basis element

$$(1\ 2\ \cdots\ n)X(T') = X(T) \mapsto (1\ 2\ \cdots\ n) \otimes X(T')$$

where $T \in CST^{\lambda+(1^n)}_{\alpha+(1^n)}$ and $T' \in CST^{\lambda}_{\alpha}$ is the tableau T with the first column removed. $\square$

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES VI: A STRAIGHTENING ALGORITHM

In the second write up I did not make clear that there are some details that need to be addressed.

Proposition 1. *dim $M_\alpha^\lambda =$ is the number of column strict tableaux of shape λ and content α.*

Given that T is the super standard tableau of shape λ, for any tableau (no restriction on the entries) S of shape λ we define $R(S)$ (the reading word of S) to be the reading of the entries in the order specified by T (that is read the rows from the bottom up). For example, if

$$S = \begin{array}{|c|c|c|c|c|}
\cline{1-1} 3 \\
\cline{1-3} 7 & 3 & 2 \\
\hline 2 & 4 & 4 & 2 & 5 \\
\hline 3 & 1 & 2 & 3 & 4 \\
\hline
\end{array}$$

then $R(S) = 31234244257323$. We have that

$$M_\alpha^\lambda = \mathcal{L}\{R(S)E_T : shape(S) = \lambda, content(S) = \alpha\}$$

(simply because this is clearly equal to $\mathcal{L}\{wE_T : content(w) = \alpha\}$).

I will show that

$$M_\alpha^\lambda = \mathcal{L}\{R(S)E_T : S \in CST_\alpha^\lambda\}$$

where CST_α^λ is the set of all column strict tableaux of shape λ and content α.

Proof. Since $R(S)E_T = R(S)N(T)P(T)$, we know that $R(S)E_T = \pm R(S')E_T$ where S' is the tableau formed by taking the columns of S and rearranging them so that they are in increasing order. For example in the example listed above we have that

$$S' = \begin{array}{|c|c|c|c|c|}
\cline{1-1} 7 \\
\cline{1-3} 3 & 4 & 4 \\
\hline 3 & 3 & 2 & 3 & 5 \\
\hline 2 & 1 & 2 & 2 & 4 \\
\hline
\end{array}$$

If there are two or more entries which are equal in the same column then $R(S)E_T = 0$.

Now assume that the entries in S are in strictly increasing in columns. Let $C(S)$ be the reading order of columns from right to left, bottom to top. (e.g. in the running example $C(S') = 45232241342337$). Now following this order there is a 'first' position where the tableau breaks from being column strict, this occurs across two columns. Consider the cells in the left column above this break point (which we will label by $a_1, a_2, \ldots, a_k$) and

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the cells in the right column below this breakpoint (which we will label by $b_1, b_2, \ldots, b_\ell$). In a diagram these cells appear as

$$\begin{array}{|c|c} \cline{1-1} a_1 & \\ \cline{1-1} a_2 & \\ \cline{1-1} \vdots & \\ \hline a_k & \multicolumn{1}{c|}{b_1} \\ \hline & \multicolumn{1}{|c|}{b_2} \\ \cline{2-2} & \multicolumn{1}{|c|}{\vdots} \\ \cline{2-2} & \multicolumn{1}{|c|}{b_\ell} \\ \cline{2-2} \end{array}$$

with

$$a_1 > a_2 > \cdots > a_k > b_1 > b_2 > \cdots > b_\ell.$$

Consider all possible interchanges of the cells in the set $\{a_1, a_2, \ldots, a_k\}$ with the cells in the set $\{b_1, b_2, \ldots, b_\ell\}$. Clearly over this set of tableau, $C(S')$ is lexicographically smallest for S' equal to the original tableau. There exists a permutation $\sigma_{SS'}$ which sends the tableau S to the tableau S'. Let

$$F = \sum_{S'} sgn(\sigma_{SS'})R(S\sigma_{SS'})N(T) = \sum_{S'} sgn(\sigma_{SS'})R(S')N(T).$$

Let τ be the transposition which exchanges a_k and b_1, then clearly we have $F = -F\tau$ and hence $F - F\tau = F + F = 2F$, or simply $F = \frac{1}{2}F(1-\tau)$. Since τ is in the row group of T, we also have

$$FP(T) = \frac{1}{2}F(1-\tau)P(T) = 0.$$

We conclude that

$$\sum_{S'} sgn(\sigma_{SS'})R(S')E_T = 0.$$

Since $C(S)$ is the lexicographically smallest element of all of the possible $C(S')$ we have a method of rewriting

$$R(S)E_T = -\sum_{S':C(S')>_{lex}C(S)} R(S')E_T.$$

We conclude that the terms where S is a column strict tableau linearly spans the space M^λ_α.

It is known that the dimension of the module M^λ is the number of column strict tableau of shape λ, therefore the spanning set must be a basis. $\square$

Example: Choose $\lambda = (3,2)$ and we will use the tableau represent the expression $R(S)E_T$. Consider the tableau

$$\begin{array}{|c|c|c} \cline{1-2} 5 & 3 & \\ \hline 4 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Then we have

$$\begin{array}{|c|c|c} \cline{1-2} 5 & 3 & \\ \hline 4 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} - \begin{array}{|c|c|c} \cline{1-2} 5 & 3 & \\ \hline 1 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array} + \begin{array}{|c|c|c} \cline{1-2} 4 & 3 & \\ \hline 1 & 5 & \multicolumn{1}{c|}{2} \\ \hline \end{array} = 0 \tag{1}$$

$$\ytableaushort{53,142} = -\ytableaushort{54,132} \tag{2}$$

$$\ytableaushort{54,132} - \ytableaushort{54,123} + \ytableaushort{53,124} = 0 \tag{3}$$

$$\ytableaushort{54,123} - \ytableaushort{45,123} + \ytableaushort{25,143} = 0 \tag{4}$$

$$\ytableaushort{25,143} - \ytableaushort{25,134} + \ytableaushort{24,135} = 0 \tag{5}$$

$$\ytableaushort{53,124} - \ytableaushort{35,124} + \ytableaushort{25,134} = 0 \tag{6}$$

$$\ytableaushort{43,152} = -\ytableaushort{45,132} \tag{7}$$

$$\ytableaushort{45,132} - \ytableaushort{45,123} + \ytableaushort{43,125} = 0 \tag{8}$$

$$\ytableaushort{43,125} - \ytableaushort{34,125} + \ytableaushort{24,135} = 0 \tag{9}$$

From (7), (8) and (9),

$$\ytableaushort{43,152} = -\ytableaushort{45,123} - \ytableaushort{34,125} + \ytableaushort{24,135} \tag{10}$$

From (2), (3), (4), (5), (6),

$$\ytableaushort{53,142} = -\ytableaushort{45,123} - \ytableaushort{24,135} + \ytableaushort{35,124} \tag{11}$$

Putting these together,

$$\ytableaushort{53,412} = -2\ytableaushort{24,135} + \ytableaushort{35,124} + \ytableaushort{34,125} \tag{12}$$

ON THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ REPRESENTATIONS INTO IRREDUCIBLE S_n MODULES VII: A DECOMPOSITION OF THE MODULE OF WORDS IN n LETTERS INTO IRREDUCIBLE S_n MODULES

The following is an excerpt from a paper 'On the $\mathfrak{S}_n$-module structure of the noncommutative harmonics' by E. Briand, M. Rosas and M. Zabrocki. Note that here we are considering the ring $\mathbb{Q}\langle X_n\rangle$ which is isomorphic to $\bigoplus_{k\geq 0}\mathbb{Q}[n]^k$. Except that the field is different, this is the space which is decomposed in earlier parts of these notes by Schur-Weyl duality.

In the following lemma we compute the graded Frobenius characteristic for the module $\mathbb{Q}\langle X_n\rangle$.

Lemma 1 (The Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$)**.**

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) = \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X].$$

Proof. For each monomial $x_{i_1}\cdots x_{i_r}$, we define its *type* $\nabla(x_{i_1}\cdots x_{i_r})$ to be the set partition of $[r]=\{1,2,\ldots,r\}$ such that a and b are in the same part of the set partition if and only if $i_a = i_b$ in the monomial. For a set partition A with at most n parts, we will let N^A equal the $\mathfrak{S}_n$ submodule of $\mathbb{Q}\langle X_n\rangle$ spanned by all monomials of type A. As $\mathfrak{S}_n$–module,

$$\mathbb{Q}\langle X_n\rangle \simeq \bigoplus_{d=0}^{n} \bigoplus_{A\,:\,\ell(A)=d} N^A$$

where the second direct sum is taken over all set partitions A with d parts.

Fix a set partition A, and let d be the number of parts of A, and $\mathbf{x}_{\vec{i}} = x_{i_1}x_{i_2}\ldots x_{i_r}$ be the smallest monomial in lex order in N^A. It involves only the variables x_1, x_2, $\ldots$, x_d. The representation N^A is the representation of $\mathfrak{S}_n$ induced by the action of the subgroup $\mathfrak{S}_d \times \mathfrak{S}_1^{n-d} \simeq \mathfrak{S}_d$ on the subspace $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_{\vec{i}}$. The representation $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_{\vec{i}}$ of $\mathfrak{S}_d$ is isomorphic to the regular representation. We use the rule for a representation R of $\mathfrak{S}_d$ induced to $\mathfrak{S}_n$,

$$\mathcal{F}rob_{\mathfrak{S}_n}(R\uparrow_{\mathfrak{S}_d}^{\mathfrak{S}_n}) = h_{n-d}[X]\mathcal{F}rob_{\mathfrak{S}_d}(R),$$

Date: October 2, 2009.

and conclude that the Frobenius characteristic of N^A is $h_{(n-d,1^d)}[X]$. Hence the graded Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$ is

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle) = \sum_{d=0}^{n} \sum_{A:\ell(A)=d} q^{|A|} h_{(n-d,1^d)}[X] = \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X].$$

□

So now, we know the Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$ by computing it through symmetric function techniques. In particular if we concentrate on the words of length k, $\mathbb{Q}\langle X_n\rangle^k$, (rather than, as it is written above, all possible words of any length), then we have

$$\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle^k) = \sum_{\mu\vdash n}\sum_{d=0}^{k} S_{k,d} K_{\mu(n-d,1^d)} s_\mu[X]$$

where $S_{n,d}$ is the Stirling number of the second(?) kind which counts the number of set partitions of size n and length d.

Moreover, if we restrict ourselves to the words of content α and denote the partition of k corresponding to the set partition A as $\lambda(A)$, then we notice that

$$\mathcal{F}rob_{\mathfrak{S}_n}\left(\bigoplus_{\lambda(A)=\alpha} N^A\right) = \sum_{\substack{A\vdash[k]\\ \lambda(A)=\alpha}} \sum_{\mu\vdash n} K_{\mu(n-\ell(\alpha),1^{\ell(\alpha)})} s_\mu[X]$$

We can also decompose the module $\bigoplus_{\lambda(A)=\alpha} N^A$ in terms of the modules

$$V_\alpha^T = \mathcal{L}\{\mathbf{x}_{\vec{i}} E_T : \lambda(\nabla(\mathbf{x}_{\vec{i}})) = \alpha\}$$

where T is some standard tableau of shape $\lambda \vdash k$. By applying the operators E^T on the right for each of these standard tableau, we have

$$\bigoplus_{\lambda(A)=\alpha} N^A = \bigoplus_{T\in ST^k} V_\alpha^T$$

where we may assume that the shape of T is less than or equal to α in dominance order.

Since what we wish to know is the decomposition numbers $d^\mu_{\lambda\alpha}$ the modules V^T_α into irreducible S_n modules we have

$$V_\alpha^T = \bigoplus_{\mu\vdash n} (M^\mu)^{\oplus d^\mu_{\lambda(T)\alpha}}$$

then we may also compute

$$\mathcal{F}rob_{\mathfrak{S}_n}\left(\bigoplus_{\lambda(A)=\alpha} N^A\right) = \sum_{\substack{T\in ST^k\\ \lambda(T)\geq\alpha}} \sum_{\mu\vdash n} d^\mu_{\lambda(T)\alpha} s_\mu[X]$$

In particular we conclude that for a fixed $\mu \vdash n$ and for $\alpha \vdash k$,

$$\sum_{\substack{A \vdash [k] \\ \lambda(A)=\alpha}} K_{\mu(n-\ell(\alpha),1^{\ell(\alpha)})} = \sum_{\substack{T \in ST^k \\ \lambda(T) \geq \alpha}} d^{\mu}_{\lambda(T)\alpha} \ .$$

This is tantalizingly close to a combinatorial interpretation for the $d^{\mu}_{\lambda(T)\alpha}$ which we would ideally like to be able to isolate from this equation.

By what I have written in previous parts of these notes I would say that we have a combinatorial interpretation for the coefficients $d^{\mu}_{\lambda(T)\alpha}$ if we can determine the coefficients $d^{\mu}_{\lambda(r^{n_r})}$ where $\ell(\lambda) < n_r$ and $\lambda_2 \geq r$ (see Conjecture 8, 10, 11 from part II of these notes).