Quiver Representations

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March 5, 2006

Overview of Talk

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The slides will be on my website www.math.toronto.edu/adouglas/quivers.

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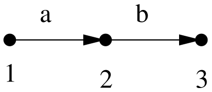
Example:



$Q_0 = \{1, 2, 3\}$, $Q_1 = \{a, b\}$, $ta = 1$, $ha = tb = 2$, $hb = 3$.■

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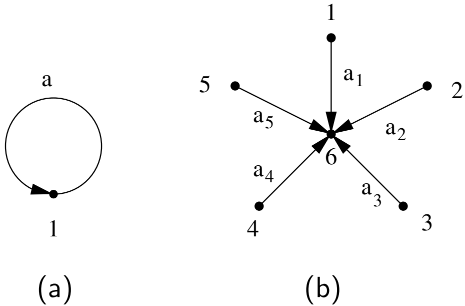

Figure: (a) Jordan quiver (b) star quiver.

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of finite dimensional K-vector spaces together with a collection

$$\{V_a : V_{ta} \longrightarrow V_{ha} | a \in Q_1\}$$

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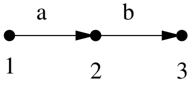

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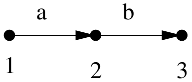

Two of its representations are

$$K \xrightarrow{1} K \xrightarrow{0} 0 \qquad\qquad K \xrightarrow{0} 0 \xrightarrow{0} 0$$

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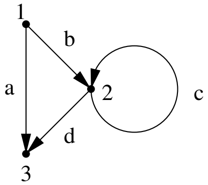

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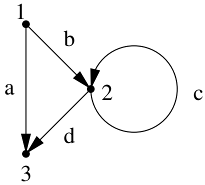

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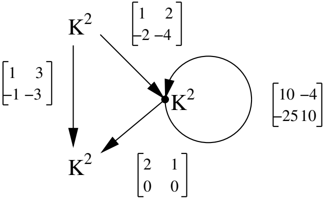

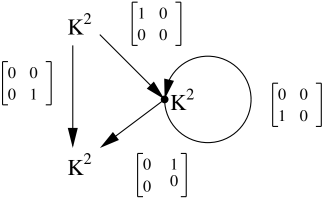

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If V and W are two representations of Q, then a morphism $F : V \longrightarrow W$ is a collection of K-linear maps

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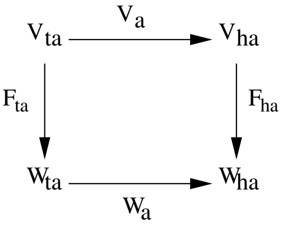

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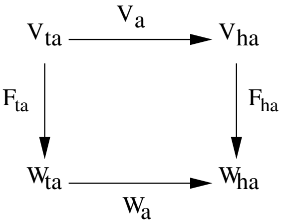
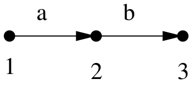


commutes for every $a \in Q_1$.

A morphism $F : V \longrightarrow W$ is an isomorphism if F_x is invertible for every $x \in Q_0$.

Example: For any representation V of Q there is always the identity morphism $1_V : V \longrightarrow V$ defined by the identity maps $(1_V)_x : V_x \longrightarrow V_x$ for any $x \in Q_0$.

Example: Recall the quiver



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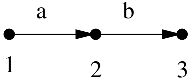

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A morphism between these representations is given by

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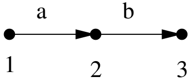

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The morphism is clearly not an isomorphsim. ■

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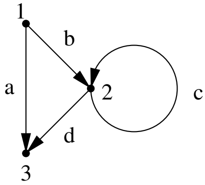

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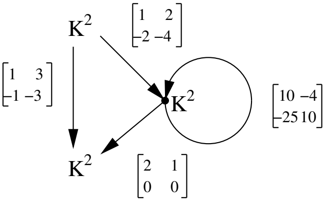

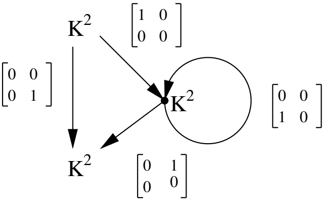

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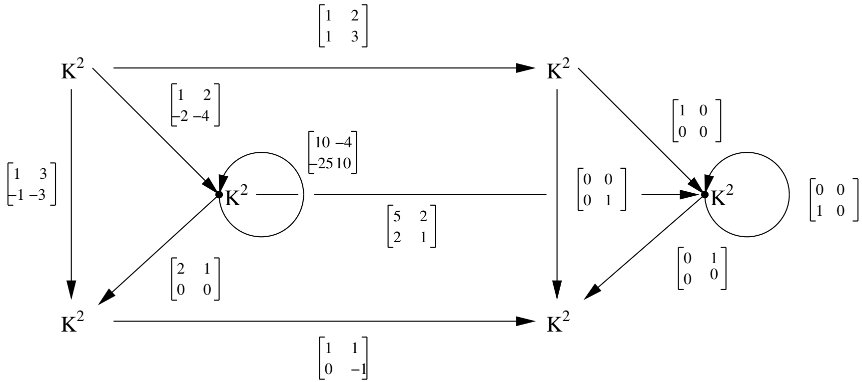

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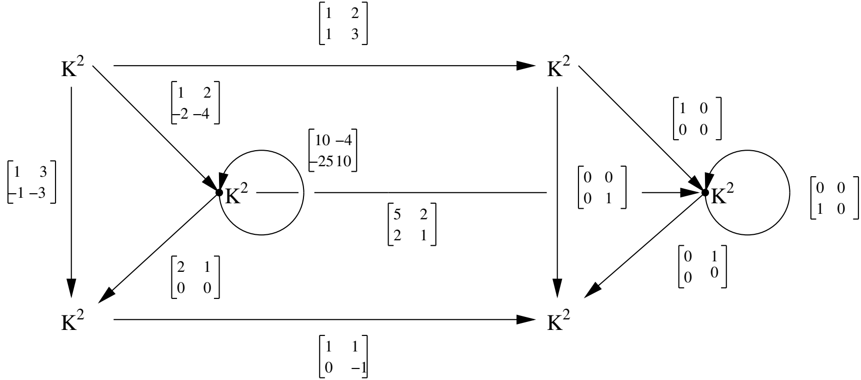

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$$(V \oplus W)_a \equiv \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} : V_{ta} \oplus W_{ta} \longrightarrow V_{ha} \oplus W_{ha}$$

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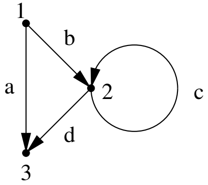

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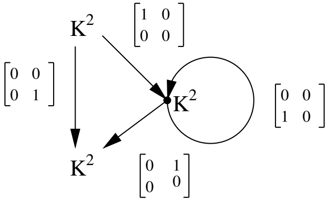

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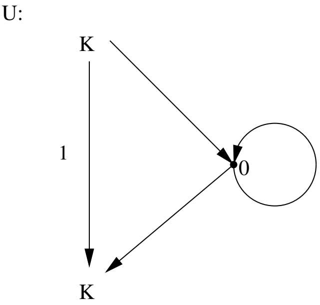

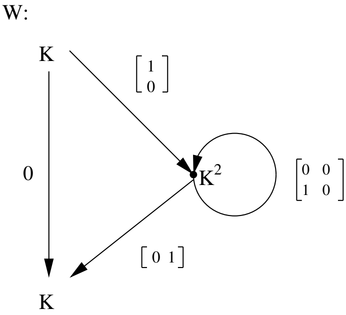

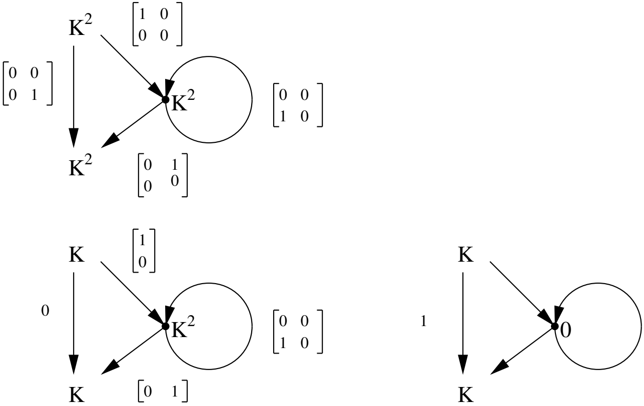

$\blacksquare$

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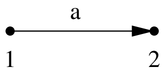
We will now attempt to solve the classification problem for certain well chosen examples. Later we will consider classification in a more general setting.

Example:



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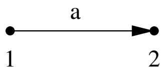

- A representation of this quiver is a collection of two finite dimensional vector spaces V_1 and V_2 together with a linear map $V_a : V_1 \longrightarrow V_2$.

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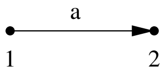

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- For a linear map $V_a : V_1 \longrightarrow V_2$ we can always choose a basis in which V_a is given by the block matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where r is the rank of V_a.

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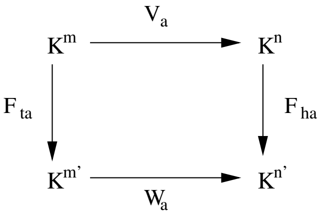

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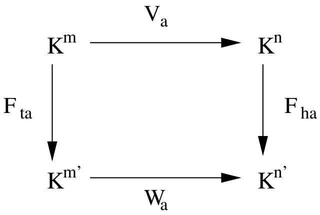

if and only if $m = m'$, $n = n'$ and V_a and W_a have the same rank.

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$$A:\ K \xrightarrow{\ 0\ } 0$$

$$B:\ 0 \xrightarrow{\ 0\ } K$$

$$C:\ K \xrightarrow{\ 1\ } K$$

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- Then, any representation Z of Q is isomorphic to

$$Z \cong A^{d_1 - r} \oplus B^{d_2 - r} \oplus C^r$$

where $d_1 = dimV_1$, $d_2 = dimV_2$ and $r = rankV_a$.

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  where $d_1 = dim V_1$, $d_2 = dim V_2$ and $r = rank V_a$.
- Thus there are 3 non-isomorphic indecomposable representations.
- Quivers that have a finite number of pairwise non-isomorphic indecomposable reps are said to be of finite type.

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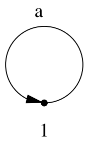

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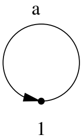

- A representation of the Jordan quiver is a finite dimensional vector spaces V_1 together with an endomorphism $V_a : V_1 \longrightarrow V_1$.

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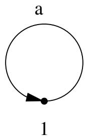

- A representation of the Jordan quiver is a finite dimensional vector spaces V_1 together with an endomorphism $V_a : V_1 \longrightarrow V_1$.
- Relative some choice of basis, we may put V_a into Jordan normal form (assume K is algebraically closed)

$$\begin{pmatrix} J_{n_1,\lambda_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_{n_r,\lambda_r} \end{pmatrix}, \quad J_{n,\lambda} = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

which is unique up to permutation of the blocks.

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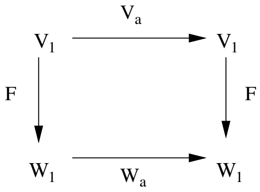

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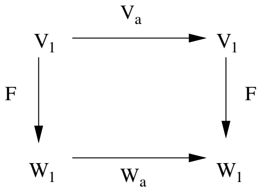

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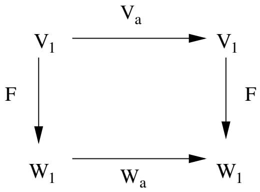

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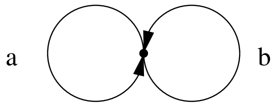
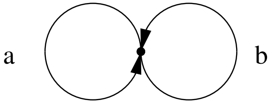
- Although there are infinitely many indecomposable reps, they can still be parameterized by a discrete parameter n (the size of the Jordan block) and a continuous parameter λ (the eigenvalue of the block).

- A quiver is of tame type if it has infinitely many isoclasses but they can be split into families, each parameterized by a single continuous parameter. ■

Example:



Example:



- A representation of this quiver is a pair $V_a : V_1 \longrightarrow V_1$ and $V_b : V_1 \longrightarrow V_1$.

- Two representations $V = \{V_a : V_1 \longrightarrow V_1, V_b : V_1 \longrightarrow V_1\}$ and $W = \{W_a : V_1 \longrightarrow V_1, W_b : W_1 \longrightarrow W_1\}$ are isomorphic if and only if

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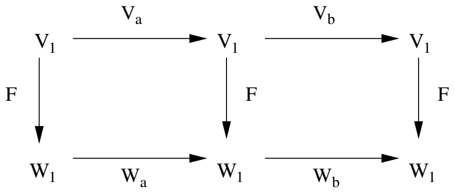

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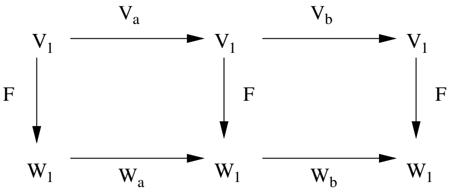

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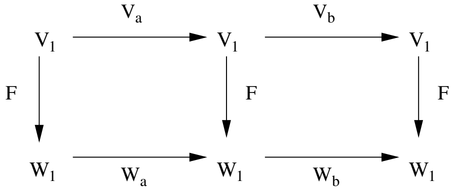

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- To classify the representations of this quiver we would have to classify all pairs of matrices (V_a, V_b) up to simultaneous conjugation.

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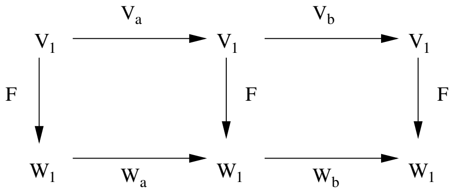

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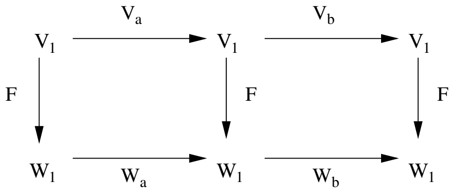

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- We call the representation theory of this quiver wild. ■

Definitions:

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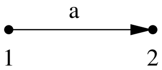

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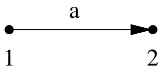

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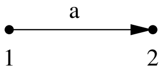

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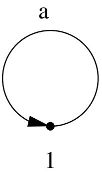

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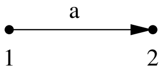

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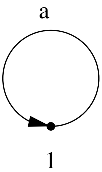

- If a quiver is of tame type we have a hope of classifying its representations.

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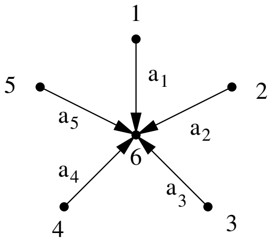

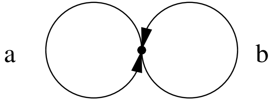

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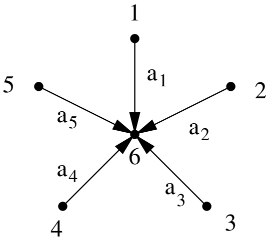

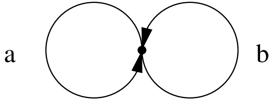

- If you could classify all pairs of matrices (A, B) up to simultaneous conjugation you could classify all quivers (and associative algebras [Drozd]) .

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For a quiver Q and a field K we can form a category $Rep_k(Q)$ whose objects are representations of Q with morphisms as defined above.

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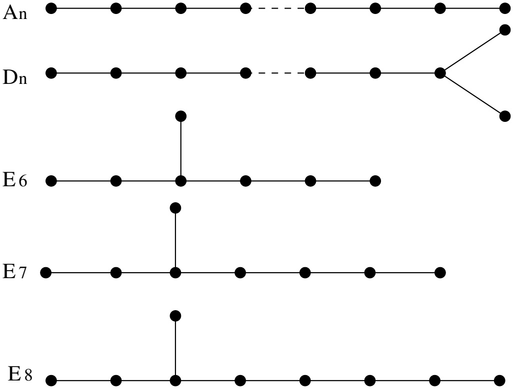

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$$V \mapsto \sum_{x \in Q_0} d_V(x)\alpha_x.$$

Gabriel's Theorem (3) : A quiver is of tame type if and only if the underlying undirected graph is a union of Dynkin graphs of type A, D or E and extended Dynkin graphs of type $\hat{A}$, $\hat{D}$ or $\hat{E}$ (with at least one extended Dynkin graphs).

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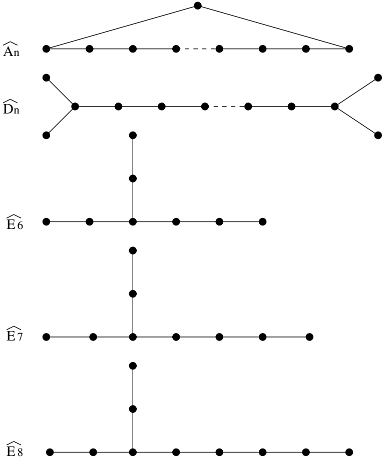

The Euler form of a quiver Q is defined to be the bilinear form on $\mathbb{Z}^{Q_0}$ given by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

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Note that the Tits form is independent of the orientation of arrow in Q.

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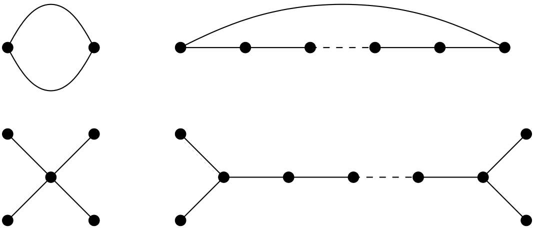

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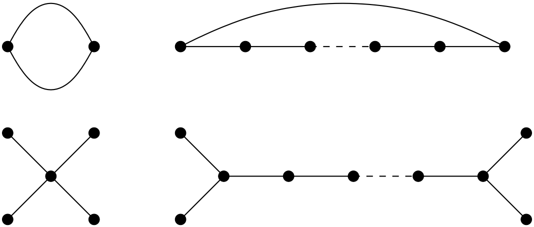

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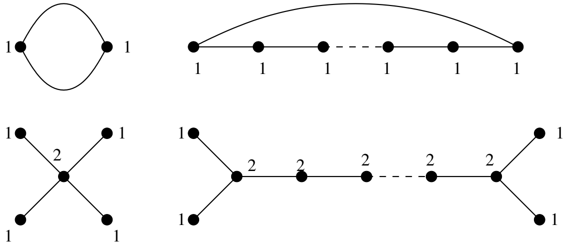

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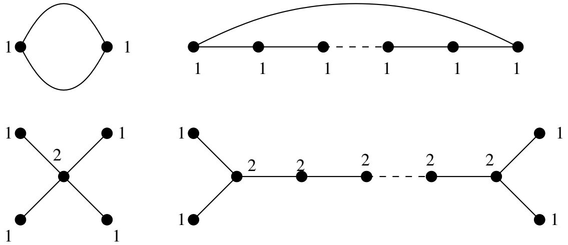

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$(B_Q(\alpha) = 1^2 + 1^2 - 1 \times 1 - 1 \times 1 = 0)$

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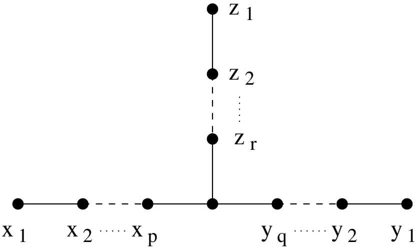

where $r \leq p \leq q$.

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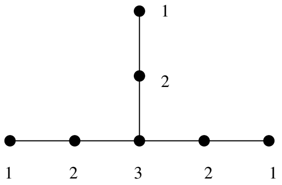

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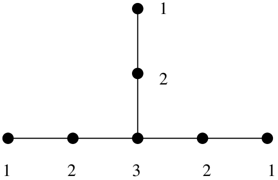

$B_Q(\alpha) =$
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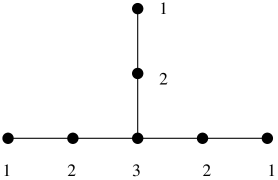

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If $r = 0$ this give A_n.

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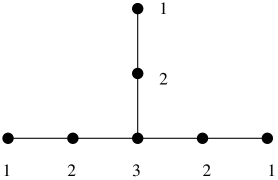

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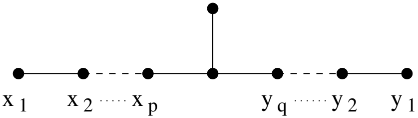
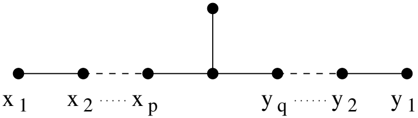
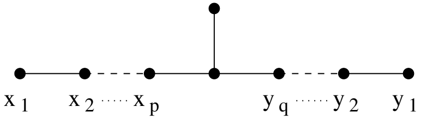

x 1
x 2 ····· x p
y q ······ y 2
y 1



3. We must have $p \leq 2$.



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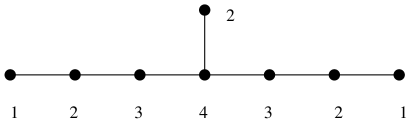

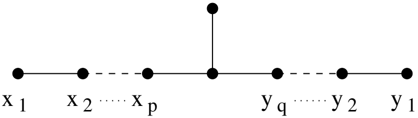

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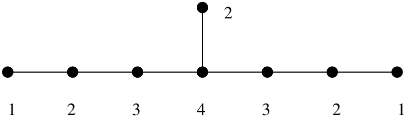
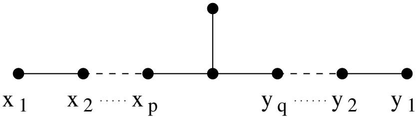


$B_Q = 0$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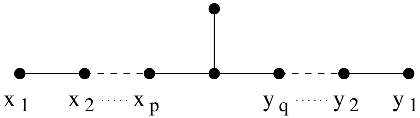
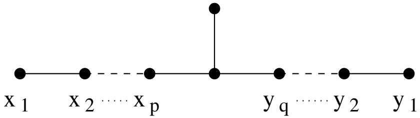


4. If $p = 1$, then q arbitrary. This gives D_n.



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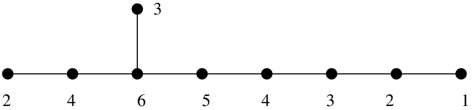

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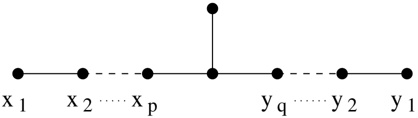

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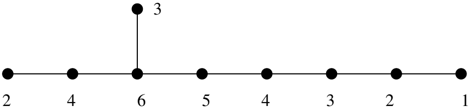
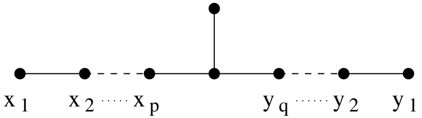


$B_Q \leq 0$.
This give E_6, E_7 or E_8.



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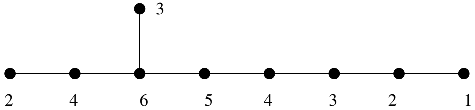

$B_Q \leq 0$.
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Thus we have only A_n, D_n, E_6, E_7 and E_8. ■

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If we fix a basis in each of the spaces V_x, then the object (V, d_V) is completely defined by the set of matrices M_a for $a \in Q_1$, where M_a is the matrix of the mapping $V_a : V_{ta} \longrightarrow V_{ha}$.

In each space V_x we change basis by means of a non-singular $m_x \times m_x$ matrix g_x.

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Then G acts on M according to $\bigstar$. And, two reps are iso iff the set of matrices $\{M_a\}$ corresponding to them lie in one orbit of G.

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Therefore, $dim(G) - dim(M) - 1 \geq 0$ implies
$B_Q(m) = \sum_{x \in Q_0} {m_x}^2 - \sum_{a \in Q_1} m_{ta} m_{ha} > 0$. ■

Proposition1: Let Q be a connected quiver. If B_Q is positive definite then the underlying graph of Q is a Dynkin graph of type A, D or E.

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Proving the other direction requires the development of reflection functors .

From Gabriel 1 we can count the number of pairwise non-isomorphic indecomposable reps for a (connected) quiver of finite type.

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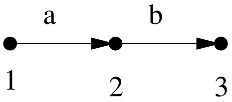
underlying graph	A_n	D_n	E_6	E_7	E_8
positive roots	$\frac{n(n+1)}{2}$	$n(n-1)$	36	63	120

Example: Let Q be the quiver of type A_3 with the following orientation



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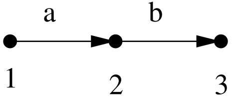

The set of positive roots of the Lie algebra of type A_3 are

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2,$$
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$$V \mapsto \sum_{x \in Q_0} d_V(x)\alpha_x.$$

The complete list of non-iso indecomposable reps is

$$K \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad a_1$$

$$0 \xrightarrow{0} K \xrightarrow{0} 0 \qquad a_2$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} K \qquad a_3$$

$$K \xrightarrow{1} K \xrightarrow{0} 0 \qquad a_1 + a_2$$

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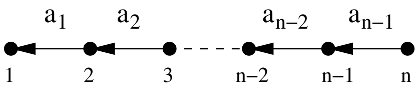
Example: Let Q be a quiver of type A_n with the following orientation



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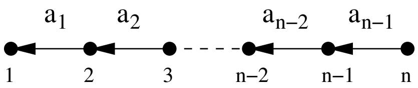

The set of positive roots of the Lie algebra of type A_n are

$$\{\sum_{i=j}^{l} \alpha_i | 1 \leq j \leq l \leq n\}.$$

The root $\sum_{i=j}^{l} \alpha_i$ with $1 \leq j \leq l \leq n$ corresponds to the unique representation V with

$$V_i = \begin{cases} K, & \text{if } j \leq i \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

$$V_{a_i} = \begin{cases} 1, & \text{if } j \leq i \leq l-1, \\ 0 & \text{otherwise.} \end{cases}$$

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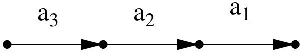

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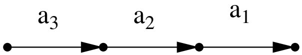

Let e_x denote trivial path with $te_x = he_x = x$ for all $x \in Q_0$.

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Let $a = a_1 a_2 ... a_r$ and $b = b_1 b_2 ... b_s$, then

$$a \cdot b = \begin{cases} a_1 a_2 ... a_r b_1 b_2 ... b_s, & ta_r = hb_1, \\ 0, & \text{otherwise.} \end{cases}$$

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KQ is an associate algebra with unit $(\sum_{x \in Q_0} e_x)$.

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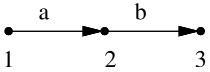
Consider the quiver Q



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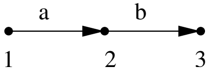

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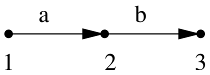

- Then, $\{e_1, e_2, e_3, a, b, ba\}$ is a K-basis of KQ.Some examples of products are $b \cdot a = ba$, $a \cdot b = 0$, $e_2 \cdot a = a$, $a \cdot e_2 = 0$, $a \cdot ba = 0$, and $ba \cdot a = 0$. ■

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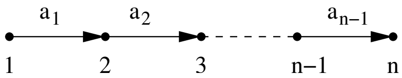

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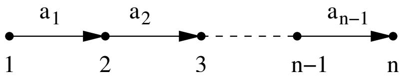

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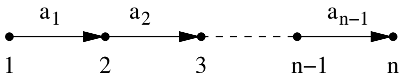

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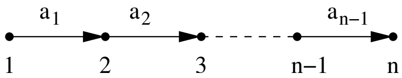

- For every $1 \leq i \leq j \leq n$ there is a unique path from i to j.
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- f is an isomorphism from KQ onto the algebra of lower triangular matrices. ■

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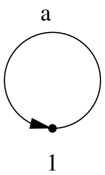

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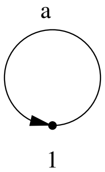

$KQ \cong K[a]$. ■

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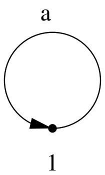

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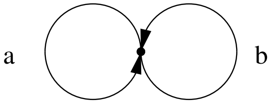

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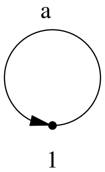

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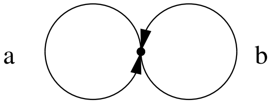

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KQ is finite dimensional iff Q has no oriented cycles.

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$$\text{left-KQ-module } V = \oplus_{x \in Q_0} V_x,$$

$$e_x \cdot v = \begin{cases} v, & v \in V_x, \\ 0, & \text{otherwise} \end{cases},$$

$$a{=}a_1 a_2 ... a_r \cdot v = \begin{cases} V_{a_1} V_{a_2} ... V_{a_r}(v), & v \in V_{ta_r}, \\ 0, & \text{otherwise.} \end{cases}$$

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Then $\frac{KQ}{\langle p_i \rangle}$ is the algebra defined by a quiver with relations.

In general, if $(Q, \langle p_i \rangle)$ is a quiver with relations, In general, if $(Q, \langle p_i \rangle)$ is a quiver with relations, we identify representations V of Q satisfying $V_{p_i} = 0$ (i.e., if $p_i = a_1 a_2$ then $V_{a_1} V_{a_2} = 0$), $\quad \forall i$, with left modules over $\frac{KQ}{\langle p_i \rangle}$.

Propostion: Let A be a finite dimensional K-algebra. Then the category of representations of A is equivalent to the category of representations of $\frac{KQ}{\mathcal{J}}$ for some quiver with relations $(Q, \mathcal{J})$.

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