Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Braided differential calculus and quantum Schubert calculus

Anatol N. Kirillov and Toshiaki Maeno

ABSTRACT. We provide a new realization of the quantum cohomology ring of a flag variety as a certain commutative subalgebra in the cross product of the Nichols-Woronowicz algebras associated to a certain Yetter-Drinfeld module over the Weyl group. We also give a generalization of some recent results by Y.Bazlov to the case of the Grothendieck ring of a flag variety of classical type.

RÉSUMÉ. Nous fournissons une nouvelle réalisation de l'anneau de la cohomologie quantique d'une variété de drapeaux comme sous-algèbre commutative dans le produit croise des algèbres de Nichols-Woronowicz associées à un certain module de Yetter-Drinfeld sur le groupe de Weyl. Nous donnons aussi une généralisation de résultats récents par Y. Bazlov au cas de l'anneau de Grothendieck d'une variété de drapeaux de type classique.

1. Introduction

The main purpose of this work is

• to construct a model of the quantum cohomology ring of the flag variety G/B corresponding to a semisimple finite-dimensional Lie group G as a quantization of Bazlov's model of the coinvariant algebra of finite Coxeter groups,

• to construct a model for the Grothendieck ring of the flag varieties of classical type, in terms of a braided (and discrete) analogue of the differential calculus.

Such a construction of a model for the classical cohomology ring of a flag variety, and more generally for the coinvariant algebra of a finite Coxeter group, as a subalgebra in a braided Hopf algebra called the Nichols-Woronowicz algebra has been invented recently by Y. Bazlov [**2**]. In the present paper we provide a new realization of the quantum cohomology ring of a flag variety as a certain commutative subalgebra in the braided cross product of the corresponding Nichols-Woronowicz algebra and its dual. We also give a generalization of some results from [**2**] to the case of the Grothendieck ring of the flag variety of classical type.

The K-theoretic counterpart of the theory of the quantum cohomology ring has been invented by Givental and Lee. In their paper [**8**], they study the quantum K-theory for the flag variety in a connection with the difference Toda system. The author hopes to report on the Nichols-Woronowicz model of the quantum Grothendieck ring of the flag variety elsewhere in the near future. A description of the quantum K-ring of the flag variety of type A in terms of generators and relations will be given in our forthcoming paper [**13**].

2. Braided differential calculus

In order to formulate our construction, we will remind of the basic notion on the braided differential calculus and the Nichols-Woronowicz algebra in this section.

DEFINITION 2.1. The category C equipped with a functor $\otimes : C \times C \to C$, a collection of isomorphisms

$$(\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W))_{U,V,W \in \mathrm{Ob}(C)},$$

Key words and phrases. braided Hopf algebras, quantum cohomology rings, flag varieties.
The authors thank Yuri Bazlov for explaining his result. This work is supported by Grant-in-Aid for Scientific Research.

an object $\mathbf{1} \in \mathrm{Ob}(C)$ and isomorphisms of functors

$$\iota_{\text{left}} : \bullet \otimes \mathbf{1} \overset{\sim}{\longrightarrow} \mathrm{id}, \quad \iota_{\text{right}} : \mathbf{1} \otimes \bullet \overset{\sim}{\longrightarrow} \mathrm{id}$$

is called a monoidal category if the following diagrams commute:
(1) (pentagon condition)

$$\begin{array}{ccccc} & & (U \otimes V) \otimes (W \otimes X) & & \\ & \nearrow & & \searrow & \\ ((U \times V) \otimes W) \otimes X & & & & U \otimes (V \otimes (W \otimes X)) \\ \downarrow & & & & \uparrow \\ (U \otimes (V \otimes W)) \otimes X & & \longrightarrow & & U \otimes ((V \otimes W) \otimes X), \end{array}$$

where all the arrows are induced by Φ,
(2) (triangle condition)

$$\begin{array}{ccc} (U \otimes \mathbf{1}) \otimes V & \overset{\Phi}{\longrightarrow} & U \otimes (\mathbf{1} \otimes V) \\ \iota \otimes \mathrm{id} \searrow & & \swarrow \mathrm{id} \otimes \iota \\ & U \otimes V. & \end{array}$$

Definition 2.2. A monoidal category $C = (C, \otimes, \Phi, \mathbf{1}, \iota)$ is called a braided category if a collection of functorial isomorphisms

$$(\Psi_{U,V} : U \otimes V \to V \otimes U)_{U,V \in \mathrm{Ob}(C)}$$

is given so that the following hexagon conditions are satisfied:

$$(\Psi \otimes \mathrm{id}) \circ \Phi^{-1} \circ (\mathrm{id} \otimes \Psi) = \Phi^{-1} \circ \Psi \circ \Phi^{-1} : U \otimes (V \otimes W) \longrightarrow (W \otimes U) \otimes V,$$

$$(\mathrm{id} \otimes \Phi) \circ \Phi \circ (\Psi \otimes \mathrm{id}) = \Phi \circ \Psi \circ \Phi : (U \otimes V) \otimes W \longrightarrow V \otimes (W \otimes U).$$

Let us take a braided category C consisting of vector spaces over a fixed field k and a braided vector space $V \in \mathrm{Ob}(C)$. Then, the braiding $\psi_V : V \otimes V \to V \otimes V$ is naturally associated to V, and the pair (V, ψ_V) is used to designate V together with the braiding ψ_V. Note that the morphism ψ_V is not necessarily an involution. Denote by ψ_i the endomorphism on the tensor product $V^{\otimes n}$ obtained by applying ψ_V on the i-th and $(i+1)$-st components of $V^{\otimes n}$. Then the braid relation

$$\psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1}$$

is a consequence of the hexagon condition.

The Nichols-Woronowicz algebra provides a natural framework to discuss the braided differential calculus. When a finite-dimensional braided vector space (V, ψ) is given, we can attach naturally a braided Hopf algebra structure to the tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$.

Definition 2.3. A k-algebra A in the braided category C is called a braided algebra if its multiplication $m : A \times A \to A$ commutes with the braiding $\psi = \psi_A$, i.e.

$$(m \otimes \mathrm{id}) \circ (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) = \psi \circ (\mathrm{id} \otimes m) : A \otimes A \otimes A \to A \otimes A.$$

The tensor algebra $T(V)$ is naturally braided by the braiding $\psi_{T(V)}$ which is uniquely characterized by the conditions:
(1) $T(V)$ is a braided algebra,
(2) $\psi_{T(V)}|_{T^1(V) \otimes T^1(V)} = \psi_V$.

Now we can discuss the braided Hopf algebra structure on the tensor algebra $T(V)$. Define the linear maps $\triangle : V \to V \otimes V$, $S : V \to V$ and $\varepsilon : V \to k$ by

$$\triangle(v) := v \otimes 1 + 1 \otimes v, \ S(v) := -v, \ \varepsilon(v) := 0.$$

Then, one can extend the maps $\triangle$, S and ε to endomorphisms on $T(V)$ so that they respectively define the coproduct, the antipode and the counit of the braided Hopf algebra. In particular, $\triangle$ is made to satisfy the condition

$$(m \otimes m) \circ (\mathrm{id} \otimes \psi \otimes \mathrm{id}) \circ (\triangle \otimes \triangle) = \triangle \circ m, \quad \text{on } T(V) \otimes T(V).$$

We call $T(V)$ the free braided Hopf algebra or the free braided group.

Definition 2.4. Let H and K be braided Hopf algebras provided with a k-linear pairing $\langle\ ,\ \rangle : H\times K \to k$. We say that H and K are dually paired if the following conditions are satisfied:

$$\langle \gamma, \kappa\kappa'\rangle = \langle \gamma_{(1)}, \kappa'\rangle\langle \gamma_{(2)}, \kappa\rangle, \ \ \langle \gamma\gamma', \kappa\rangle = \langle \gamma', \kappa_{(1)}\rangle\langle \gamma, \kappa_{(2)}\rangle,$$
$$\langle \gamma, 1\rangle = \varepsilon_H(\gamma), \ \ \langle 1, \kappa\rangle = \varepsilon_K(\kappa), \ \ \langle S_H(\gamma), \kappa\rangle = \langle \gamma, S_K(\kappa)\rangle,$$

where we use Sweedler's notation $\triangle(a) = a_{(1)}\otimes a_{(2)}$. If the conditions above are satisfied, the pairing $\langle\ ,\ \rangle$ is called a duality pairing.

Let V^* be the dual vector space of V. Then it has the natural braiding ψ^* dual to ψ. It is nontrivial problem to extend the natural pairing $\langle\ ,\ \rangle : V^*\times V \to k$ to the duality pairing between the braided Hopf algebras $T(V^*)$ and $T(V)$. The construction due to Woronowicz [**22**] guarantees the possibility of such an extension of the pairing $\langle\ ,\ \rangle$. Note that from the braid relation one can define the endomorphism Ψ_w on $V^{\otimes n}$ associated to an element w in the symmetric group S_n with a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$, $s_i = (i, i+1)$, as $\Psi_w := \psi_{i_1}\cdots\psi_{i_l}$. The *Woronowicz symmetrizer* is defined as $\sigma_n(\psi) := \sum_{w\in S_n}\Psi_w$. Then the pairing $\langle\ ,\ \rangle : V^*\times V \to k$ can be extended to the one between $(V^*)^{\otimes n}$ and $V^{\otimes n}$ for each $n\geq 2$ by the formula

$$\langle \alpha_1\otimes\cdots\otimes\alpha_n, v_1\otimes\cdots\otimes v_n\rangle := (\alpha_n\otimes\cdots\otimes\alpha_1)(\sigma_n(\psi)(v_1\otimes\cdots\otimes v_n)), \ \ \alpha_i\in V^*, \ v_j\in V.$$

Proposition 2.1. *The free braided Hopf algebras $T(V^*)$ and $T(V)$ are dually paired with respect to the pairing $\langle\ ,\ \rangle : T(V^*)\times T(V) \to k$.*

The dually paired braided Hopf algebras $T(V^*)$ and $T(V)$ are not appropriate objects to perform the braided differential calculus on them, since the kernel of the duality pairing is big in general. The *Nichols-Woronowicz algebra* is a braided Hopf algebra which is obtained as a quotient of the free braided Hopf algebra by the kernel of the duality pairing. Such a construction is due to Majid [**17**]. Note that the kernels

$$I(V^*) := \{\xi\in T(V^*) \mid \langle\xi, x\rangle = 0, \ \forall x\in T(V)\},$$
$$I(V) := \{x\in T(V) \mid \langle\xi, x\rangle = 0, \ \forall \xi\in T(V^*)\}$$

are Hopf ideals.

Definition 2.5. The Nichols-Woronowicz algebras $\mathbf{B}(V^*)$ and $\mathbf{B}(V)$ are the dually paired braided Hopf algebras defined to be the quotients of the free braided Hopf algebras by $I(V^*)$ and $I(V)$ respectively:

$$\mathbf{B}(V^*) := T(V^*)/I(V^*), \ \ \mathbf{B}(V) := T(V)/I(V).$$

The following equivalent definition is due to Andruskiewitsch and Schneider [**1**]:

Definition 2.6. The Nichols-Woronowicz algebra $\mathbf{B}(V)$ is the graded braided Hopf algebra characterized by the conditions:
(1) $\mathbf{B}^0(V) = k$,
(2) $V = \mathbf{B}^1(V) = \{x\in\mathbf{B}(V) \mid \triangle(x) = x\otimes 1 + 1\otimes x\}$,
(3) $\mathbf{B}(V)$ is generated by $\mathbf{B}^1(V)$ as an algebra.

Each element $v\in\mathbf{B}^1(V)$ acts on $\mathbf{B}(V^*)$ as a *twisted derivation* $\overleftarrow{D}_v$ from the right:

$$\overleftarrow{D}_v : \mathbf{B}(V^*) \overset{\triangle}{\longrightarrow} \mathbf{B}(V^*)\otimes\mathbf{B}(V^*) \overset{\mathrm{id}\otimes\langle\ ,v\rangle}{\longrightarrow} \mathbf{B}(V^*)\otimes k = \mathbf{B}(V^*).$$

The twisted derivation $\overleftarrow{D}_v$ satisfies the twisted Leibniz rule

$$(fg)\overleftarrow{D}_v = f(g\overleftarrow{D}_v) + f\triangleleft\psi^{-1}(g\otimes\overleftarrow{D}_v),$$

where $f\triangleleft\psi^{-1}(g\otimes\overleftarrow{D}_v) = \sum_i (f\overleftarrow{D}_{v_i})g_i$ if $\psi^{-1}(g\otimes v) = \sum_i v_i\otimes g_i$. This action extends to the left action of the opposite algebra $\mathbf{B}(V)^{op}$ on $\mathbf{B}(V)$. The braided cross product $\mathbf{B}(V)^{op}\rtimes\mathbf{B}(V^*)$ with respect to the action by the twisted derivations can be identified with the algebra of the braided differential operators acting on $\mathbf{B}(V^*)$. In other words, the algebra structure of $\mathbf{B}(V)^{op}\rtimes\mathbf{B}(V^*)$ is given by the multiplication rule

$$(u\otimes x)\cdot(v\otimes y) = u(\psi^{-1}(x\otimes v_{(1)})\triangleleft v_{(2)})y$$

on $\mathbf{B}(V)^{op}\otimes\mathbf{B}(V^*)$, see [**18**] for details.

At the end of this section, we introduce an important example of the braided categories, which is called the category of the *Yetter-Drinfeld modules.* Let Γ be a finite group.

Definition 2.7. A k-vector space V is called a Yetter-Drinfeld module over Γ, if
(1) V is a Γ-module,
(2) V is Γ-graded, i.e. $V = \bigoplus_{g \in \Gamma} V_g$, where V_g is a linear subspace of V,
(3) for $h \in \Gamma$ and $v \in V_g$, $h(v) \in V_{hgh^{-1}}$.

One of the importance of the category ${}_{\Gamma}^{\Gamma}YD$ of the Yetter-Drinfeld modules over a fixed group Γ is that it is naturally braided. The tensor product of V and W in ${}_{\Gamma}^{\Gamma}YD$ is again a Yetter-Drinfeld module with the Γ-action $g(v \otimes w) = g(v) \otimes g(w)$ and the Γ-grading $(V \otimes W)_g = \bigoplus_{h,h' \in \Gamma,\ hh'=g} V_h \otimes W_{h'}$. The braiding between V and W is defined by $\psi_{V,W}(v \otimes w) = g(w) \otimes v$, for $v \in V_g$ and $w \in W$.

3. Nichols-Woronowicz model of quantum Schubert calculus

Let G be a connected, simply-connected and semi-simple complex Lie group. Fix a Borel subgroup B of G. Denote by Δ the set of roots, which is decomposed into the disjoint union $\Delta = \Delta_+ \sqcup (-\Delta_+)$ by choosing the set of positive roots Δ_+ corresponding to B. Our main interest is a combinatorial structure of the (quantum) cohomology ring of the flag variety G/B. It is well-known that the cohomology ring of the flag variety is isomorphic to the quotient ring of the ring of polynomial functions on the Cartan subalgebra $\mathfrak{h}$ by the ideal generated by the fundamental invariants $f_1, \ldots, f_r$, $r = \mathrm{rk}\mathfrak{h}$, of the Weyl group W, i.e.

$$H^*(G/B, \mathbf{Q}) \cong \mathrm{Sym}_{\mathbf{Q}}\mathfrak{h}^*/(f_1, \ldots, f_r).$$

On the other hand, the Schubert classes Ω_w, $w \in W$, corresponding to the dual of the cycles $\overline{Bw_0wB/B}$ form a linear basis of $H^*(G/B, \mathbf{Q})$. Then the fundamental problems of the Schubert calculus are stated as follows:

Problem 3.1. (1) Find the natural polynomial representative for the Schubert class Ω_w in the coinvariant algebra $\mathrm{Sym}_{\mathbf{Q}}\mathfrak{h}^*/(f_1, \ldots, f_r)$.
(2) Determine the structure constants c_{uv}^w in the multiplication rule

$$\Omega_u \Omega_v = \sum_{w \in W} c_{uv}^w \Omega_w.$$

The answer to the first problem (1) is given for example by the polynomials due to Bernstein, Gelfand and Gelfand [**3**] for general root system, and Schubert polynomials defined by Lascoux and Schützenberger [**14**] for the root system of type A. The latter have nice combinatorial properties. As for the second problem (2), the structure constants c_{uv}^w are complicated in general. However, for special choices of the element $u, v, w \in W$, some combinatorial descriptions of the constants c_{uv}^w, such as Pieri's formula, are known.

The origin of the model of the cohomology ring of the flag variety in terms of a certain noncommutative algebra defined by the data of the root system is the work by Fomin and Kirillov [**5**]. They have introduced an associative $\mathbf{Q}$-algebra $\mathcal{E}_n$, for the root system of type A_{n-1}, generated by the symbols

$$[i,j] = -[j,i], \ \ 1 \leq i, j \leq n, i \neq j,$$

subject to the quadratic relations:
(1) $[i,j]^2 = 0$,
(2) $[i,j][k,l] = [k,l][i,j]$, if $\{i,j\} \cap \{k,l\} = \emptyset$,
(3) $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

Define the *Dunkl element* $\theta_1, \ldots, \theta_n$ in $\mathcal{E}_n$ by

$$\theta_i := \sum_{j \neq i} [i,j].$$

Then one can check the commutativity $\theta_i\theta_i - \theta_j\theta_i = 0$, $\forall i, j$, from the quadratic relations above.

Theorem 3.1. (Fomin and Kirillov [**5**]) *The subalgebra generated by the Dunkl elements is isomorphic to the cohomology ring of the flag variety Fl_n of type A_{n-1}. The isomorphism is given by*

$$\begin{array}{rccc} \mathcal{E}_n \supset & \mathbf{Q}[\theta_1, \ldots, \theta_n] & \rightarrow & H^*(Fl_n), \\ & \theta_1 + \cdots + \theta_i & \mapsto & \Omega_{s_i}. \end{array}$$

The key tool which connects the algebra $\mathcal{E}_n$ to the Schubert calculus is the following *Bruhat representation.*

Definition 3.2. The Bruhat representation of $\mathcal{E}_n$ is defined to be the representation on the vector space $\bigoplus_{w\in W} \mathbf{Q}\cdot w$ by

$$[i,j]w = \begin{cases} ws_{ij}, & \text{if } l(ws_{ij}) = l(w)+1, \\ 0, & \text{otherwise,} \end{cases}$$

where $i<j$ and s_{ij} is the transposition of i and j.

The algebra $\mathcal{E}_n$ admits a natural quantum deformation which corresponds to the quantum cohomology ring of the flag variety. The quantum cohomology ring $QH^*(G/B)$ of the flag variety G/B also has a structure of a quotient ring of the polynomial ring $\mathrm{Sym}_{\mathbf{Q}}\mathfrak{h}^* \otimes \mathbf{Q}[q_1,\dots,q_r]$, where $q_1,\dots,q_r$ are deformation parameters corresponding to the simple roots. The generators $\tilde{f}_1,\dots,\tilde{f}_r$ of the defining ideal of $QH^*(G/B)$ are explicitly determined by Givental and Kim [**7**] for the root system of type A, and by Kim [**9**] for general root systems. Roughly speaking, they are the conserved quantities of the Toda system. Denote by R the polynomial ring $\mathbf{Q}[q_1,\dots,q_{n-1}]$.

Definition 3.3. The quantum deformed quadratic algebra $\tilde{\mathcal{E}}_n$ is an R-algebra defined by the same symbols and relations as those for the algebra $\mathcal{E}_n$ except that the relation (1) for $\mathcal{E}_n$ is replaced by
(1)'

$$[i,j]^2 = \begin{cases} q_i, & \text{if } i=j-1, \\ 0, & \text{if } i<j-1. \end{cases}$$

The quantized version of the Bruhat representation of $\tilde{\mathcal{E}}_n$ is also defined on $\bigoplus_{w\in W} R\cdot w$. The action of the generator $[i,j]$, $i<j$, is given by

$$[i,j]w = \begin{cases} ws_{ij}, & \text{if } l(ws_{ij}) = l(w)+1, \\ q_i q_{i+1}\cdots q_{j-1} ws_{ij}, & \text{if } l(ws_{ij}) = l(w)-2(j-i)+1, \\ 0, & \text{otherwise,} \end{cases}$$

The Dunkl elements θ_i in the quantized algebra $\tilde{\mathcal{E}}_n$ is defined as before. The following theorem was first conjectured in [**5**] and later proved by Postnikov [**21**].

Theorem 3.4. *The subalgebra generated by the Dunkl elements is isomorphic to the quantum cohomology ring of the flag variety Fl_n of type A_{n-1}.*

Their description of the (quantum) cohomology ring Fl_n in terms of the algebra $\mathcal{E}_n$ (or $\tilde{\mathcal{E}}_n$) is of use to consider Problem 2.1 combinatorially. See [**5**] and [**21**] for the detail on this point. A generalization to other root systems is treated in [**10**].

The algebra $\mathcal{E}_n$ is defined by generators and relations, so it is a problem to understand its meaning conceptually. The importance of the (braided) Hopf algebra structure of $\mathcal{E}_n$ has been pointed out by [**6**], [**20**] and other works. Now it is conjectured that the algebra $\mathcal{E}_n$ is a kind of the Nichols-Woronowicz algebra. Bazlov [**2**] has constructed a model of the cohomology ring of the flag variety G/B by using a Nichols-Woronowicz algebra $\mathbf{B}_W$ defined below instead of $\mathcal{E}_n$. When we work on the algebra $\mathcal{E}_n$, all the considerations are based on the defining relations and the Bruhat representation. On the other hand, the results on the Nichols-Woronowicz algebra should come from the method of the braided differential calculus. Hence, the argument for the algebra $\mathbf{B}_W$ is completely different from that for $\mathcal{E}_n$.

Let us define a Yetter-Drinfeld module $V=V_W$ over the Weyl group W. We consider a vector space V generated by the symbols $[\alpha]=-[-\alpha]$, $\alpha\in\Delta$:

$$V = \bigoplus_{\alpha\in\Delta} \mathbf{Q}\cdot[\alpha]/([\alpha]+[-\alpha]).$$

The action of $w\in W$ on V is given by $w[\alpha]=[w(\alpha)]$. If we set the W-degree of the symbol $[\alpha]$ to be the reflection s_α, then V becomes a Yetter-Drinfeld module over W. The braiding $\psi : V\otimes V \to V\otimes V$ is given by $\psi([\alpha]\otimes[\beta]) = s_\alpha([\beta])\otimes[\alpha]$. The braided vector space V is identified with its dual V^* via a W-invariant inner product on V. We denote by $\mathbf{B}_W$ the Nichols-Woronowicz algebra associated to the Yetter-Drinfeld module V.

Remark 3.5. It is conjectured that the Nichols-Woronowicz algebra $\mathbf{B}_W$ for A_{n-1} should be isomorphic to the Fomin-Kirillov quadratic algebra $\mathcal{E}_n$. This conjecture is now confirmed up to $n=6$.

Consider a W-homomorphism $\mu_0 : \mathfrak{h}^* \to V$. The homomorphism μ_0 can be written as

$$\mu_0(x) = \sum_{\alpha\in\Delta_+} c_\alpha(\alpha, x)[\alpha],$$

by using a set of W-invariant constants $(c_\alpha)_{\alpha\in\Delta}$. Then the following result corresponds to the commutativity of the Dunkl elements.

Proposition 3.1. *The image of μ_0 generates a commutative subalgebra in $\mathbf{B}_W$.*

Then μ_0 can be extended to an algebra homomorphism $\mu : \mathrm{Sym}_{\mathbf{Q}}\mathfrak{h}^* \to \mathbf{B}_W$.

Theorem 3.6. (Bazlov [**2**]) *If μ_0 is injective, the image of μ is isomorphic to the cohomology ring $H^*(G/B, \mathbf{Q})$.*

Remark 3.7. Bazlov proved the theorem above for arbitrary finite Coxeter groups and for their coinvariant algebras (over $\mathbf{R}$).

The braided differential operator $\overleftarrow{D}_\alpha = \overleftarrow{D}_{[\alpha]}$ plays an important role for the proof of Theorem 2.3. Indeed, the following properties
(1) $\mu(f)\overleftarrow{D}_\alpha = c_\alpha\mu(\partial_\alpha f)$,
(2) $\cap_{\alpha\in\Delta_+}\mathrm{Ker}(\overleftarrow{D}_\alpha) = \mathbf{B}_W^0(= \mathbf{Q})$
imply the result. Here, we denote by ∂_α the divided difference operator on $\mathrm{Sym}_{\mathbf{Q}}\mathfrak{h}^*$:

$$\partial_\alpha(f) := \frac{f - s_\alpha(f)}{\alpha}.$$

We introduce a quantum deformed version of Bazlov's construction. Let $R = \mathbf{Q}[q^{\alpha^\vee}|\alpha \in \Delta_+]$, where the parameters q^a satisfy the condition $q^{a+b} = q^a q^b$. We denote by $\mathbf{B}_{W,R}$ the scalar extension $R \otimes \mathbf{B}_W$. Since the twisted derivations $\overleftarrow{D}_\alpha$ satisfy the Coxeter relations, one can define the operators $\overleftarrow{D}_w$ for any elements $w \in W$ by $\overleftarrow{D}_w = \overleftarrow{D}_{\alpha_1} \cdots \overleftarrow{D}_{\alpha_l}$ for a reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_l}$.

Definition 3.8. Let $(c_\alpha)_{\alpha\in\Delta}$ be a set of nonzero constants with the condition $c_\alpha = c_{w\alpha}$, $w \in W$. For each root $\alpha \in \Delta_+$, we define an element $\widetilde{[\alpha]}$ in the algebra of braided differential operators $\mathbf{B}_{W,R}^{op} \bowtie \mathbf{B}_{W,R}$ by

$$\widetilde{[\alpha]} := \begin{cases} c_\alpha[\alpha] + d_\alpha q^{\alpha^\vee}\overleftarrow{D}_{s_\alpha}, & \text{if } l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1, \\ c_\alpha[\alpha], & \text{otherwise.} \end{cases},$$

where $d_\alpha = (c_{\alpha_1} \cdots c_{\alpha_l})^{-1}$.

Let $\tilde{\mu}_0$ be a W-homomorphism $\mathfrak{h}_R \to \mathbf{B}_{W,R}^{op} \otimes (R \oplus V_R)$ given by

$$\tilde{\mu}_0(x) = \sum_{\alpha\in\Delta_+} (\alpha, x)\widetilde{[\alpha]}.$$

The image of $\tilde{\mu}_0$ again generates a commutative subalgebra in $\mathbf{B}_{W,R}^{op} \bowtie \mathbf{B}_{W,R}$, so it can be extended to an algebra homomorphism $\tilde{\mu} : \mathrm{Sym}_R\mathfrak{h}_R^* \to \mathbf{B}_{W,R}^{op} \bowtie \mathbf{B}_{W,R}$. Now we can state our main result:

Theorem 3.9. ([**11**]) *The image of $\tilde{\mu}$ is isomorphic to the quantum cohomology ring of the flag variety G/B.*

The key fact to prove this theorem is that the action of the operator $\tilde{\mu}_0(x)$ on $\mathrm{Im}(\mu)$ coincides with the quantization operator by Fomin, Gelfand and Postnikov [**4**] for A_{n-1} and by Maré [**19**] for other root systems.

4. Model of the Grothendieck ring

The Nichols-Woronowicz model of the Grothendieck ring $K(G/B)$ of the holomorphic vector bundles on the flag variety G/B has also been constructed for the classical root systems and G_2 in [**12**]. In this section, we briefly show the construction of the model of $K(G/B)$ for the root system of type B_n.

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{h}^*$, and $\{\pm e_i \pm e_j, \pm e_i \mid 1 \le i, j \le n, i \ne j\}$ be a standard realization of the root system $\Delta = \Delta(B_n)$. For simplicity, we use the symbols $[i,j]$, $\overline{[i,j]}$ and $[i]$ to denote $[e_i - e_j]$, $[e_i + e_j]$ and $[e_i]$ in $\mathbf{B}_W$ respectively. Then the elements $h_{ij} := 1 + [i,j]$, $g_{ij} := 1 + \overline{[i,j]}$ and

$h_i := 1 + [i]$ are solutions of the Yang-Baxter equations:
(1) $h_{ij}h_{kl} = h_{kl}h_{ij}$, $g_{ij}g_{kl} = g_{kl}g_{ij}$, for $\{i,j\} \cap \{k,l\} = \emptyset$,
$h_ih_j = h_jh_i$, $h_{ij}g_{ij} = g_{ij}h_{ij}$,
(2) $h_{ij}h_{ik}h_{jk} = h_{jk}h_{ik}h_{ij}$, $h_{ij}g_{ik}g_{jk} = g_{jk}g_{ik}h_{ij}$,
(3) $h_{ij}h_ig_{ij}h_j = h_jg_{ij}h_ih_{ij}$.
The equations (1), (2) and (3) are respectively corresponding to the subsystems of types $A_1 \times A_1$, A_2 and B_2.

Definition 4.1. We define the multiplicative Dunkl elements or the Ruijsenaars-Schneider-Macdonald elements $\Theta_1^B, \ldots, \Theta_n^B$ of type B_n by the formula

$$\Theta_i^B := h_{i-1\,i}^{-1}h_{i-2\,i}^{-1}\cdots h_{1\,i}^{-1}\cdot h_i \cdot g_{1\,i}g_{2\,i}\cdots g_{n\,i}\cdot h_i \cdot h_{i\,n}h_{i\,n-1}\cdots h_{i\,i+1}.$$

The multiplicative Dunkl elements Θ_i^D (resp. Θ_i^A) of type D_n (resp. A_{n-1}) are obtained by the specialzation $h_i \mapsto 1$ (resp. $g_{ij} \mapsto 1$ and $h_i \mapsto 1$).

Remark 4.2. The multiplicative Dunkl elements have been also introduced by Lenart and Yong [**15**], [**16**] for the root system of type A.

The commutativity $\Theta_i^B\Theta_j^B = \Theta_j^B\Theta_i^B$ follows from the Yang-Baxter relations.

Theorem 4.3. ([**12**]) *The subalgebra in the Nichols-Woronowicz algera $\mathbf{B}_{B_n}$ generated by the multiplicative Dunkl elements $\Theta_1^B, \ldots, \Theta_n^B$ is isomorphic to the Grothendieck ring $K(G/B)$ of the flag variety G/B of type B_n.*

Corollary 4.1. The following identity in the algebra $\mathbf{B}_{B_n}$ holds:

$$\sum_{j=1}^{n}(\Theta_j^B + (\Theta_j^B)^{-1})^k = n\cdot 2^k$$

for all $k \in \mathbf{Z}_{\geq 0}$.

Remark 4.4. The results for D_n and A_{n-1} are obtained by the specializations $h_i \mapsto 1$, $\forall i$, and $h_i \mapsto 1$, $g_{ij} \mapsto 1$, $\forall i,j$, respectively.

The (small) quantum K-ring $QK(Fl_n)$ of the flag variety Fl_n has the following expression by generators and relations:

$$QK(Fl_n) \cong \mathbf{Z}[q_1,\ldots,q_{n-1}][X_1,\ldots,X_n]/(\varphi_k^q(X),\ k=1,\ldots,n),$$

where

$$\varphi_k^q(X) = \sum_{I\subset\{1,\ldots n\},|I|=k}\prod_{i\in I}X_i\prod_{i\notin I,i+1\in I}(1-q_i) - \binom{n}{k}.$$

Let us introduce the quantized multiplicative Dunkl elements by substituting $\widetilde{[ij]}$ defined in Definition 3.8 for $[ij]$ in the definition of Θ_i^A. Here, we put $c_\alpha = 1$. More precisely, we define the quantized multiplicative Dunkl elements $\widetilde{\Theta}_i^A$, $i = 1,\ldots,n$, of type A_{n-1} by the formula

$$\widetilde{\Theta}_i^A = (1-q_{i-1})\tilde{h}_{i-1\,i}^{-1}\tilde{h}_{i-2\,i}^{-1}\cdots\tilde{h}_{1\,i}^{-1}\cdot\tilde{h}_{i\,n}\tilde{h}_{i\,n-1}\cdots\tilde{h}_{i\,i+1},$$

where $\tilde{h}_{ij} := 1 + \widetilde{[ij]} = 1 + [ij] + q_i\cdots q_{j-1}\overleftarrow{D}_{s_{ij}}$, $i<j$.

Theorem 4.5. ([**13**]) *The equalities*

$$\varphi_k^q(\widetilde{\Theta}_1^A,\ldots,\widetilde{\Theta}_n^A) = 0,\ k=1,\ldots n,$$

hold in the algebra $\mathbf{B}_{A_{n-1},R}^{op} \bowtie \mathbf{B}_{A_{n-1},R}$.

References

[1] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras,* New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., **43**, Cambridge Univ. Press, Cambridge, 2002, 1-68.
[2] Y. Bazlov, *Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups,* preprint, math.QA/0409206.
[3] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells and cohomology of the spaces* G/P, Russian Math. Surveys, **28** (1973), 1-26.
[4] S. Fomin, S. Gelfand and A. Postnikov, *Quantum Schubert polynomials,* J. Amer. Math. Soc., **10** (1997), 565-596.
[5] S. Fomin and A. N. Kirillov, *Quadratic algebras, Dunkl elements and Schubert calculus,* Advances in Geometry, (J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan, and P. Xu, eds.) Progress in Math., **172**, Birkhäuser, 1995, 147-182.
[6] S. Fomin and C. Procesi, *Fibered quadratic Hopf algebras related to Schubert calculus,* J. Algebra, **230** (2000), 174-183.
[7] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices,* Commun. Math. Phys., **168** (1995), 609-641.
[8] A. Givental and Y.-P. Lee, *Quantum* K*-theory on flag manifolds, finite-difference Toda lattices and quantum groups,* Invent. Math., **151** (2003), 193-219.
[9] B. Kim, *Quantum cohomology of flag manifolds* G/B *and quantum Toda lattices,* Ann. of Math., **149** (1999), 129-148.
[10] A. N. Kirillov and T. Maeno, *Noncommutative algebras related with Schubert calculus on Coxeter groups,* European J. Combin., **25** (2004), 1301-1325.
[11] A. N. Kirillov and T. Maeno, *A note on quantization operators on Nichols algebra model for Schubert calculus on Weyl groups,* Lett. Math. Phys., **72** (2005). 233-241.
[12] A. N. Kirillov and T. Maeno, *On some noncommutative algebras related to* K*-theory of flag varieties, part I,* Int. Math. Res. Notices **2005**, no. 60, 3753-3789.
[13] A. N. Kirillov and T. Maeno, *A note on quantum* K*-theory of flag varieties,* in preparation.
[14] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert,* C. R. Acad. Sci. Paris, **294** (1982), 447-450.
[15] C. Lenart, *The* K*-theory of the flag variety and the Fomin-Kirillov quadratic algebra*, J. Algebra, **285** (2005), 120–135.
[16] C. Lenart and A. Yong, *Lecture notes on the* K*-theory of the flag variety and the Fomin-Kirillov quadratic algebra,* `http://www.math.umn.edu/~ayong/`
[17] S. Majid, *Free braided differential calculus, braided binomial theorem, and the braided exponential map,* J. Math. Phys., **34** (1993), 4843-4856.
[18] S. Majid, *Foundations of quantum group theory,* Cambridge University Press, Cambridge, 1995.
[19] A.-L. Maré, *The combinatorial quantum cohomology ring of* G/B, J. Alg. Combin., **21** (2005), 331-349.
[20] A. Milinski and H.-J. Schneider, *Pointed indecomposable Hopf algebras over Coxeter groups,* Contemp. Math. **267** (2000), 215-236.
[21] A. Postnikov, *On a quantum version of Pieri's formula,* Advances in Geometry, (J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan and P. Xu, eds.) Progress in Math., **172** Birkhäuser, 1995, 371-383.
[22] S. L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups),* Commun. Math. Phys., **122** (1989), 125-170.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `kirillov@kurims.kyoto-u.ac.jp`

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `maeno@math.kyoto-u.ac.jp`