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Dual Graded Graphs and Fomin's r-correspondences associated to the Hopf Algebras of Planar Binary Trees, Quasi-symmetric Functions and Noncommutative Symmetric Functions (EXTENDED ABSTRACT)

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Abstract. Fomin (1994) introduced a notion of duality between two graded graphs on the same set of vertices. By a construction similar to the plactic monoid, Hivert, Novelli and Thibon (2001) introduced a monoid structure on the set of binary search trees, the Robinson-Schensted insertion algorithm being replaced by the binary search tree insertion algorithm. Using this monoid they gave a new construction of the algebra of Planar Binary Trees of Loday-Ronco. In this construction, one can build pairs of graded graphs of which we study the duality as in Fomin's setting. We observe that the sylvester congruence defining this algebra is in fact an r-correspondence as defined by Fomin. We also observe graph duality in the algebras of noncommutative symmetric functions, and quasi symmetric functions, and we identify an r-correspondence of two graded graphs built in these algebras, with the hypoplactic congruence introduced by Krob and Thibon (1997). We also present a combinatorial description of the Schensted-Fomin algorithm for dual graded graphs and we use this description to give a proof of a bijection between pairs of paths in any pair of dual graded graphs and permutations of the symmetric group. We conclude with the statement of a possible connection between graded graphs duality and the construction of dual Hopf algebras.

Résumé. Fomin (1994) a introduit une notion de dualité entre graphes gradués. D'autre part, par une construction analogue à celle définissant le monoïde plaxique, Hivert, Novelli et Thibon (2001) introduisent une structure de monoïde sur l'ensemble des arbres binaires de recherche, la congruence plaxique étant remplacée dans cette construction par l'algorithme d'insertion dans un arbre binaire de recherche, encore appelé congruence sylvestre par ces auteurs. Cette construction donne lieu à une nouvelle réalisation de l'algèbre de Hopf des arbres binaires de Loday-Ronco. Dans cette algèbre, il est possible de construire des paires de graphes gradués dont nous étudions la dualité au moyen d'un isomorphisme avec des graphes définis par Fomin. Nous identifions par la suite la congruence sylvestre à une r-correspondance que nous définissons dans ces graphes. La congruence hypoplaxique introduite par Krob et Thibon (1997) est quant à elle identifiée à une r-correspondance définie sur des graphes en dualité dans les algèbres des fonctions symétriques non commutatives et des fonctions quasi-symétriques. Nous donnons aussi une description combinatoire de l'algorithme de Schensted-Fomin pour les graphes gradués en dualité, et nous l'utilisons pour faire une preuve d'une bijection entre paires de chemins dans de tels graphes, et les permutations du groupe symétrique. Compte tenu du mode de construction des graphes étudiés, nous concluons par une possible relation entre graphes gradués en dualité et construction d'algèbres de Hopf duales.

1. Introduction and preliminary definitions

The Young lattice is defined on the set of partitions of integers, with covering relations given by the natural inclusion order. This lattice is associated to the operation of multiplication of Schur functions [**6**] s_λ by s_1, where there is an edge connecting λ and μ if s_μ appears with a nonzero coefficient in the expansion of $s_1 s_\lambda$. The distributive lattice nature of this graph was generalized by S. Fomin (1994) with the introduction of graph duality [**7**]. With this extension he introduced a generalization of the classical Robinson-Schensted algorithm, giving a general scheme for establishing bijective correspondences between pairs of paths in dual

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graded graphs both starting at a vertex of rank 0 and having a common end point of rank n, on the one hand, and permutations of the symmetric group $\mathfrak{S}_n$ on the other hand.

Later, Krob and Thibon (1997), Hivert, Novelli and Thibon (2001) showed that using two congruence relations on words, namely the hypoplactic congruence [**2**] and the binary search tree insertion algorithm [**1, 3**], one can realize as polynomials (commutative or not), two pairs of dual Hopf algebras. The first pair of algebras is the dual pair formed by the algebra of quasi-symmetric functions (*QSym*) and the algebra of noncommutative symmetric functions (**Sym**). The second is made of the algebra of planar binary trees (**PBT**) of Loday-Ronco [**5**] and its dual ($\mathbf{PBT}^*$) which is isomorphic to itself. In those algebras one builds pairs of graded graphs analogue to the Young lattice, and associated to the operations of multiplication. One aim of this paper is first to prove the duality of these pairs of graphs, next to find natural r-correspondences [**7**] associated to those graphs, and last to show that these correspondences convert the Schensted-Fomin algorithm for dual graded graphs into a parallel version of the hypoplactic insertion algorithm and sylvester insertion algorithm respectively.

The paper is organized as follows: we first recall definitions related to graph duality, and then we examine the case of **PBT** in the second section, **Sym** and *QSym* are treated in the next section. In the fourth section we present a combinatorial description of the Schensted-Fomin algorithm for dual graded graphs. We also use this algorithmic description to give a proof of the bijection between pairs of paths in dual graded graphs and permutations of the symmetric group. Last we apply this algorithm to the graphs of the previous sections. Now let us introduce graph duality.

Definition 1.1. A graded graph [**7**] is a triple $G = (P, \rho, E)$ where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function and E is a multi-set of edges (x, y) satisfying $\rho(y) = \rho(x) + 1$.

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common set of vertices and a common rank function.

Definition 1.2. An oriented graded graph [**7**] $G = (P, \rho, E_1, E_2)$ is defined by directing the G_1-edges, E_1 up (in the direction of increasing rank) and the G_2-edges, E_2 down (in the direction of decreasing rank).

Let $G = (P, \rho, E_1, E_2)$ be an oriented graded graph and $\mathbb{K}$ a field of characteristic zero, define $\mathbb{K}P$ as the vector space formed by linear combinations of vertices of P. One can now define two linear operators U (Up) and D (Down) acting on $\mathbb{K}P$ as follows:

$$U\,x \;=\; \sum_{(x,y)\,\in\,E_1} m_1(x,y)\;y \qquad ; \qquad D\,y \;=\; \sum_{(x,y)\,\in\,E_2} m_2(x,y)\;x \tag{1.1}$$

where $m_i(x, y)$ is the multiplicity or the weight of the edge (x, y) in E_i.

Definition 1.3. G_1 and G_2 are said to be dual [**7**] if U and D satisfy the commutation relation:

$$D_{n+1}\,U_n = U_{n-1}\,D_n + I_n \tag{1.2}$$

where U_n (resp. D_n) denote the restriction of the operator U (resp. D) to the n^{th} level of the graph, and I_n the identical operator at the same level.

Generalizations of this definition are also found in [**7**], notably the case of an r-duality with $r > 1$ where the commutation relation generalizes to:

$$D_{n+1}\,U_n = U_{n-1}\,D_n + r\,I_n \quad \text{and} \quad D_{n+1}\,U_n = U_{n-1}\,D_n + r_n\,I_n \tag{1.3}$$

A well-known example of a graded graph is the Young lattice of partitions of integers, which describes the multiplication of Schur functions s_λ by s_1 (Fig. 1). This is a first and natural example of graph duality in relation with the operation of multiplication in two dual Hopf algebras. In fact, the Young lattice is a self-dual graded graph or distributive lattice. Its duality expresses the fact that for any partition λ, there is one more partition obtained by adding a single part to λ than by deleting a single part from λ, and for two partitions λ and μ there are as many partitions simultaneously contained by λ and μ than those simultaneously containing λ and μ. On the other hand, the collection of Schur functions span a self-dual Hopf algebra, that is the algebra of symmetric functions [**6**]. So the self-dual Hopf algebra of symmetric functions is described by the Young lattice which is a self-dual graded graph.

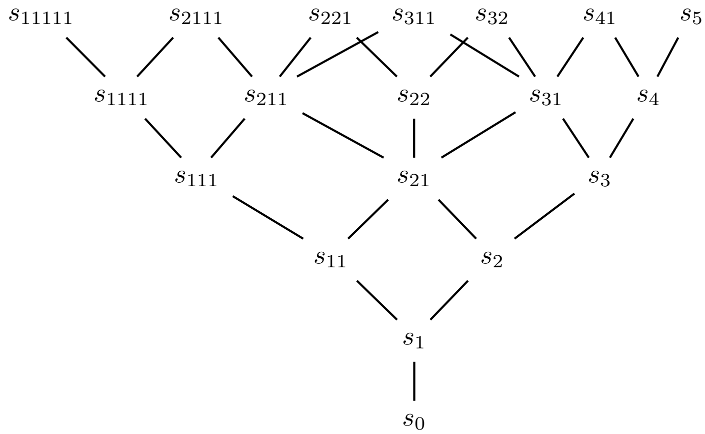


FIGURE 1. The Young graph: multiplication of Schur functions s_λ by s_1

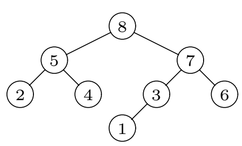
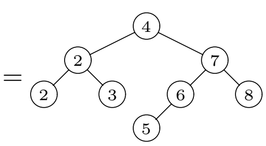
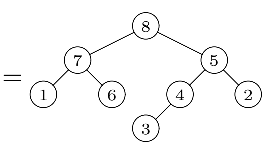
2. Dual graded graphs in PBT

This section is devoted to **PBT**, the Hopf algebra of planar binary trees, for which Loday and Ronco [**5**] gave an explicit embedding as a subalgebra of the convolution algebra of permutations, via the construction of the decreasing tree of a permutation. After recalling the construction of this algebra, we will describe a second example of graph duality in relation with the operation of multiplication in two dual Hopf algebras. In all that follows, we will be considering only words on a totally ordered alphabet, for instance $A = \{1, 2, 3, \cdots\}$.

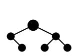
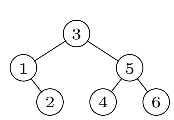
2.1. Definitions.

DEFINITION 2.1. A decreasing tree $\mathcal{T}$ is a labeled binary tree such that the label of each internal node is greater than the labels of all the nodes in its subtrees.

Let w be a word with no repetition of letters. Its *decreasing tree* $\mathcal{T}(w)$ is obtained as follows: its root is labeled with the greatest letter n of w, and if $w = u\,n\,v$, where u and v are words with no repetition of letters, then the left subtree of $\mathcal{T}(w)$ is $\mathcal{T}(u)$ and its right subtree is $\mathcal{T}(v)$. Another tree associated to a word w is its right strict binary search tree, this is a labeled binary tree labeled with w's letters such that for each internal node, its label is greater or equal to the labels of the nodes in its left subtree and strictly smaller than the labels of the nodes in it's right subtree. The binary search tree associated to a word w will be denoted $\mathcal{P}(w)$. It is obtained by applying the well-known binary search tree insertion algorithm [**1**] to w, but *reading w from right to left*. During this insertion process one can use a second tree denoted $\mathcal{Q}(w)$ to record the positions in w of the letters inserted at each step. $\mathcal{Q}(w)$ coincides with $\mathcal{T}(std(w)^{-1})$ where $std(w)$ is the standardized word of w. The user not familiar with the standardization process may consult [**3**] for definitions. For example let us consider the two words $w_1 = 25481376$ and $w_2 = 28567324$, then we have:

$\mathcal{T}(w_1) =$  ; $\mathcal{P}(w_2) =$  ; $\mathcal{Q}(w_2) =$ 

The map $w \mapsto (\mathcal{P}(w), \mathcal{Q}(w))$ is known as the *sylvester correspondence* and is associated to a congruence, the sylvester congruence, defined on words on the alphabet A by: $u \equiv_{syv} v \Leftrightarrow \mathcal{P}(u) = \mathcal{P}(v)$. See [**3**] for a plactic-like characterization on words. The *sylvester canonical permutation* associated to an unlabeled binary tree T is the right-to-left postfix reading of the only binary search tree that is the left-to-right infix labeling of T. The sylvester canonical permutation of a permutation σ is the right-to-left postfix reading of $\mathcal{P}(\sigma)$. For example let us consider:

$T =$  ; the labeling is  and the sylvester canonical permutation is $\sigma_T = 645213$.

645213 is also the sylvester canonical permutation of the permutation 465213. Now let us recall the definition of the algebra of free quasi-symmetric functions. This definition is needed to introduce **PBT**.

DEFINITION 2.2. Let σ be a permutation. The *Free Quasi-Ribbon* $\mathbb{F}_\sigma$ is the noncommutative polynomial

$$\mathbb{F}_\sigma \quad = \sum_{w\,:\,std\,(w)\,=\,\sigma^{-1}} w$$

where $std(w)$ denotes the standardized word of w, and w runs over the words on the alphabet A. The free quasi-ribbons span a subalgebra of the free associative algebra. This subalgebra is the algebra of free quasi-symmetric functions (**FQSym**), and its multiplication rule is the following:

$$\mathbb{F}_\alpha\,\mathbb{F}_\beta = \sum_{\sigma\,\in\,(\,\alpha\,\sqcup\!\sqcup\,\beta\,[|\alpha|]\,)} \mathbb{F}_\sigma$$

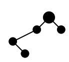
where $\alpha \sqcup\!\sqcup \beta\,[|\alpha|]$ is the shifted shuffle of the two permutations α and β. The user not familiar with the standardization process and shuffles may consult [**3**] for definitions. The dual basis of the $\mathbb{F}_\sigma$ are the $\mathbb{G}_\sigma$ defined by:

$$\mathbb{G}_\sigma \quad = \quad \mathbb{F}_{\sigma^{-1}} \quad = \sum_{w\,:\,std\,(w)\,=\,\sigma} w$$

An embedding of **PBT** in **FQSym** is given as the linear span of the $(\mathbf{P}_T)$ defined [**3**] by:

$$\mathbf{P}_T \quad = \sum_{w\,:\,shape\,(\mathcal{T}(std(w)))\,=\,T} w \quad = \sum_{\sigma\,:\,shape\,(\mathcal{P}(\sigma))\,=\,T} \mathbb{F}_\sigma \tag{2.1}$$

where T is an unlabeled binary tree, σ a permutation, the shape of a labeled tree being the corresponding unlabeled tree. For example:

$\mathbf{P}$  $\quad = \quad \mathbf{P}_{52134} \quad = \quad \mathbb{F}_{21354} + \mathbb{F}_{21534} + \mathbb{F}_{25134} + \mathbb{F}_{52134}$

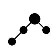
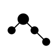
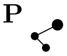
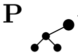
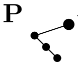
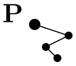
The multiplication rule in **PBT** is given by:

$$\mathbf{P}_{T_1}\,\mathbf{P}_{T_2} \quad = \sum_{T\,\in\,\text{shuffle}\,(T_1,T_2)} \mathbf{P}_T \tag{2.2}$$

where shuffle(T_1, T_2) is the set of unlabeled binary trees whose canonical sylvester permutations appear in $\sigma_1 \sqcup\!\sqcup \sigma_2\,[|\sigma_1|]$, σ_i being the canonical sylvester permutations associated to T_i. For example:

$$12 \sqcup\!\sqcup 21[2] \;=\; 12 \sqcup\!\sqcup 43 \;=\; (1243 + 1423 + 4123) \quad + \quad (1432 + 4132 + 4312)$$

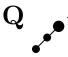
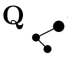
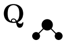
so we will have:

$\mathbf{P}$  $\mathbf{P}$  $=$ $\mathbf{P}$  $+$ $\mathbf{P}$  ; and one can also check that $\mathbf{P}_\bullet$ $\mathbf{P}$  $=$ $\mathbf{P}$  $+\,\mathbf{P}$  $+\,\mathbf{P}$ 

The dual basis of the $(\mathbf{P}_T)$ are the $(\mathbf{Q}_T)$ defined by $\mathbf{Q}_T = \pi\,(\mathbf{G}_{\sigma_T})$ where $\pi \;:\; \mathbb{C}\langle A\rangle \longrightarrow \mathbb{C}\langle A\rangle/\equiv_{sylv}$ is the canonical projection sending a sum of permutations to the sum of the corresponding sylvester canonical permutations. The multiplication rule in $\mathbf{PBT}^*$ is given by:

$$\mathbf{Q}_{T_1}\,\mathbf{Q}_{T_2} \quad = \sum_{T\,\in\,\text{Conv}\,(T_1,T_2)} \mathbf{Q}_T$$

where Conv$\,(T_1, T_2)$ is defined as follows: let σ_i be the canonical sylvester permutation associated to T_i, then Conv$\,(T_1, T_2)$ is the set of unlabeled binary trees whose canonical sylvester permutations appear in the convolution product $\mathbf{G}_{\sigma_1}\mathbf{G}_{\sigma_2}$. For example,

$\mathbb{G}_1\mathbb{G}_{12} = \mathbb{G}_{123} + \mathbb{G}_{213} + \mathbb{G}_{312}$, so one will have $\mathbf{Q}_\bullet\mathbf{Q}$  $=$ $\mathbf{Q}$  $+\,\mathbf{Q}$  $+\,\mathbf{Q}$ 

Using the multiplication rules in **PBT** and $\mathbf{PBT}^*$, it is possible to build a pair of graded graphs (Fig. 2 and Fig. 3) whose set of vertices of degree n are the binary trees of size n. In those graphs, there is an edge between T and T' if T' appears in the product $\mathbf{P}_\bullet\,\mathbf{P}_T$ (resp. $\mathbf{Q}_\bullet\,\mathbf{Q}_T$), where $\bullet$ is the tree of size 1. All edges are weighted 1 since there are no multiplicities in the products in **PBT** and $\mathbf{PBT}^*$. A second pair of graphs describes the right multiplication by $\mathbf{P}_\bullet$ and $\mathbf{Q}_\bullet$ respectively. See [**3**] for related figures.

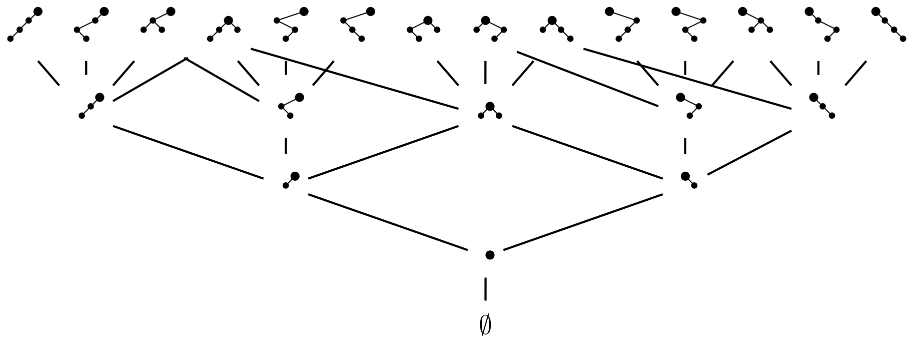


FIGURE 2. $\Gamma^{left}_{Q_\bullet}$: left multiplication by $\mathbf{Q}_\bullet$ in **PBT**

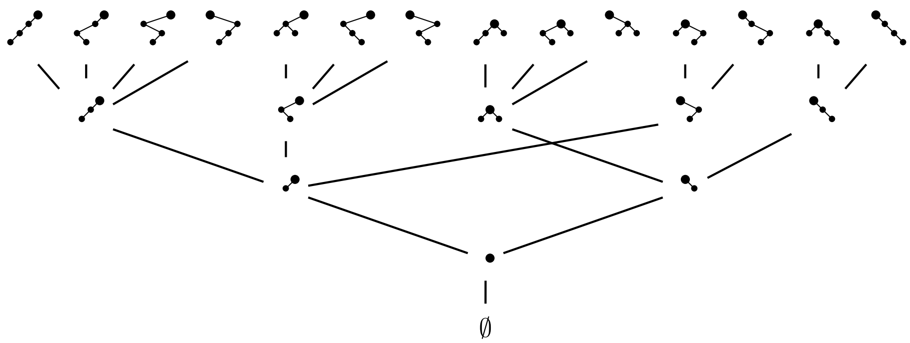


FIGURE 3. $\Gamma^{left}_{P_\bullet}$: left multiplication by $\mathbf{P}_\bullet$ in **PBT**

2.2. Graph's duality.

It was already stated in [**3**], but without proof, that the graphs $\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$ above could be in duality. We prove this using the fact that they are isomorphic to two dual graded graphs studied by Fomin [**7**] and known as the lattice of binary trees and the bracket tree (Fig. 4 and Fig. 5). The lattice of binary trees is defined as follows: it's vertices of rank n are the syntactically correct formulae defining different versions of calculation of a non-associative product of $n+1$ entries. So any vertex of rank n is a valid sequence of $n-1$ opening and $n-1$ closing brackets inserted into $x_1 . x_2 \cdots x_n$. In the bracket tree, two vertices are linked if one results from the other by deleting the first entry, and then removing subsequent unnecessary brackets, and renumbering the new expression.

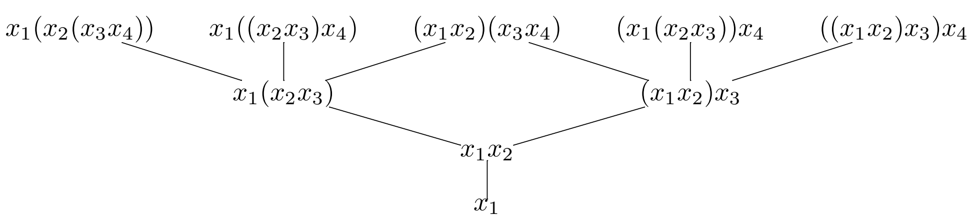


FIGURE 4. The lattice of binary trees

REMARK 2.3. There is a one-to-one correspondence between unlabeled binary trees and bracketed expressions. In this correspondence, an unlabeled binary tree is identified with the expression obtained by completing the tree, adding one leaf to any node having a single child-node, and two leaves to any childless node. Then label the leaves of the resulting *complete* unlabeled binary tree according to the left-to-right infix order. If the final tree is empty then the expression is x_1, or else the expression is obtained by recursively reading its left and right subtrees in that order.

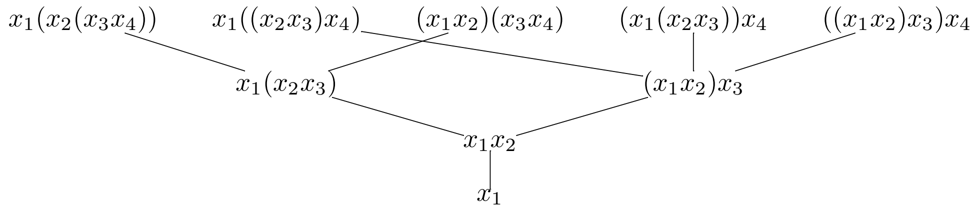
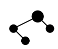
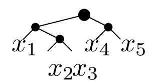


Figure 5. The bracket tree, dual of the lattice of binary trees

An example to illustrate the correspondence described in remark 2.3 is the following:

$T =$  ; the labeling is 

and the expression: $(x_1(x_2x_3))(x_4x_5)$

Proposition 2.1. *$\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$ are respectively isomorphic to the lattice of binary trees and it's dual, the bracket tree.*

The proof of **Proposition 2.1** is made using a combinatorial description of the covering relations in the lattice of binary trees and in the bracket tree, identifying each bracketed expression with an unlabeled binary tree as described in **Remark 2.3**. These relations are:

(1) In the lattice of binary trees, a tree T is covered by the set of trees obtained from it by addition of a single node, in all possible ways.
(2) In the bracket tree, a tree T' covers a single tree T obtained from T' by deleting it's left-most node if any, or its root otherwise, and replacing the deleted node by its own right subtree if any.

It can then be shown that performing $\mathbf{Q}_\bullet \mathbf{Q}_T$ and $\mathbf{P}_\bullet \mathbf{P}_T$ respectively corresponds exactly to applying the above operations to T. Hence from Fomin's statement that the lattice of binary trees is dual to the bracket tree, we have:

Corollary 2.1. *$\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$ are dual as defined by Fomin.*

3. Dual graded graphs in Sym and *QSym*

In the same way as in **PBT**, one can build two graded graphs in the algebras of noncommutative symmetric functions (**Sym**), and of quasi symmetric functions (*QSym*). They are associated to the operation of multiplication of ribbon Schur functions (resp., of quasi-ribbon functions) by R_1 (resp., F_1), see [**2**] for definitions. This gives us a third example of graph duality arising from multiplications in two dual Hopf algebras. The graphs are illustrated below (Fig. 6 and Fig. 7).

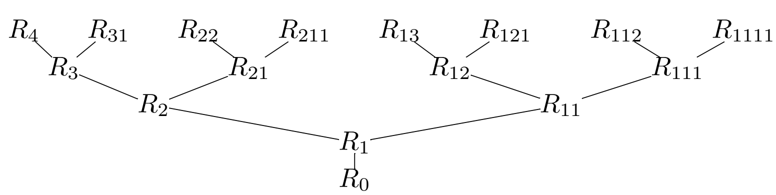


Figure 6. $\Gamma^{right}_{R_1}$: right multiplication by R_1 in **Sym**

Investigating the duality of the two graphs defined above, we found that they are isomorphic to two dual graded graphs studied by Fomin [**7**] and known as the lifted binary tree and Binword (Fig. 8 and Fig. 9). Their vertices are words on the alphabet $\{0, 1\}$. In the first graph, a word w is covered by the two words $w.0$ and $w.1$ (where . denotes the usual concatenation of words), except 0 which is only covered by 1. In Binword, there exists an edge from u to v if u is obtained by deleting a single letter (but not the first) from v, and in addition, there is an edge from 0 to 1.

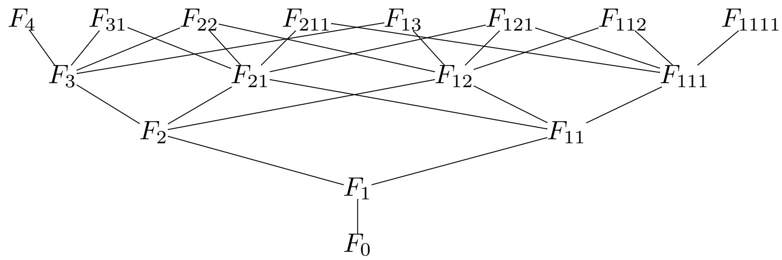


FIGURE 7. Γ_{F_1} : multiplication by F_1 in *QSym*

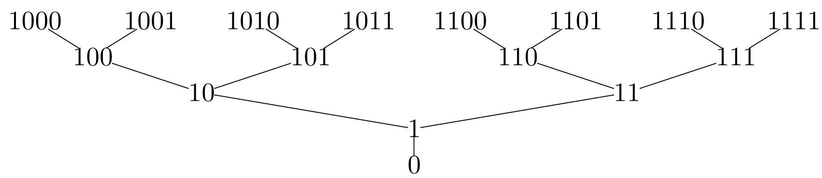


FIGURE 8. The lifted binary tree

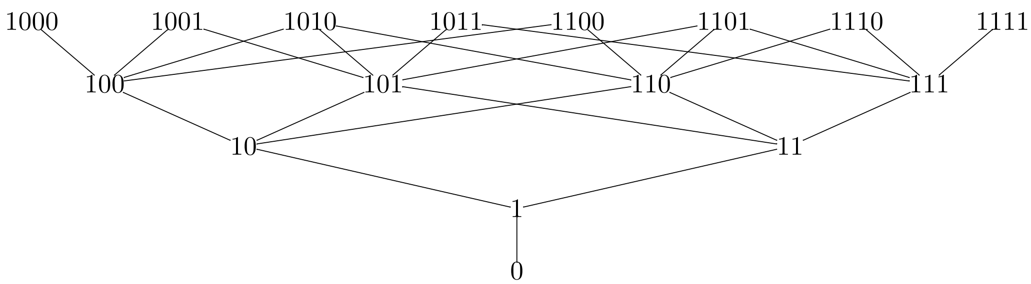
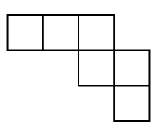
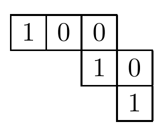


FIGURE 9. Binword, a dual of the lifted binary tree

REMARK 3.1. There is a one-to-one correspondence between compositions of an integer n and the vertices of rank n in the lifted binary tree. A composition I is identified with the word w_I obtained by filling its ribbon diagram from left to right and from top to bottom, with 1 in the first box and in any box following a descent, 0 elsewhere.

$I = (3,2,1) \quad =$  $\quad ; \quad w_I \quad =$ 

$\quad = \quad 100101$

PROPOSITION 3.1. *$\Gamma^{right}_{R_1}$ and Γ_{F_1} are isomorphic to the lifted binary tree and it's dual binword, respectively.*

COROLLARY 3.1. *$\Gamma^{right}_{R_1}$ and Γ_{F_1} are dual as defined by Fomin.*

4. Schensted-Fomin algorithm for dual graded graphs

In all that follows and unless otherwise stated, G_1 and G_2 will denote two graded graphs in r-duality, with a zero denoted $\widehat{0}$.

4.1. r-correspondences.

As introduced in [**8**], r-correspondences are bijective realizations of equation (1.2) and its generalizations (1.3). Let ϕ be a bijective map associating pairs (b_1, b_2) to triples (a_1, a_2, α) where a_1 and b_1 are edges in

G_1, a_2 and b_2 are edges in G_2 such that $start\,(a_1) = start\,(a_2)$, $end\,(b_1) = end\,(b_2)$, and $\alpha \in \{0, 1, ..., r\}$. The map ϕ is said to be an r-correspondence if the following conditions are satisfied:

(i) if $\phi(b_1, b_2) = (a_1, a_2, \alpha)$ then $end(a_1) = start(b_2)$ and $end(a_2) = start(b_1)$
(ii) if b_1 and b_2 are degenerated ($b_1 = b_2 = (x_0, x_0)$) then $\phi(b_1, b_2) = (b_1, b_2, 0)$

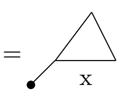
An important lemma is the following:

Lemma 4.1. *There exists an r-correspondence between two graded graphs G_1 and G_2 if and only if G_1 and G_2 are in r-duality* [**7**].

One goal of Fomin's construction is to use r-correspondences to establish bijective maps between pairs of paths in G_1 and G_2 both starting at $\widehat{0}$ and having a common end point of rank n, and permutations of the symmetric group $\mathfrak{S}_n$. In this section we define two natural r-correspondences associated to the pairs of dual graded graphs of section 2 and 3. Using the Schensted-Fomin algorithm for dual graded graphs, we will later see that these r-correspondences are parallel versions of the hypoplactic and sylvester insertion algorithms. A combinatorial description of this algorithm is given in section 4.2.

4.1.1. *A natural r-correspondence in **PBT**'s graphs.*
The following is an algorithm to find $a_1 = (t, x)$, $a_2 = (t, y)$ and α, when $b_1 = (y, z)$ and $b_2 = (x, z)$ are given, $r = 1$ in this case.

Function *getAr*:

Inputs: x, y, z;
Outputs: t, α;
Begin
  If $x = z$ **then** $t = y$ **and** $\alpha = 0$
  Else if $y = z$ **then** $t = x$ **and** $\alpha = 0$
  Else if $x \neq y$ **then**
    $t =$ (y without its left-most node, replaced by its own right subtree if any) **and** ($\alpha = 0$) ; (1)
  Else if $z = x' :=$ 

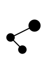
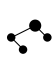
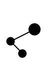
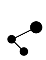
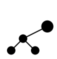
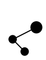
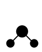
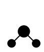
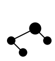
 (that is x + one node added to the left of its left-most node) **then** (2)
    ($t = x$) **and** ($\alpha = 1$) ; (3)
  Else
    $t =$ (y without its left-most node) **and** ($\alpha = 0$) ; (4)
  End_if
End.

(1): for this correspondence to be well defined, one should prove that t is covered by x in $\Gamma^{left}_{Q_\bullet}$. Indeed, (b_1, b_2) is in this case a DU-path from y to x and it is the only DU-path from y to x since $\Gamma^{left}_{P_\bullet}$ is a tree. And since $\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$ are dual graphs (**corollary 2.1**), there is a single UD-path from y to x, necessary having t as middle point.
(2): this serves to define x'.
(4): (b_1, b_2) is the 2^{nd} DU-loop of the form (x, x); it will match the unique UD-loop (x, x). The first is processed in (3).

Of course this algorithm is invertible.

Proposition 4.1. *The previous algorithm defines an r-correspondence in $\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$.*

Now a few examples to illustrate this correspondence.

	y	x	z	t	α	**comments**
1.					0	($x \neq y$) $\Rightarrow$ ($t = y$ without its left-most node)
2.					1	($x = y$ and $z = x'$) $\Rightarrow$ ($t = x$)
3.					0	($x = y$ and $z \neq x'$) $\Rightarrow$ ($t = y$ without its left-most node)

We can do the same for the two graphs $\Gamma^{right}_{R_1}$ and Γ_{F_1} that we have defined in **Sym** and *QSym*. In this case, we observe that the natural choice of an r-correspondence in $\Gamma^{right}_{R_1}$ and Γ_{F_1} converts the Schensted-Fomin algorithm for dual graded graphs (applied to those graphs) into a parallel version of the hypoplactic insertion algorithm. The construction is the following.

4.1.2. *A natural r-correspondence in* ***Sym*** *'s and QSym 's graphs.*
The following algorithm is an adaptation of the one described in [**8**] for the lattice of binary trees and the bracket tree, using our **Proposition 3.1**. Of course this algorithm is invertible.

Function *getAr*:

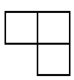
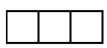
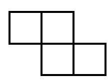
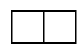
Inputs: x, y, z ;
Outputs: t, α ;
Begin
    If $x = z$ **then** $t = y$ **and** $\alpha = 0$
    Else if $y = z$ **then** $t = x$ **and** $\alpha = 0$
    Else if $(x \neq y)$ or (the last box of z does not follow a descent) **then** (1)
        $t =$ (x without its last box) **and** $(\alpha = 0)$;
    Else /* $x = y$ *and the last box of* z *follows a descent* */ (2)
        $(t = x)$ **and** $(\alpha = 1)$;
    End_if
End.

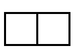
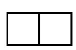
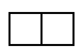
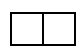
(1): that is w_z ends with 0.
(2): that is w_z ends with 1. w_z is defined in **Proposition 3.1**.

Proposition 4.2. *The previous algorithm defines an r-correspondence in* $\Gamma^{right}_{R_1}$ *and* Γ_{F_1}.

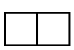
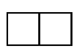
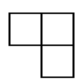
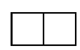
Now a few examples to illustrate this correspondence.

(i): Two cases where $(x \neq y)$ or (the last box of z does not follow a descent)

$y =$  $= 21$; $x =$  $= 3$; $z =$  $= 22$; $t =$  $= 2$; $\alpha = 0$

$y =$  $= 2$; $x =$  $= 2$; $z =$  $= 2$; $t =$  $= 2$; $\alpha = 0$

(ii): A case where $(x = y)$ and (the last box of z follows a descent)

$y =$  $= 2$; $x =$  $= 2$; $z =$  $= 21$; $t =$  $= 2$; $\alpha = 1$

4.2. Schensted-Fomin algorithm for dual graded graphs.

This algorithm was introduced in [**8**]. Given an r-correspondence ϕ, the algorithm establishes a bijective correspondence between pairs of paths in G_1 and G_2 starting at $\widehat{0}$ and having a common end point of rank n, and permutations of $\mathfrak{S}_n$. *We describe a combinatorial version of this algorithm and apply it to the two pairs of graded graphs whose duality has been studied in the previous sections. We also use this description to give a simpler proof of a bijection between permutations and pairs of paths in dual graded graphs, for the case* $r = 1$. The two paths (inputs) are given as two sequences of vertices $v = (v_0, v_1, \cdots, v_n)$ and $w = (w_0, w_1, \cdots, w_n = v_n)$. We'll be using a double entry matrix $M_{\phi,\sigma}$ initialized with v on its last column and w on its last line. In this matrix, lines and columns of odd indices will form an $(n+1) \times (n+1)$ matrix M_ϕ representing a correspondence table for ϕ, while those of even indices will form an $n \times n$ matrix M_σ representing the permutation σ generated from the two paths v and w.

Given (k, l), evaluating $\sigma(k, l)$ requires the values of $M_\phi(k, l-1)$ and $M_\phi(k-1, l)$, and will inform us of the value of $M_\phi(k-1, l-1)$. To do this, we set $z = M_\phi(k, l)$, $x = M_\phi(k, l-1)$, $y = M_\phi(k-1, l)$ and define t to be equal to $M_\phi(k-1, l-1)$. So x, y, and z are known values, and t is being searched for. Setting $b_1 = (y, z)$ and $b_2 = (x, z)$, one can evaluate $\phi(b_1, b_2)$ which can be expressed as $[a_1 := (t, x), a_2 := (t, y), \alpha]$. Now t is known and it only remains to set $\sigma(k, l) = \alpha$ and we are done. This is illustrated below:

$$M_{\phi,\sigma} \quad = \quad \begin{pmatrix} & & & & & & \ddots & \vdots \\ & & & \mathbf{t = ?} & & \mathbf{y} & \cdots & v_k \\ & & & & \sigma_{k,l} = \mathbf{?} & & \ddots & \\ & & & \mathbf{x} & & \mathbf{z} & \cdots & v_{k+1} \\ & & & \vdots & & \vdots & \ddots & \vdots \\ w_0 & w_1 & \cdots & w_l & & w_{l+1} & \cdots & v_n \end{pmatrix}$$

M_ϕ (M in short) and σ will be filled from their $(n,n)^{th}$ component to the $(0,0)^{th}$ for M_ϕ, or $(1,1)^{th}$ for σ, following diagonals:

Step 1: use $x = w_{n-1}$, $y = v_{n-1}$ and $z = v_n = w_n$ to compute $M_{n-1,n-1} = t$ and $\sigma_{n,n} = \alpha$, satisfying $(a_1, a_2, \alpha) = \phi(b_1, b_2)$ with $b_1 = (y,z)$, $b_2 = (x,z)$, $a_1 = (t,x)$ and $a_2 = (t,y)$.

$$M_{\phi,\sigma} \quad = \quad \begin{pmatrix} & & & & & & v_0 \\ & & & & & & v_1 \\ & & & & & & v_2 \\ & & & & & & \vdots \\ & & & & & & v_{n-2} \\ & & & & M_{n-1,\,n-1} & & v_{n-1} \\ & & & & & \sigma_{n,n} & \\ w_0 & w_1 & \cdots & w_{n-2} & w_{n-1} & & v_n \end{pmatrix}$$

Step 2: $M_{n-1,\,n-2}$, $\sigma_{n,n-1}$, $M_{n-2,\,n-1}$ and $\sigma_{n-1,n}$ are computed.

Step 2n-1: M_ϕ and σ are completely filled. The number $2n-1$ of steps is determined by the total number of diagonals in M_ϕ which is an $(n+1) \times (n+1)$ matrix.

Below is a computer implementable description of the algorithm:

Function *permutation_from_paths*:

Inputs: v, w, ϕ
Outputs: σ
Temporary variables: $M_\phi, b_1, b_2, a_1, a_2, \alpha, L, L_0, k, l$;
Begin
  $n = \text{length}(v) - 1$; /* *length(v): number of consecutive points defining v, that is 1 more than the number of edges in the path represented by v* */
  For all k **from** 0 **to** n **do**
    $M_\phi(k,n) = v(k)$ **and** $M_\phi(n,k) = w(k)$;
  End_loop.
  $KL = \{ (n,n) \}$; /* *first pair of indices to process* */
  While $L \neq \{\}$ **do**
    $L_0 = \{\}$; /* *the empty set* */
    For all (k,l) in L **do**
      $x = M_\phi(k, l-1)$ **and** $y = M_\phi(k-1, l)$ **and** $z = M_\phi(k,l)$;
      $b_1 = (y,z)$ **and** $b_2 = (x,z)$;
      $(a_1, a_2, \alpha) = \phi(b_1, b_2)$;
      $M_\phi(k-1, l-1) = t$; /* *common origin to the edges* a_1 *and* a_2 */
      $\sigma(k,l) = \alpha$;
      If $k > 1$ **then** $L_0 = L_0 \cup \{ (k-1, l) \}$;
      If $l > 1$ **then** $L_0 = L_0 \cup \{ (k, l-1) \}$;
    End_loop.
    $L = L_0$; /* *next pairs of indices to process* */
  End_loop.
End.

It seems not obvious from this description that M_σ is indeed a permutation matrix. Below is a simple proof in the case of a simple duality ($r=1$), together with the proof of the bijection between permutations and pairs of paths. The reasoning remains valid only for $r=1$.

Proof. Let M'_ϕ stands for M_ϕ where each vertex is replaced by its rank in the graph. One first observes that M'_ϕ is an $(n+1)\times(n+1)$ matrix satisfying the following conditions: the first line and the first column are initialized with 0 while the last ones are initialized with integer entries increasing from 0 to n ; entries increase at most by 1 on lines and columns ; and for any 2×2 sub-matrix the difference of the sums on the first and second diagonals is 0 or 1. This is formally equivalent to:

$$\left\{\begin{array}{ll} m(i,n)=i\ ;\ m(n,j)=j & i,j=0,\cdots,n \\ m(i,j+1)-m(i,j)\ \in\ \{0,1\} & \\ m(i+1,j)-m(i,j)\ \in\ \{0,1\} & \\ m(i+1,j+1)+m(i,j)-m(i+1,j)-m(i,j+1)\ \in\ \{0,1\} & \end{array}\right. \tag{4.1}$$

Next one uses the definition and properties of ϕ to see that:

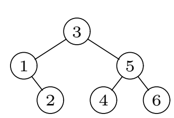
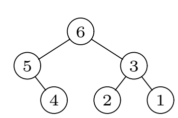
$$M_\sigma(i,j)=M'_\phi(i+1,j+1)+M'_\phi(i,j)-M'_\phi(i+1,j)-M'_\phi(i,j+1) \tag{4.2}$$

Finally, one establishes a bijective map between matrices satisfying (4.1) on the one hand, and permutations of the symmetric group $\mathfrak{S}_n$ on the other hand, using (4.2) to determine the permutation matrix associated to any matrix satisfying (4.1). So the described algorithm (*permutation_from_paths*) sends a pair of paths to a permutation.

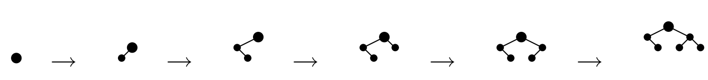
As for the proof of the bijection between permutations and pairs of paths, first notice that the above algorithm is naturally invertible. Given a permutation σ, initialize the first line and the first column of $M_{\phi,\sigma}$ with the common zero of the two graphs. Then fill the permutation matrix M_σ with 1's and 0's, and starting from the upper left corner, fill M_ϕ using ϕ^{-1}. Hence establishing a bijective correspondence between pairs of paths in G_1 and G_2 starting at $\widehat{0}$ and having a common end point of rank n, on the one hand, and permutations of $\mathfrak{S}_n$, on the other hand. □

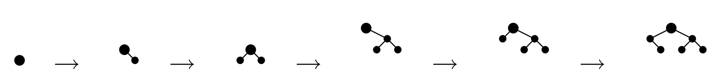
Now let us apply this algorithm to the graphs and r-correspondences we studied in the previous sections.

4.2.1. *Schensted-Fomin algorithm applied to **PBT**'s graphs.*
We identify any permutation $\alpha\in\mathfrak{S}_n$ with two *natural* paths v_α and w_α, the first in $\Gamma^{left}_{Q_\bullet}$ and the second in $\Gamma^{left}_{P_\bullet}$. These two paths are both paths from the empty tree $\emptyset$ to $T_\alpha=\mathrm{shape}(\mathcal{P}(\alpha))$. The path in $\Gamma^{left}_{Q_\bullet}$ is the sequence of shapes of partial binary search trees corresponding to inserting the last k letters of α, for $k=0..n$. As for the path in $\Gamma^{left}_{P_\bullet}$, the shapes defining it correspond to selecting in α only letters greater than k, for $k=n..0$. For example, let us consider $\alpha=645213$, applying the sylvester correspondence to α leads to:

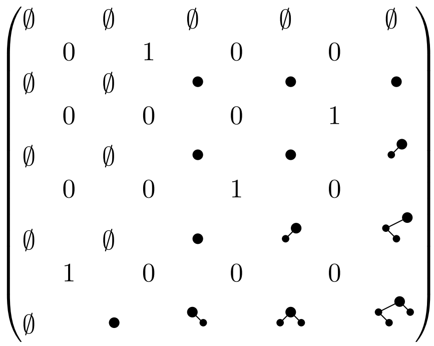
$\mathcal{P}(\alpha)$ =  ; $\mathcal{Q}(\alpha)$ = $\mathcal{T}(\alpha^{-1})$ = 

The two natural paths are then:

v_α = $\emptyset$ → 

w_α = $\emptyset$ → 

Now let us choose a smaller example to which we will apply the Schensted-Fomin algorithm for dual graded graphs, say $\gamma = 4213$. Below are $M_{\phi,\sigma}$; γ and σ:



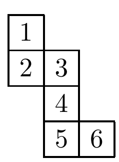
$$; \begin{pmatrix} 0 & 0 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} & 0 \\ 0 & \mathbf{0} & 0 & 1 \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 1 \\ 0 & \mathbf{0} & 1 & 0 \\ \mathbf{1} & 0 & 0 & 0 \end{pmatrix} $$

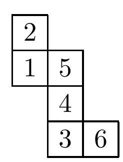
$\gamma = 4213$ and $\sigma = 4132$, so the permutation produced by the Schensted-Fomin algorithm in **PBT** using our natural r-correspondence (see **4.1.1**), differs from the initial permutation. But one observation can be made on the relation between the two permutations: *one is obtained from the other by reflexion on the second diagonal*. It is known [**8**] that a natural choice of an r-correspondence in the Young lattice converts the Schensted-Fomin algorithm for dual graded graphs into a parallel version of the Robinson-Schensted algorithm.

PROPOSITION 4.3. *Our natural choice of r-correspondence in $\Gamma^{left}_{Q_\bullet}$ and $\Gamma^{left}_{P_\bullet}$ converts the Schensted-Fomin algorithm for dual graded graphs into a parallel version of the sylvester insertion algorithm.*

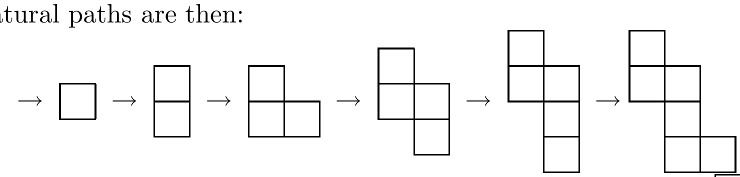
4.2.2. *Schensted-Fomin algorithm applied to **Sym**'s and QSym's graphs.*

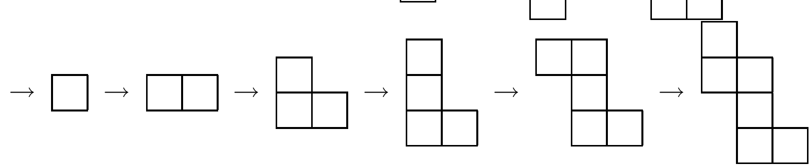
We identify any permutation $\alpha \in \mathfrak{S}_n$ with two *natural* paths v_α and w_α, the first in $\Gamma^{right}_{R_1}$ and the second in Γ_{F_1}. These two paths are both paths from the empty composition $\emptyset$ to the recoil composition of α. The paths in $\Gamma^{right}_{R_1}$ is the sequence of recoil compositions of restrictions of α to $[1..k]$, for $k = 0\,..\,n$. As for the path in Γ_{F_1}, it is made of descent compositions of restrictions of α^{-1} to $[k..n]$, for $k = (n+1)\,..\,1$. For example, consider $\alpha = 215436$, applying the hypoplactic insertion algorithm [**4**] leads to the following quasi-ribbon and ribbon diagrams:

$Q_r(\alpha) =$ 

; $R(\alpha) = Q_r(\alpha^{-1}) =$ 

The two natural paths are then:

$v_\alpha = \emptyset \rightarrow$ 

$w_\alpha = \emptyset \rightarrow$ 

Now let us choose a smaller example to which we will apply the Schensted-Fomin algorithm for dual graded graphs, say $\gamma = 1243$. Below are $M_{\phi,\sigma}$; γ and σ:

$$\left(\begin{array}{ccccccccc}
\emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset \\
 & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{1} & \\
\emptyset & & \emptyset & & \emptyset & & \emptyset & & 1 \\
 & \mathbf{0} & & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & \\
\emptyset & & \emptyset & & \emptyset & & 1 & & 2 \\
 & \mathbf{1} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & \\
\emptyset & & 1 & & 1 & & 2 & & 3 \\
 & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & & \mathbf{0} & \\
\emptyset & & 1 & & 11 & & 21 & & 31
\end{array}\right) \; ; \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; ; \; \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\gamma = 1243$ and $\sigma = 3421$, so the permutation produced by the Schensted-Fomin algorithm in **Sym** and *QSym* using the adaptation (see **4.1.2**) of Fomin's natural r-correspondence, differs from the initial permutation. Once more, an observation can be made on the relation between them: *one is obtained from the other by reflexion in a central vertical line.*

Proposition 4.4. *The natural choice of an r-correspondence in $\Gamma_{R_1}^{right}$ and Γ_{F_1} converts the Schensted-Fomin algorithm for dual graded graphs into a parallel version of the hypoplactic insertion algorithm.*

5. Conclusion

As suggested by the three examples studied in this paper, which are not isolated cases since numerous other examples can be found in some other algebras, there seems to be a strong connection between dual graded graphs and the construction of some dual Hopf algebras. For more examples, in the Hopf algebra of free quasi-symmetric functions (**FQSym**) we've consider the two pairs of graded graphs describing the operations $\mathbb{F}_1\mathbb{F}_\sigma$ and $\mathbb{G}_\sigma\mathbb{G}_1$ for the first pair, $\mathbb{F}_\sigma\mathbb{F}_1$ and $\mathbb{G}_\sigma\mathbb{G}_1$ for the second pair. From explicit computations on finite realizations of those graphs, we believe that they are also examples of graph duality arising from multiplication in dual Hopf algebras. Finally, another interesting example is the algebra of free symmetric functions denoted **FSym**, providing a realization of the algebra of tableaux introduced by Poirier and Reutenauer [**9**] as a subalgebra of the free associative algebra. In this case, the Hopf algebra duality may be identified with the duality of two graphs the duality of which is established in [**7**]: the Schensted graph and the SYT-Tree. Their vertices are Young tableaux.

So dual graded graphs could be viewed as the description of the multiplication rules for products of basis elements by the ones of rank 1, in two dual Hopf algebras constructed by means of a congruence relation on words. The congruence itself could be obtained by the Schensted-Fomin algorithm using a certain r-correspondence in those graphs. This is clearly observed in the construction of the algebra of symmetric functions (*Sym*), the algebra of noncommutative symmetric functions (**Sym**), the algebra of quasi-symmetric functions (*QSym*) and the algebra of planar binary trees (**PBT**), using the plactic, hypoplactic and syslvester congruences respectively.

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