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Restricted Patience Sorting and Barred Pattern Avoidance

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Abstract. Patience Sorting is a combinatorial algorithm that can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm. In recent work the authors have shown that Patience Sorting provides an algorithmic description for permutations avoiding the barred (generalized) permutation pattern $3{-}\bar{1}{-}42$. Motivated by this and a recently formulated geometric form for Patience Sorting in terms of certain intersecting lattice paths, we study the related themes of restricted input and avoidance of similar barred permutation patterns. One such result is to characterize those permutations for which Patience Sorting is an invertible algorithm as the set of permutations simultaneously avoiding the barred patterns $3{-}\bar{1}{-}42$ and $3{-}\bar{1}{-}24$. We then enumerate this avoidance set, which involves convolved Fibonacci numbers.

Résumé. *Patience Sorting* est un algorithme combinatoire que l'on peut comprendre comme étant une version itérée, non-récursive de la correspondence de Schensted. Dans leur travail récent les auteurs ont démontré que *Patience Sorting* donne une description algorithmique des permutations évitant le motif barré (généralisé) $3{-}\bar{1}{-}42$. Motivés par ceci et par une forme récemment formulée de *Patience Sorting* en termes de certaine parcours du treillis intersectants, nous étudions les thèmes connexe d'input restreinte et permutations qui évitent de similaire motifs barrés. Un de nos résultats est de caractériser les permutations pour lesquelles *Patience Sorting* est un algorithme inversible comme étant l'ensemble des permutations évitant simultanément les motifs barrés $3{-}\bar{1}{-}42$ and $3{-}\bar{1}{-}24$. Nous énumérons ensuite cet ensemble, qui utilise des convolutions des nombres de Fibonacci.

1. Introduction

The term *Patience Sorting* was introduced in 1962 by C. L. Mallows [**12, 13**] while studying a card sorting algorithm invented by A. S. C. Ross. Given a shuffled deck of cards $\sigma = c_1 c_2 \cdots c_n$ (which we take to be a permutation $\sigma \in \mathfrak{S}_n$), Ross proposed the following algorithm:

Step 1 Use what Mallows called a "patience sorting procedure" to form the subsequences $r_1, r_2, \ldots, r_m$ of σ (called *piles*) as follows:

- Place the first card c_1 from the deck into a pile r_1 by itself.
- For each remaining card c_i ($i = 2, \ldots, n$), consider the cards $d_1, d_2, \ldots, d_k$ atop the piles $r_1, r_2, \ldots, r_k$ that have already been formed.
  - If $c_i > \max\{d_1, d_2, \ldots, d_k\}$, then put c_i into a new right-most pile r_{k+1} by itself.
  - Otherwise, find the left-most card d_j that is larger than c_i and put the card c_i atop pile r_j.

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Step 2 Gather the cards up one at a time from these piles in ascending order.

We call **Step 1** of the above algorithm *Patience Sorting* and denote by $R(\sigma) = \{r_1, r_2, \ldots, r_m\}$ the *pile configuration* associated to the permutation $\sigma \in \mathfrak{S}_n$. Moreover, given any pile configuration R, one forms its *reverse patience word* $RPW(R)$ by listing the piles in R "from bottom to top, left to right" (i.e., by reversing the so-called "far-eastern reading"). In [**5**] these words are characterized as being exactly the elements of the avoidance set $S_n(3\text{-}\bar{1}\text{-}42)$. That is, they are permutation avoiding the generalized pattern 2-31 unless every occurrence of 2-31 is contained within an occurrence of the generalized pattern 3-1-42. (A review of generalized permutation patterns can be found in Section 1.2 below).

We illustrate the formation of $R(\sigma)$ and $RPW(R)$ in the following example.

Example 1.1. Let $\sigma = 64518723 \in \mathfrak{S}_8$. Then we form the pile configuration $R(\sigma)$ as follows:

Form a new pile with **6**:
$$\begin{array}{l} \\ \mathbf{6} \end{array}$$

Then place **4** atop 6:
$$\begin{array}{l} \mathbf{4} \\ 6 \end{array}$$

Form a new pile with **5**:
$$\begin{array}{ll} 4 & \\ 6 & \mathbf{5} \end{array}$$

Add **1** to the left-most pile:
$$\begin{array}{ll} \mathbf{1} & \\ 4 & \\ 6 & 5 \end{array}$$

Form a new pile with **8**:
$$\begin{array}{lll} 1 & & \\ 4 & & \\ 6 & 5 & \mathbf{8} \end{array}$$

Then place **7** atop 8:
$$\begin{array}{lll} 1 & & \\ 4 & & \mathbf{7} \\ 6 & 5 & 8 \end{array}$$

Add **2** to the middle pile:
$$\begin{array}{lll} 1 & & \\ 4 & \mathbf{2} & 7 \\ 6 & 5 & 8 \end{array}$$

Finally, place **3** atop 7:
$$\begin{array}{lll} 1 & & \mathbf{3} \\ 4 & 2 & 7 \\ 6 & 5 & 8 \end{array}$$

Then, by reading up the columns of $R(\sigma)$ from left to right, $RPW(R(\sigma)) = 64152873 \in S_8(3\text{-}\bar{1}\text{-}42)$.

Given $\sigma \in \mathfrak{S}_n$, the formation of $R(\sigma)$ can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm for interposing values into the rows of a standard Young tableau (see [**2**]). In [**5**] the authors augment the formation of $R(\sigma)$ so that the resulting extension of Patience Sorting becomes a full non-recursive analogue of the celebrated Robinson-Schensted-Knuth (or RSK) Correspondence. As with RSK, this Extended Patience Sorting Algorithm (given as Algorithm 1.2 in Section 1.1 below) takes a simple idea — that of placing cards into piles — and uses it to build a bijection between elements of the symmetric group $\mathfrak{S}_n$ and certain pairs of combinatorial objects. In the case of RSK, one uses the Schensted Insertion Algorithm to build a bijection with (unrestricted) pairs of standard Young tableau having the same shape (see [**16**]). However, in the case of Patience Sorting, one achieves a bijection between permutations and (somewhat more restricted) pairs of pile configurations having the same shape. We denote this latter bijection by $\sigma \overset{PS}{\longleftrightarrow} (R(\sigma), S(\sigma))$ and call $R(\sigma)$ (resp. $S(\sigma)$) the *insertion piles* (resp. *recording piles*) corresponding to σ. Collectively, we also call $(R(\sigma), S(\sigma))$ the *stable pair* of pile configurations corresponding to σ and characterize such pairs in [**5**] using a somewhat involved pattern avoidance condition on their reverse patience words.

Barred (generalized) permutation patterns like $3\text{-}\bar{1}\text{-}42$ arise quite naturally when studying Patience Sorting. We discuss and enumerate the avoidance classes for several related patterns in Section 2. Then, in Section 3, we examine properties of Patience Sorting under restricted input that can be characterized using such patterns. One such characterization, discussed in Section 3.1, is for the crossings in the initial iteration of the Geometric Patience Sorting Algorithm given by the authors in [**6**]. This geometric form for the Extended Patience Sorting Algorithm is naturally dual to Viennot's Geometric RSK (originally defined in [**19**]) and gives, among other things, a geometric interpretation for the stable pairs of $3\text{-}\bar{1}\text{-}42$-avoiding permutations corresponding to a permutation under Extended Patience Sorting. However, unlike Viennot's geometric form for RSK, the shadow lines in Geometric Patience Sorting are allowed to cross. While a complete characterization for these crossings is given in [**6**] in terms of the pile configurations formed, this new result is the first step in providing a characterization for the permutations involved in terms of barred pattern avoidance.

We close this introduction by describing both the Extended and Geometric Patience Sorting Algorithms. We also briefly review the notation of generalized permutation patterns.

1.1. Extended and Geometric Patience Sorting. Mallows' original "patience sorting procedure" can be extended to a full bijection between the symmetric group $\mathfrak{S}_n$ and certain restricted pairs of pile configurations using the following algorithm (which was first introduced in [**5**]):

ALGORITHM 1.2 (Extended Patience Sorting Algorithm). Given $\sigma = c_1 c_2 \cdots c_n \in \mathfrak{S}_n$, inductively build *insertion piles* $R(\sigma) = \{r_1, r_2, \ldots, r_m\}$ and *recording piles* $S(\sigma) = \{s_1, s_2, \ldots, s_m\}$ as follows:

- Place the first card c_1 from the deck into a pile r_1 by itself, and set $s_1 = \{1\}$.
- For each remaining card c_i ($i = 2, \ldots, n$), consider the cards $d_1, d_2, \ldots, d_k$ atop the piles $r_1, r_2, \ldots, r_k$ that have already been formed.
  - If $c_i > \max\{d_1, d_2, \ldots, d_k\}$, then put c_i into a new pile r_{k+1} by itself and set $s_{k+1} = \{i\}$.
  - Otherwise, find the left-most card d_j that is larger than c_i and put the card c_i atop pile r_j while simultaneously putting i at the bottom of pile s_j.

Note that the pile configurations that comprise a resulting stable pair must have the same "shape", which we define as follows:

DEFINITION 1.3. Given a pile configuration $R = \{r_1, r_2, \ldots, r_m\}$ on n cards, we call the composition $\gamma = (|r_1|, |r_2|, \ldots, |r_m|)$ of n the *shape* of R and denote this by $\mathrm{sh}(R) = \gamma \models n$.

The idea behind Algorithm 1.2 is that we are using the auxiliary pile configuration $S(\sigma)$ to implicitly label the order in which the elements of the permutation $\sigma \in \mathfrak{S}_n$ are added to the usual Patience Sorting pile configuration $R(\sigma)$ (which we now call the "insertion piles" of σ in this context by analogy to RSK). It is clear that this information then allows us to uniquely reconstruct σ by reversing the order in which the cards were played. As with normal Patience Sorting, we visualize the pile configurations $R(\sigma)$ and $S(\sigma)$ by listing their constituent piles vertically as illustrated in the following example.

EXAMPLE 1.4. Given $\sigma = 64518723 \in \mathfrak{S}_8$ from Example 1.1 above, we simultaneously form the following pile configurations with shape $\mathrm{sh}(R(\sigma)) = \mathrm{sh}(S(\sigma)) = (3, 2, 3)$ under Extended Patience Sorting (Algorithm 1.2):

$$R(\sigma) = \begin{matrix} 1 & & 3 \\ 4 & 2 & 7 \\ 6 & 5 & 8 \end{matrix} \quad \text{and} \quad S(\sigma) = \begin{matrix} 1 & & 5 \\ 2 & 3 & 6 \\ 4 & 7 & 8 \end{matrix}$$

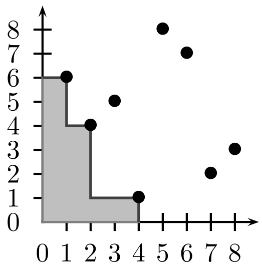
Note that the insertion piles $R(\sigma)$ are the same as the pile configuration formed in Example 1.1 and that $RPW(S(64518723)) = 42173865 \in S_8(3\text{-}\bar{1}\text{-}42)$.

In order to now describe a natural geometric form for this Extended Patience Sorting Algorithm, we begin with the following fundamental definition.

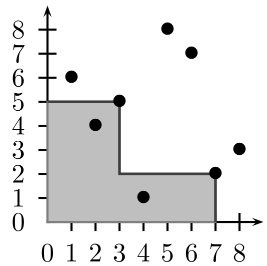
DEFINITION 1.5. Given a lattice point $(m, n) \in \mathbb{Z}^2$, we define the *(southwest) shadow* of (m, n) to be the quarter space $U(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \le m,\ y \le n\}$.

As with the northeasterly-oriented shadows that Viennot used when building his geometric form for RSK (see [**19**]), the most important use of these southwesterly-oriented shadows is in building shadowlines (which is illustrated in Figure 1(a)):

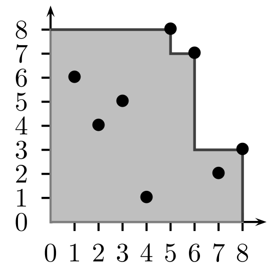
DEFINITION 1.6. The *(southwest) shadowline* of $(m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k) \in \mathbb{Z}^2$ is defined to be the boundary of the union of the shadows $U(m_1, n_1), U(m_2, n_2), \ldots, U(m_k, n_k)$.



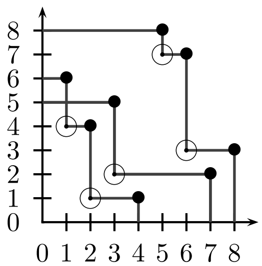
(a) Shadowline $L_1^{(0)}(64518723)$.



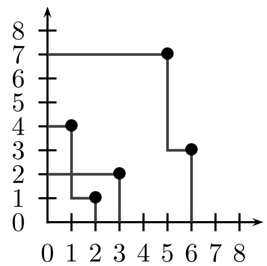
(b) Shadowline $L_2^{(0)}(64518723)$.



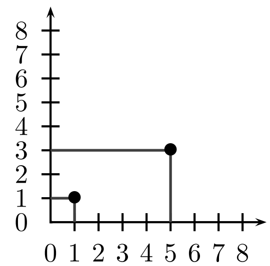
(c) Shadowline $L_3^{(0)}(64518723)$.



(d) Salient points ($\odot$) for $D^{(0)}(64518723)$.



(e) Shadow Diagram $D^{(1)}(64518723)$.



(f) Shadow Diagram $D^{(2)}(64518723)$.

Figure 1. Examples of Shadowline and Shadow Diagram Construction.

In particular, we wish to associate to each permutation a certain collection of (southwest) shadowlines called its *shadow diagram*. However, unlike the northeasterly-oriented shadowlines used to define the northeast shadow diagrams of Geometric RSK [**19**], these southwest shadowlines are allowed to intersect as illustrated in Figure 1(d)–(e). (We characterize those permutations having intersecting shadowlines under Definition 1.7 in Theorem 3.6 below.)

Definition 1.7. The *(southwest) shadow diagram* $D^{(0)}(\sigma)$ of $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ consists of the (southwest) shadowlines $D^{(0)}(\sigma) = \{L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_k^{(0)}(\sigma)\}$ formed as follows:

- $L_1^{(0)}(\sigma)$ is the shadowline for those lattice points $(x, y) \in \{(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)\}$ such that the shadow $U(x, y)$ does not contain any other lattice points.
- While at least one of the points $(1, \sigma_1), (2, \sigma_2), \ldots, (n, \sigma_n)$ is not contained in the shadowlines $L_1^{(0)}(\sigma), L_2^{(0)}(\sigma), \ldots, L_j^{(0)}(\sigma)$, define $L_{j+1}^{(0)}(\sigma)$ to be the shadowline for the points

$$(x, y) \in A := \{(i, \sigma_i) \mid (i, \sigma_i) \notin \bigcup_{k=1}^{j} L_k^{(0)}(\sigma)\}$$

  such that the shadow $U(x, y)$ does not contain any other lattice points from the set A.

In other words, we define a shadow diagram by inductively eliminating points in the permutation diagram until every point has been used to define a shadowline (as illustrated in Figure 1(a)–(c)).

One can prove (see [**5**]) that the ordinates (i.e., y-coordinates) of the points used to define each shadowline in the shadow diagram $D^{(0)}(\sigma)$ are exactly the *left-to-right minima subsequences* (a.k.a. *basic subsequences*) in the permutation $\sigma \in \mathfrak{S}_n$. These are defined as follows:

DEFINITION 1.8. Let $\pi = \pi_1\pi_2\cdots\pi_l$ be a partial permutation on the set $[n] = \{1, 2, \ldots, n\}$. Then the *left-to-right minima* (resp. *maxima*) *subsequence* of π consists of those components π_j of π such that $\pi_j = \min\{\pi_i \mid 1 \leq i \leq j\}$ (resp. $\pi_j = \max\{\pi_i \mid 1 \leq i \leq j\}$).

We then inductively define the left-to-right minima (resp. maxima) subsequences $s_1, s_2, \ldots, s_k$ of the permutation σ by taking s_1 to be the left-to-right minima (resp. maxima) subsequence for σ itself and then each remaining subsequence s_i to be the left-to-right minima (resp. maxima) subsequence for the partial permutation obtained by removing the elements of $s_1, s_2, \ldots, s_{i-1}$ from σ.

Finally, one can produce a sequence $D(\sigma) = (D^{(0)}(\sigma),\ D^{(1)}(\sigma),\ D^{(2)}(\sigma),\ \ldots)$ of shadow diagrams for a given permutation $\sigma \in \mathfrak{S}_n$ by recursively applying Definition 1.7 to the southwest corners (called *salient points*) of a given set of shadowlines (as illustrated in Figure 1(d)–(f)). The only difference is that, with each iteration, newly formed shadowlines can only connect salient points along the same pre-existing shadowline. One can then uniquely reconstruct the pile configurations $R(\sigma)$ and $S(\sigma)$ from these shadowlines by taking their intersections with the x- and y-axes in a certain canonical order (as detailed in [**6**]).

DEFINITION 1.9. We call $D^{(k)}(\sigma)$ the k^{th} *iterate* of the *exhaustive shadow diagram* $D(\sigma)$ for the permutation $\sigma \in \mathfrak{S}_n$.

1.2. Generalized Pattern Avoidance. We first recall the following definition:

DEFINITION 1.10. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$ with $m \leq n$. Then we say that σ *contains* the *(classical) permutation pattern* π if there exists a subsequence $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ of σ (meaning $i_1 < i_2 < \cdots < i_m$) such that the word $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ is order-isomorphic to π. I.e., each $\sigma_{i_j} < \sigma_{i_{j+1}}$ if and only if $\pi_j < \pi_{j+1}$.

Note, though, that the elements in the subsequence $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ are not required to be contiguous in σ. This motivates the

DEFINITION 1.11. A *generalized permutation pattern* is a classical permutation pattern π in which one assumes that every element in the subsequence $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ of σ must be taken contiguously unless a dash is inserted between the corresponding order-isomorphic elements of the pattern π.

Finally, if σ does not contain a subsequence that is order-isomorphic to π, then we say that σ *avoids* the pattern π. This motivated the

DEFINITION 1.12. Given any collection $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}$ of permutation patterns (classical or generalized), we denote by

$$S_n(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}) = \bigcap_{i=1}^{k} S_n(\pi^{(i)}) = \bigcap_{i=1}^{k} \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \pi^{(i)}\}$$

the *avoidance set* of permutations $\sigma \in \mathfrak{S}_n$ such that σ simultaneously avoids each of the patterns $\pi^{(1)}, \pi^{(2)}, \ldots,$ $\pi^{(k)}$. Furthermore, the set

$$\bigcup_{n\geq 1} S_n(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})$$

is called the *(pattern) avoidance class* with *basis* $\{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\}$.

More information about permutation patterns in general can be found in [**4**].

2. Barred and Unbarred Generalized Pattern Avoidance

An important further generalization of the notion of generalized permutation pattern requires that the context in which the occurrence of a generalized pattern occurs be taken into account. The resulting concept of *barred permutation patterns*, along with the accompanying notation, first arose within the study of stack-sortability of permutations by J. West [**20**]. Given how naturally these barred patterns now arise in the study of Patience Sorting (as illustrated in both [**5**] and Section 3 below), we initiate their systematic study in this section.

DEFINITION 2.1. A *barred (generalized) permutation pattern* β is a generalized permutation pattern in which overbars are used to indicate that barred values cannot occur at the barred positions. As before, we denote by $S_n(\beta^{(1)}, \ldots, \beta^{(k)})$ the set of all permutations $\sigma \in \mathfrak{S}_n$ that simultaneously avoid $\beta^{(1)}, \ldots, \beta^{(k)}$ (i.e., permutations that contain no subsequence that is order-isomorphic to any of the $\beta^{(1)}, \ldots, \beta^{(k)}$).

EXAMPLE 2.2. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n \notin S_n(3\text{-}\bar{5}\text{-}2\text{-}4\text{-}1)$ contains an occurrence of the barred permutation pattern 3-$\bar{5}$-2-4-1 if it contains an occurrence of the generalized pattern 3-2-4-1 (i.e., contains a subsequence $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$ that is order-isomorphic to the classical pattern 3241) in which no value larger than the element playing the role of "4" is allowed to occur between the elements playing the roles of "3" and "2". This is one of the two basis elements for the pattern avoidance class used to characterize the set of 2-stack-sortable permutations [**10, 11, 20**]. (The other pattern is 2-3-4-1, i.e., the classical pattern 2341.)

Despite the added complexity involved in avoiding barred permutation patterns, it is still sometimes possible to characterize the avoidance class for a barred permutation pattern in terms of an unbarred generalized permutation pattern. The following theorem gives such a characterization for the pattern 3-$\bar{1}$-42. (Note, though, that there is no equivalent characterization for such barred permutation patterns as 1$\bar{3}$-42 and 3-$\bar{5}$-2-4-1.)

THEOREM 2.3. *Let* $B_n = \frac{1}{e}\sum_{k\geq 0}\frac{k^n}{k!}$ *denote the* n^{th} *Bell number. Then*
(1) $S_n(3\text{-}\bar{1}\text{-}42) = S_n(3\text{-}\bar{1}\text{-}4\text{-}2) = S_n(23\text{-}1)$
(2) $|S_n(3\text{-}\bar{1}\text{-}42)| = B_n$

PROOF. (Sketch)
As in [**7**], we see that each of these sets consists of permutations having the form

$$\sigma = \sigma_1 a_1 \sigma_2 a_2 \ldots \sigma_k a_k,$$

where $a_k > a_{k-1} > \cdots > a_2 > a_1$ are the successive right-to-left minima of σ (reversing the order of the elements in Definition 1.8) and where each segment $\sigma_i a_i$ is a decreasing subsequence. □

REMARK 2.4. We emphasize the following important consequences of Theorem 2.3.

(1) Even though $S_n(3\text{-}\bar{1}\text{-}42) = S_n(23\text{-}1)$ by Theorem 2.3(1), it is more natural to use avoidance of the barred pattern 3-$\bar{1}$-42 in studying Patience Sorting. As shown in [**5**] and elaborated upon in Section 3 below, $S_n(3\text{-}\bar{1}\text{-}42)$ is the set of equivalence classes of $\mathfrak{S}_n$ modulo the transitive closure of the relation 3-$\bar{1}$-42 $\sim$ 3-$\bar{1}$-24. (I.e., two permutations $\sigma, \tau \in \mathfrak{S}_n$ are equivalent if the elements creating an occurrence of one of these patterns in σ form an occurrence of the other pattern in τ.) Moreover, each permutation $\sigma \in \mathfrak{S}_n$ in a given equivalence class has the same pile configuration $R(\sigma)$ under Patience Sorting, a description of which is significantly more difficult to describe for occurrences of the unbarred generalized permutation pattern 23-1.

(2) Marcus and Tardos proved in [**14**] that the avoidance set $S_n(\pi)$ for any classical pattern π grows at most exponentially fast as $n \to \infty$. (This was previously known as the Stanley-Wilf Conjecture.) The Bell numbers, though, satisfy $\log B_n = n(\log n - \log\log n + O(1))$ and so exhibit superexponential growth. (See [**18**] for more information about Bell numbers.) While it was previously known that the Stanley-Wilf Conjecture does not extend to generalized permutation patterns (see, e.g., [**7**]), it took Theorem 2.3(2) (originally proven in [**5**] using Patience Sorting) to provide the first verification that one also cannot extend the Stanley-Wilf Conjecture to barred generalized permutation patterns.

A further abstraction of barred permutation pattern avoidance (called *Bruhat-restricted avoidance*) was recently given by A. Woo and A. Yong in [**21**]. The result in Theorem 2.3(2) has led A. Woo to conjecture to the second author that the Stanley-Wilf ex-Conjecture also does not extend to this new notion of pattern avoidance.

We conclude this section with a simple corollary to Theorem 2.3 that gives similar equivalences and enumerations for some barred permutation patterns that also arise naturally in the study of Patience Sorting (see Proposition 3.2 and Theorem 3.6 in Section 3 below).

COROLLARY 2.5. *Using the notation in Theorem 2.3,*

(1) $S_n(31\text{-}\bar{4}\text{-}2) = S_n(3\text{-}1\text{-}\bar{4}\text{-}2) = S_n(3\text{-}12)$
(2) $S_n(\bar{2}\text{-}41\text{-}3) = S_n(\bar{2}\text{-}4\text{-}1\text{-}3) = S_n(2\text{-}4\text{-}1\text{-}\bar{3}) = S_n(2\text{-}41\text{-}\bar{3})$
(3) $|S_n(\bar{2}\text{-}41\text{-}3)| = |S_n(31\text{-}\bar{4}\text{-}2)| = |S_n(3\text{-}\bar{1}\text{-}42)| = B_n$.

PROOF. (Sketches)

(1) Take reverse complements in $S_n(3\text{-}\bar{1}\text{-}42)$ and apply Theorem 2.3.
(2) Similar to (1). (Note that (2) is also proven in [**1**].)
(3) This follows from the fact that the patterns 3-1-$\bar{4}$-2 and $\bar{2}$-4-1-3 are inverses of each other. □

3. Patience Sorting under Restricted Input

3.1. Patience Sorting on Restricted Permutations. The similarities between the Extended Patience Sorting Algorithm (Algorithm 1.2) and RSK applied to permutations is perhaps most observable in the following simple proposition:

PROPOSITION 3.1. *Let* $\imath_k = 1\text{-}2\text{-}\cdots\text{-}k = 12\cdots k$ *and* $\jmath_k = k\text{-}\cdots\text{-}2\text{-}1 = k\cdots 21$ *be the classical monotone permutation patterns. Then there is*

(1) *a bijection between* $S_n(\imath_{k+1})$ *and pairs of pile configurations having the same composition shape* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n$ *but with at most* k *piles (i.e.,* $m \leq k$*).*
(2) *a bijection between* $S_n(\jmath_{k+1})$ *and pairs of pile configurations having the same composition shape* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n$ *but with no pile having more than* k *cards in it (i.e.,* $\gamma_i \leq k$ *for each* $i = 1, 2, \ldots, m$*).*

PROOF. (Sketches)

(1) Given $\sigma \in \mathfrak{S}_n$, a bijection is formed in [**2**] between the set of piles $R(\sigma) = \{r_1, r_2, \ldots, r_k\}$ formed under Patience Sorting and the components of a particular distinguished longest increasing subsequence in σ. Since avoiding the monotone pattern $\imath_{k+1}$ is equivalent to restricting the length of the longest increasing subsequence in a permutation, the result then follows.
(2) Follows from (1) by reversing each of the permutations in $S_n(\imath_{k+1})$ in order to form $S_n(\jmath_{k+1})$. □

Proposition 3.1 states that Patience Sorting can be used to efficiently compute the length of both the longest increasing and longest decreasing subsequences in a given permutation. In particular, one can compute these lengths without needing to examine every subsequence of a permutation, just as with RSK. However, while both RSK and Patience Sorting can be used to implement this computation in $O(n\log(n))$ time, an extension of this technique is given in [**3**] that also simultaneously tabulates all of the longest increasing or decreasing subsequences without incurring any additional asymptotic computational cost.

As mentioned in Section 2 above, Patience Sorting also has immediate connections to certain barred permutation patterns:

PROPOSITION 3.2.

(1) $S_n(3\text{-}\bar{1}\text{-}42) = \{RPW(R(\sigma)) \mid \sigma \in \mathfrak{S}_n\}$. *In particular, given* $\sigma \in S_n(3\text{-}\bar{1}\text{-}42)$, *the entries in each column of the insertion piles* $R(\sigma)$ *(when read from bottom to top) occupy successive positions in the permutation* σ.
(2) $S_n(\bar{2}\text{-}41\text{-}3) = \{RPW(R(\sigma^{-1})) \mid \sigma \in \mathfrak{S}_n\}$. *In particular, given* $\sigma \in S_n(\bar{2}\text{-}41\text{-}3)$, *the columns of the insertion piles* $R(\sigma)$ *(when read from top to bottom) contain successive values.*

PROOF. Part (1) is proven in [**5**], and part (2) follows immediate by taking inverses in (1). □

As an immediate corollary, we can characterize an important category of classical permutation patterns in terms of barred permutation patterns.

DEFINITION 3.3. Given a composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \models n$, the *(classical) layered permutation pattern* $\pi_\gamma \in \mathfrak{S}_n$ is the permutation

$$\gamma_1 \cdot (\gamma_1 - 1) \cdots 1 \cdot (\gamma_1 + \gamma_2) \cdot (\gamma_1 + \gamma_2 - 1) \cdots (\gamma_1 + 1) \cdots n \cdot (n-1) \cdots (\gamma_1 + \gamma_2 + \cdots + \gamma_{m-1} + 1).$$

EXAMPLE 3.4. Given $\gamma = (3, 2, 3) \models 8$, the corresponding layered pattern is $\pi_{(3,2,3)} = \widehat{321}\widehat{54}\widehat{876} \in \mathfrak{S}_8$ (following the notation in [**15**]). Moreover, applying Extended Patience Sorting (Algorithm 1.2) to $\pi_{(3,2,3)}$:

$$R(\pi_{(3,2,3)}) = \begin{matrix} 1 & & 6 \\ 2 & 4 & 7 \\ 3 & 5 & 8 \end{matrix} \quad \text{and} \quad S(\pi_{(3,2,3)}) = \begin{matrix} 1 & & 6 \\ 2 & 4 & 7 \\ 3 & 5 & 8 \end{matrix}$$

Note in particular that $\pi_{(3,2,3)}$ satisfies both of the conditions in Proposition 3.2, which illustrates the following characterization of layered patterns:

COROLLARY 3.5. *$S_n(3\text{-}\bar{1}\text{-}42, \bar{2}\text{-}41\text{-}3)$ is the set of layered patterns in $\mathfrak{S}_n$.*

PROOF. Apply Proposition 3.2 noting that $S_n(3\text{-}\bar{1}\text{-}42, \bar{2}\text{-}41\text{-}3) = S_n(23\text{-}1, 31\text{-}2)$ (as considered in [**8**]). □

As a consequence of this interaction between Patience Sorting and barred permutation patterns, we can now explicitly characterize those permutations for which the initial iteration of Geometric Patience Sorting (as defined in Section 1.1 above) yields non-crossing lattice paths.

THEOREM 3.6. *The set $S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$ consists of all reverse patience words having non-intersecting shadow diagrams. (I.e., no shadowlines cross in the 0^{th} iterate shadow diagram.) Moreover, given a permutation $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$, the values in the bottom rows of $R(\sigma)$ and $S(\sigma)$ increase from left to right.*

PROOF. From Theorem 2.3 and Corollary 2.5, $R(S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)) = R(S_n(23\text{-}1, 3\text{-}12))$ consists exactly of set partitions of $[n] = \{1, 2, \ldots, n\}$ whose components can be ordered so that both the minimal and maximal elements of the components simultaneously increase. (These are called *strongly monotone partitions* in [**9**]).

Let $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 31\text{-}\bar{4}\text{-}2)$. Since σ avoids $3\text{-}\bar{1}\text{-}42$, we have that $\sigma = RPW(R(\sigma))$ by Proposition 3.2. Thus, the i^{th} shadowline $L_i^{(0)}(\sigma)$ of σ is the boundary of the union of shadows with generating points in decreasing segments $\sigma_i a_i$, $i \in [k]$, where $\sigma_i a_i$ are as in the proof of Theorem 2.3. Let b_i be the i^{th} left-to-right maximum of σ. Then b_i is the left-most (i.e. maximal) entry of $\sigma_i a_i$, so $\sigma_i a_i = b_i \sigma_i' a_i$ for some decreasing subsequence σ_i'. Note that σ_i' may be empty so that $b_i = a_i$.

Since b_i is the i^{th} left-to-right maximum of σ, it must be at the bottom of the i^{th} column of $R(\sigma)$ (similarly, a_i is at the top of the i^{th} column). So the bottom rows of both $R(\sigma)$ and $S(\sigma)$ must be in increasing order.

Now consider the i^{th} and j^{th} shadowlines $L_i^{(0)}(\sigma)$ and $L_j^{(0)}(\sigma)$ of σ, respectively, where $i < j$. We have that $b_i < b_j$ from which the initial horizontal segment of the i^{th} shadowline is lower than that of the j^{th} shadowline. Moreover, a_i is to the left of b_j, so the remaining segment of the i^{th} shadowline is completely to the left of the remaining segment of the j^{th} shadowline. Thus, $L_i^{(0)}(\sigma)$ and $L_j^{(0)}(\sigma)$ do not intersect. □

In [**6**] the authors actually give the following stronger result:

THEOREM 3.7. *Each iterate $D_{SW}^{(m)}(\sigma)$ $(m \geq 0)$ of $\sigma \in \mathfrak{S}_n$ is free from crossings if and only if every row in both $R(\sigma)$ and $S(\sigma)$ is monotone increasing from left to right.*

However, this only characterizes the output of the Extended Patience Sorting Algorithm involved. As such, Theorem 3.6 provides the first step toward characterizing those permutations that result in non-crossing lattice paths under Geometric Patience Sorting.

We conclude this section by noting that, while the strongly monotone condition implied by simultaneously avoiding 3-$\bar{1}$-42 and 31-$\bar{4}$-2 is necessary to alleviate such crossings, it is clearly not sufficient. (The problem lies with what we call "polygonal crossings" in the shadow diagrams in [**6**], which occur in permutations like $\sigma = 45312$.) Thus, to avoid crossings at all iterations of Geometric Patience Sorting, we need to impose further "ordinally increasing" conditions on the set partition associated to a given permutation under Patience Sorting. In particular, in addition to requiring just the minima and maxima elements in the set partition to increase as in the strongly monotone partitions encountered in the proof of Theorem 3.6, it is necessary to require that every record value simultaneously increase under an appropriate ordering of the blocks. That is, under a single ordering of these blocks, we must simultaneously have that the largest elements in each block increase, then the next largest elements, then the next-next largest elements, and so on. E.g., the partition $\{\{5,3,1\},\{6,4,2\}\}$ of the set $[6] = \{1,2,\ldots,6\}$ satisfies this condition.

3.2. Invertibility of Patience Sorting. It is clear that the pile configurations corresponding to two permutations under the Patience Sorting Algorithm need not be distinct in general (e.g., $R(3142) = R(3412)$). As proven in [**5**], two permutations give rise to the same pile configuration under Patience Sorting if and only if they have the same left-to-right minima subsequences (e.g., 3142 and 3412 both have the left-to-right minima subsequences 31 and 42). In this section we characterize permutations having distinct pile configurations under Patience Sorting in terms of certain barred permutation patterns. We then establish a non-trivial enumeration for the resulting avoidance sets.

THEOREM 3.8. *A pile configuration pile R has a unique preimage $\sigma \in \mathfrak{S}_n$ under Patience Sorting if and only if $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$.*

PROOF. (Sketch)

It is clear that every pile configuration R has at least one preimage, namely its reverse patience word $\sigma = RPW(R)$. By Proposition 3.2, reverse patience words are exactly those permutations that avoid the barred pattern 3-$\bar{1}$-42. Furthermore, as shown in [**5**], two permutations have the same insertion piles under Extended Patience Sorting (Algorithm 1.2) if and only if one can be obtained from the other by a sequence of order-isomorphic exchanges 3-$\bar{1}$-24 $\rightsquigarrow$ 3-$\bar{1}$-42 or 3-$\bar{1}$-42 $\rightsquigarrow$ 3-$\bar{1}$-24. (I.e., the occurrence of one pattern is reordered to form an occurrence of the other pattern.) Thus, it is easy to see that R has a unique preimage σ if and only if σ has no occurrence of 3-$\bar{1}$-42 or 3-$\bar{1}$-24. □

Given this pattern avoidance characterization of invertibility, we have the following recurrence relation for the number of permutations having distinct pile configurations under Patience Sorting:

LEMMA 3.9. *Set $f(n) = |S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)|$ and, for $k \leq n$,*

$$f(n,k) = |\{\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24) \,:\, \sigma(1) = k\}|.$$

Then $f(n) = \sum_{k=1}^{n} f(n,k)$, and we have the following recurrence relation for $f(n,k)$:

$$f(n,0) = 0 \quad \text{for} \quad n \geq 1 \tag{3.1}$$

$$f(n,1) = f(n,n) = f(n-1) \quad \text{for} \quad n \geq 1 \tag{3.2}$$

$$f(n,2) = 0 \quad \text{for} \quad n \geq 3 \tag{3.3}$$

$$f(n,k) = f(n,k-1) + f(n-1,k-1) + f(n-2,k-2) \quad \text{for} \quad n \geq 3 \tag{3.4}$$

subject to the initial conditions $f(0) = f(0,0) = 1$.

PROOF. First note that Equation (3.1) is the obvious boundary condition for $k = 0$.

Now suppose that the first letter of $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ is $\sigma(1) = 1$ or n. Then $\sigma(1)$ cannot be part of any occurrence of 3-$\bar{1}$-42 or 3-$\bar{1}$-24 in σ. Thus, deletion of $\sigma(1)$, and subtraction of 1 from each component if $\sigma(1) = 1$, yields a bijection with $S_{n-1}(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ so that Equation (3.2) follows.

Similarly, suppose that the first letter of $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ is $\sigma(1) = 2$. Then the first column of $R(\sigma)$ must be $\begin{smallmatrix}1\\2\end{smallmatrix}$ regardless of where 1 occurs in σ. Therefore, $R(\sigma)$ has a unique preimage σ if and only if $\sigma = 21 \in \mathfrak{S}_2$ so that Equation (3.3) follows.

Finally, suppose that $\sigma \in S_n(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ with $3 \le k \le n$. Since σ avoids 3-$\bar{1}$-42, σ is a RPW by Proposition 3.2, and hence the left prefix of σ from k to 1 is a decreasing subsequence. Let σ' be the permutation obtained by interchanging the values k and $k-1$ in σ. Then the only instances of the patterns 3-$\bar{1}$-42 and 3-$\bar{1}$-24 in σ' must involve both k and $k-1$. Note that the number of σ for which no instances of these patterns are created by interchanging k and $k-1$ is exactly $f(n, k-1)$.

There are then two cases in which an instance of the barred pattern 3-$\bar{1}$-42 or 3-$\bar{1}$-24 will be created in σ' by this interchange:

Case 1. If $k-1$ occurs between $\sigma(1) = k$ and 1 in σ, then $\sigma(2) = k-1$, so interchanging k and $k-1$ creates an instance of the pattern 23-1 via the subsequence $(k-1, k, 1)$ in σ'. Thus, by Theorem 2.3, σ' contains 3-$\bar{1}$-42 from which $\sigma' \in S_n(3\text{-}\bar{1}\text{-}42)$ if and only if $k-1$ occurs after 1 in σ. Note also that if $\sigma(2) = k-1$, then deleting k yields a bijection with permutations in $S_{n-1}(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ that start with $k-1$. So the number of permutations counted in Case 1 is exactly $f(n-1, k-1)$.

Case 2. If $k-1$ occurs to the right of 1 in σ, then σ' both contains the subsequence $(k-1, 1, k)$ and avoids the pattern 3-$\bar{1}$-42, so it must also contain the pattern 3-$\bar{1}$-24. If an instance of 3-$\bar{1}$-24 in σ' involves both $k-1$ and k, then $k-1$ and k must play the roles of "3" and "4", respectively. If the value ℓ preceding k is not 1, then the subsequence $(k-1, 1, \ell, k)$ is an instance of 3-1-24, so $(k-1, \ell, k)$ is not an instance of 3-$\bar{1}$-24. Therefore, for σ' to contain 3-$\bar{1}$-24, k must follow 1 in σ', and so $k-1$ follows 1 in σ. If the letter preceding 1 is some $m < k$, then the subsequence $(m, 1, k-1)$ is an instance of 3-$\bar{1}$-24 in σ, which is impossible. Therefore, k must precede 1 in σ, from which σ must start with the initial segment $(k, 1, k-1)$. But then deleting the values k and 1 and then subtracting 1 from each component yields a bijection with permutations in $S_{n-2}(3\text{-}\bar{1}\text{-}42, 3\text{-}\bar{1}\text{-}24)$ that start with $k-2$. It follows that the number of permutations counted in Case 2 is then exactly $f(n-2, k-2)$, which yields Equation (3.4). □

If we denote by

$$\Phi(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(n, k) x^n y^k$$

the bivariate generating function for the sequence $\{f(n, k)\}_{n \ge k \ge 0}$, then Equation (3.4) implies that

$$(1 - y - xy - x^2y^2)\Phi(x, y) = 1 - y - xy + xy^2 - xy^2\Phi(xy, 1) + xy(1 - y - xy)\Phi(x, 1).$$

Moreover, using the kernel method, one can show that

$$x + 1 + \frac{\sqrt{1 + 2x + 5x^2} - x - 1}{2} \cdot F(x) - F\left(\frac{\sqrt{1 + 2x + 5x^2} - x - 1}{2x}\right) = 0$$

where $F(x) = \sum_{n \ge 0} f(n) x^n = \Phi(x, 1)$ is the generating function for the sequence $\{f(n)\}_{n \ge 0}$.

We conclude with the following main enumerative result about invertibility of Patience Sorting.

Theorem 3.10. *Denote by F_n the n^{th} Fibonacci number (with $F_0 = F_1 = 1$) and by*

$$a(n, k) = \sum_{\substack{n_1, \ldots, n_k \ge 0 \\ n_1 + \cdots + n_k = n-k-2}} F_{n_1} F_{n_2} \ldots F_{n_k}$$

convolved Fibonacci numbers for $n \ge k + 2$ (where $a(n, k) := 0$ otherwise). Then, defining

$$X = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \\ \vdots \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = (a(n,k))_{n,k\geq 0} = \begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ a(2,0) & 0 & 0 & & & \\ a(3,0) & a(3,1) & 0 & 0 & & \\ a(4,0) & a(4,1) & a(4,2) & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

we have that $X = (\mathbf{I} - \mathbf{A})^{-1} F$, *where* $\mathbf{I}$ *is the infinite identity matrix and* $\mathbf{A}$ *is lower triangular.*

PROOF. (Sketch)

From Equations (3.1)–(3.4), we can conjecture an equivalent recurrence where (3.3) and (3.4) are replaced by the following equation (here δ_{nk} is the Kronecker delta function):

$$f(n,k) = \sum_{m=0}^{k-3} c(k,m) f(n-k+m) + \delta_{nk} F_{k-2}, \quad n \geq k \geq 2. \tag{3.5}$$

For this relation to hold, the coefficients $c(k,m)$ must satisfy the following recurrence relation:

$$c(k,m) = c(k-1,m-1) + c(k-1,m) + c(k-2,m), \quad k \geq 2,$$

or, equivalently,

$$c(k-1,m-1) = c(k,m) - c(k-1,m) - c(k-2,m), \quad k \geq 2,$$

with $c(2,0) = 1$ and $c(k,m) = 0$ in the case that $k < 2$, $m < 0$ or $m > k-2$. This implies that the generating function for the sequence $\{c(k,m)\}_{k\geq 0}$ (for each $m \geq 0$) is

$$\sum_{n\geq 0} c(k,m) x^k = \frac{x^{m+2}}{(1-x-x^2)^{m+1}}.$$

It follows that the coefficients $c(k,m) = a(k,m)$ in Equation (3.5) are convolved Fibonacci numbers [**17**] forming the so-called skew Fibonacci-Pascal triangle in the matrix $\mathbf{A} = (a(k,m))_{k,m\geq 0}$. In particular, the sequence of nonzero entries in column $m \geq 0$ of $\mathbf{A}$ is the m^{th} convolution of the sequence $\{F_n\}_{n\geq 0}$.

Combining the expansion of $f(n,n)$ from Equation (3.5) with Equation (3.2), we obtain

$$f(n) = \sum_{m=0}^{n-2} a(n,m) f(m) + F_{n-1},$$

which is equivalent to the matrix equation $X = \mathbf{A}X + F$. Since $\mathbf{I} - \mathbf{A}$ is clearly invertible, the result follows. □

Due to space restrictions, we omit a direct bijective proof of Theorem 3.10 that will be included in the full article.

REMARK 3.11. Note that $\mathbf{A}$ is a strictly lower triangular matrix with zero sub-diagonal. From this it follows that multiplication of a matrix $\mathbf{B}$ by $\mathbf{A}$ shifts the position of the highest nonzero diagonal in $\mathbf{B}$ down by two rows, so $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{n\geq 0} \mathbf{A}^n$ as a Neumann series, and thus all nonzero entries of $(\mathbf{I} - \mathbf{A})^{-1}$ are positive integers.

Finally, one can explicitly compute

$$\mathbf{A} = \begin{bmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 1 & 1 & 0 & 0 & & & & & \\ 2 & 2 & 1 & 0 & 0 & & & & \\ 3 & 5 & 3 & 1 & 0 & 0 & & & \\ 5 & 10 & 9 & 4 & 1 & 0 & 0 & & \\ 8 & 20 & 22 & 14 & 5 & 1 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \implies (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ 1 & 1 & 0 & 1 & & & & & \\ 3 & 2 & 1 & 0 & 1 & & & & \\ 7 & 6 & 3 & 1 & 0 & 1 & & & \\ 21 & 16 & 10 & 4 & 1 & 0 & 1 & & \\ 66 & 50 & 30 & 15 & 5 & 1 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

from which the first few values of the sequence $\{f(n)\}_{n\geq 0}$ are immediately calculable as

$$1, 1, 2, 4, 9, 23, 66, 209, 718, 2645, 10373, 43090, 188803, 869191, 4189511, 21077302, 110389321 \ldots .$$

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