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Flag Vectors of Polytopes: An Overview

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A convex polytope is the convex hull of a finite set of points in $\mathbf{R}^d$. A d-dimensional polytope has faces of dimension 0 through $d-1$; each face is itself a convex polytope. The faces (along with $\emptyset$ and P itself), ordered by inclusion, form a lattice. This talk is concerned with a study of the face lattices of convex polytopes.

Of historical importance is the problem of characterizing the face vectors of polytopes; these vectors give the number of faces of each dimension. The characterization of face vectors of 3-dimensional polytopes was done by Steinitz a century ago. For 4-dimensional polytopes the problem is still open. The biggest advance since Steinitz was the characterization of face vectors of simplicial polytopes (where all faces are simplices) by Stanley, and Billera and Lee in 1980.

The face vector is apparently not robust enough for attempts at characterization by combinatorial and algebraic techniques. We turn instead to the *flag vector* of a polytope. For a d-dimensional polytope P, and $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{0, 1, \ldots, d-1\}$, $f_S(P)$ is the number of chains of faces $\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$ with $\dim F_i = s_i$. The *flag vector* of P is the length 2^d vector $(f_S(P))_{S \subset \{0,1,\ldots,d-1\}}$. In the cases of 3-dimensional polytopes and simplicial polytopes the flag vector is determined linearly by the face vector; in general it can be viewed as an extension of the face vector.

Richard Stanley (1979) studied flag vectors of Cohen-Macaulay posets, a class that contains face lattices of convex polytopes. Bayer and Billera (1985) proved the generalized Dehn-Sommerville equations, the complete set of linear equations satisfied by the flag vectors of all convex polytopes. Since then a wide variety of approaches have been used in the study of flag vectors.

A crucial ingredient in the characterization of face vectors of simplicial polytopes is the connection with toric varieties. In the nonsimplicial case, the middle perversity intersection homology of the toric variety gives an h-vector, linearly dependent on the flag vector. Results from algebraic geometry translate into linear inequalities on the flag vector (Stanley 1987). Another main source of linear inequalities is the cd-index of a polytope, discovered by Jonathan Fine (1985). The cd-index is a vector linearly equivalent to the flag vector; it can be viewed as a reduction of the flag vector by the generalized Dehn-Sommerville equations.

Rigidity theory, shellings, and co-algebras have been used to generate inequalities on flag vectors of polytopes. The talk will survey results and highlight techniques. Some results pertain to special classes of polytopes, such as cubical polytopes and zonotopes. Others hold for more general classes of combinatorial objects, such as general graded posets, Eulerian posets, and Gorenstein* lattices.

We are still, apparently, far from a characterization of flag vectors of polytopes. In fact, we do not even know if the closed convex cone of flag vectors is finitely generated. This is an area of active research. It has exposed interesting connections with other areas of combinatorics, algebra and geometry.

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