Structure relation and raising/lowering operators for orthogonal polynomials

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Classical orthogonal polynomials

Orthogonal polynomials $\{p_n(x)\}$:
three-term recurrence relation

$$x\,p_n(x) = A_n\,p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x).$$

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- derivative again OP
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These three properties are generated by a pair of
shift operators:
One lowers the degree and raises the parameters.
The other raises the degree and lowers the parameters.

Example

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, orthogonal with respect to $(1-x)^\alpha(1+x)^\beta\,dx$ on $(-1,1)$.

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$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \mu_n\,P_{n-1}^{(\alpha+1,\beta+1)}(x), \qquad (1)$$

$$(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}(1-x)^{\alpha+1}(1+x)^{\beta+1}\,P_{n-1}^{(\alpha+1,\beta+1)}(x) = \nu_n\,P_n^{(\alpha,\beta)}(x). \qquad (2)$$

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- *eigenfunctions of 2nd order differential operator*: compose (1) and (2).
- *derivative again OP*: by (1).
- *Rodrigues formula*: iterate (2).

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- *Rodrigues formula* : iterate (2).

Wanted: operators lowering or raising degree without parameter shift.

structure relation

The classical OP's $\{p_n(x)\}$ satisfy:

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- structure relation ($\pi(x)$ polynomial of degree ≤ 2)

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$$\pi(x)p_n'(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \qquad (4)$$

- lowering relation

$$\pi(x)\,p_n'(x) - (\alpha_n x + \beta_n)p_n(x) = \gamma_n p_{n-1}(x), \qquad (5)$$

- raising relation

$$\pi(x)\,p_n'(x) - (\tilde\alpha_n x + \tilde\beta_n)p_n(x) = \tilde\gamma_n p_{n+1}(x). \qquad (6)$$

(5) and (6) obtained by eliminating a term from (3) and (4). However, lowering and raising operators dependent on n.

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structure relation, continued

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Semi-classical orthogonal polynomials are OP's $\{p_n(x)\}$ which satisfy the more general structure relation

$$\pi(x)\,p_n'(x) = \sum_{j=n-s}^{n+t} a_{n,j}p_j(x)$$

($\pi(x)$ a polynomial; s, t independent of n).

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and being eigenfunctions of some explicit operator D, symmetric w.r.t. $d\mu$: $\quad Dp_n = \lambda_n p_n. \quad$ Put $\gamma_n := \lambda_{n+1} - \lambda_n$.

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Definition

structure operator $L := [D, X] = DX - XD$.

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Theorem

L is skew-symmetric w.r.t. $d\mu$, and we have structure equation

$$Lp_n = \gamma_n A_n p_{n+1} - \gamma_{n-1} C_n p_{n-1}.$$

Lowering and raising relations

By elimination of term from

$$Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1},$$
$$Lp_n = \gamma_n A_n p_{n+1} - \gamma_{n-1} C_n p_{n-1},$$

we get a lowering and raising relation:

$$-\gamma_n (x - B_n) p_n(x) + (Lp_n)(x) = -(\gamma_n + \gamma_{n-1}) C_n p_{n-1}(x),$$
$$\gamma_{n-1}(x - B_n) p_n(x) + (Lp_n)(x) = (\gamma_n + \gamma_{n-1}) A_n p_{n+1}(x).$$

Example, Hermite

Example

Hermite polynomials $H_n(x)$,
orthogonal w.r.t. $e^{-x^2}\,dx$ on $(-\infty,\infty)$.

$$
\begin{aligned}
(DH_n)(x) &:= \tfrac{1}{2}H_n''(x) - x\,H_n'(x) = -n\,H_n(x),\\
X\,H_n &= \tfrac{1}{2}H_{n+1} + nH_{n-1},
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$$(LH_n)(x) := ([D,X]H_n)(x) =$$
$$H_n'(x) - x\,H_n(x) = -\tfrac{1}{2}H_{n+1}(x) + nH_{n-1}(x).$$

Example, Laguerre

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Laguerre polynomials $L_n^\alpha(x)$,
orthogonal w.r.t. $e^{-x}x^\alpha\,dx$ on $(0,\infty)$.

$$\begin{aligned}(DL_n^\alpha)(x) :=& x\frac{d^2}{dx^2}L_n^\alpha(x)+(\alpha+1-x)\frac{d}{dx}\,L_n^\alpha(x)=-n\,L_n^\alpha(x),\\ X\,L_n^\alpha =&(n+1)L_{n+1}^\alpha+(2n+\alpha+1)L_n^\alpha+(n+\alpha)L_{n-1}^\alpha,\end{aligned}$$

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\end{aligned}
$$

$$
\begin{aligned}
(L\,L_n^\alpha)(x) :=& ([D,X]L_n^\alpha)(x)\\
=& 2x\,\frac{d}{dx}L_n^\alpha(x) + (\alpha+1-x)\,L_n^\alpha(x)\\
=& -(n+1)L_{n+1}^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x).
\end{aligned}
$$

Example

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

$$D := \tfrac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \tfrac{1}{2}\big(\beta-\alpha-(\alpha+\beta+2)x\big)\frac{d}{dx},$$

$$\lambda_n = -\tfrac{1}{2}n(n+\alpha+\beta+1), \quad \gamma_n = -\tfrac{1}{2}(2n+\alpha+\beta+2).$$

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Structure operator:

$$(Lf)(x) := (1-x^2)f'(x) - \tfrac{1}{2}\big(\alpha-\beta+(\alpha+\beta+2)x\big)f(x)$$

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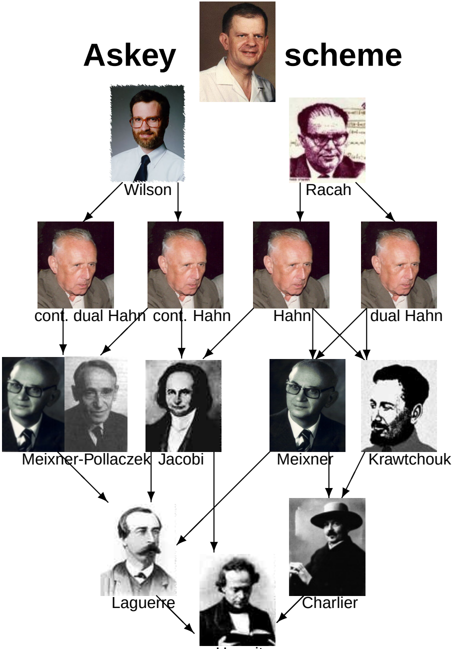
Structure relation:

$$\Big((1-x^2)\frac{d}{dx} - \tfrac{1}{2}\big(\alpha-\beta+(\alpha+\beta+2)x\big)\Big)P_n^{(\alpha,\beta)}(x) =$$
$$-\frac{(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1}P_{n+1}^{(\alpha,\beta)}(x) + \frac{(n+\alpha)(n+\beta)}{2n+\alpha+\beta+1}P_{n-1}^{(\alpha,\beta)}(x).$$

Askey scheme

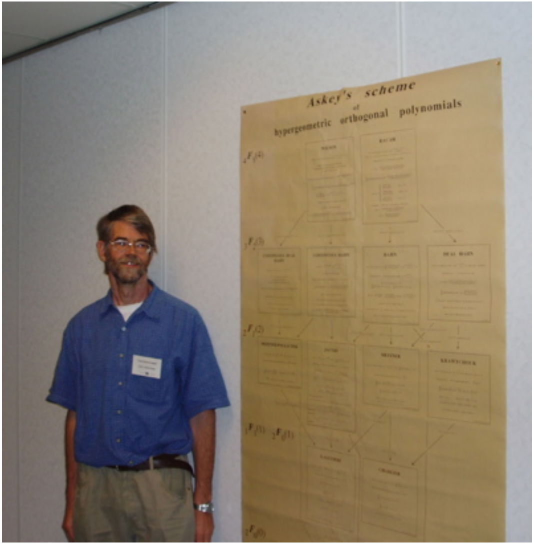


There was in 1977 a meeting at Oberwolfach on combinatorics run by Foata. I gave a talk about many of classical type orthogonal polynomials and it fell flat. Few there appreciated it.

Later in the week Mike Hoare, a physicist then at Bedford College, talked about some very nice work he and Mizan Rahman had done. In this talk he had an overhead of the polynomials they had dealt with, starting with Hahn polynomials at the top and moving down to limiting cases with arrows illustrating the limits which they had used. The audience did not seem to care much about the probability problem, but they were very excited about the chart he had shown and wanted copies. If there was that much interest in his chart, I thought that it should be extended to include all of the classical type polynomials which had been found.

Askey in Oberwolfach, 1977

Jacques Labelle's Askey tableau poster



Eigenfunctions of structure operator

OP's $\{p_n(x)\}$, $\int_a^b p_n(x)\,p_m(x)\,d\mu(x) = \omega_n^{-1}\,\delta_{n,m}$.

Write structure relation as

$$L_x(p_n(x)) = M_n(p_n(x)),$$

L_x skew symmetric operator on $L^2([a,b], d\mu)$, M_n skew symmetric operator on $l^2(\mathbb{N}, \omega_n)$.

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Formally we expect eigenfunctions $\phi_\lambda(x)$ of L_x and $q_n(\lambda)$ of M_n:

$$L_x(\phi_\lambda(x)) = i\lambda\phi_\lambda(x), \quad M_n(q_n(\lambda)) = i\lambda\,q_n(\lambda) \quad (q_n/q_0 \text{ OP's in } \lambda)$$

such that

$$\int_a^b p_n(x)\,\phi_{-\lambda}(x)\,d\mu(x) = q_n(\lambda).$$

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- Laguerre: $\phi_\lambda(x) = e^{\frac{1}{2}x} x^{\frac{1}{2}(i\lambda-\alpha-1)}$,
  $q_n(\lambda) = i^{-n} 2^{\frac{1}{2}(\alpha+1-i\lambda)} \Gamma(\frac{1}{2}(\alpha+1-i\lambda)) P_n^{(\frac{1}{2}\alpha+\frac{1}{2})}(\frac{1}{2}\lambda; \frac{1}{2}\pi)$
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- Jacobi:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)(1-x)^{\frac{1}{2}(\alpha-1+i\lambda}(1+x)^{\frac{1}{2}(\beta-1-i\lambda)}\,dx$$

$$= \text{stuff} \times p_n(\tfrac{1}{2}\lambda; \tfrac{\alpha+1}{2}, \tfrac{\beta+1}{2}, \tfrac{\alpha+1}{2}, \tfrac{\beta+1}{2})$$

(continuous Hahn)

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Possibly related to K (LNM 1171, 1985) and Groenevelt (IMRN, 2003).

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with Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$ on top.

Askey-Wilson polynomials

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with Askey-Wilson polynomials $p_n(x\,;a,b,c,d\mid q)$ on top.

$$p_n[z] = p_n\big(\tfrac{1}{2}(z+z^{-1})\big)$$
$$:= \frac{(ab,ac,ad\,;q)_n}{a^n}\; {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

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Orthogonal w.r.t. inner product

$$\langle f, g\rangle := \frac{1}{4\pi i} \oint_C f[z]\, g[z]\, w(z)\, \frac{dz}{z},$$
$$w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty},$$

where C is unit circle traversed in positive direction.

Askey-Wilson polynomials, continued

Second order q-difference operator D:

$$Dp_n = \lambda_n p_n, \quad \text{where}$$

$$\tfrac{1}{2}(1-q^{-1})(Df)[z] = v(z)\,f[qz] - \big(v(z)+v(z^{-1})\big)f[z] + v(z^{-1})\,f[q^{-1}z],$$

$$v(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)},$$

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Structure operator $L := [D, X]$, where $(Xf)[z] := \frac{1}{2}(z+z^{-1})f[z]$:

$$(Lf)[z] = \frac{u[z]\,f[qz] - u[z^{-1}]\,f[q^{-1}z]}{z-z^{-1}}, \quad \text{where}$$

$$u[z] = (1-az)(1-bz)(1-cz)(1-dz)\,z^{-2}.$$

Zhedanov's algebra $AW(3)$

Zhedanov (1991),
"Hidden symmetry" of Askey-Wilson polynomials.
Defines algebra $AW(3)$ with generators K_0, K_1, K_2 and relations

$$[K_0, K_1]_q = K_2,$$
$$[K_1, K_2]_q = C_0 K_0 + B K_1 + D_0,$$
$$[K_2, K_0]_q = B K_0 + C_1 K_1 + D_1,$$

with q-commutator $[X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$
and with structure constants B, C_0, D_0, C_1, D_1 .

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Constants can be chosen such that relations are realized in terms of operators D and X for Askey-Wilson polynomials:

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Constants can be chosen such that relations are realized in terms of operators D and X for Askey-Wilson polynomials:

$$K_0 = \tfrac{1}{2}(1 - q^{-1})D + 1 + q^{-1}abcd, \qquad K_1 = X.$$

Then K_2 is q-structure operator.

$AW(3)$: q-structure relation

Suppose that $AW(3)$ acts on a vector space spanned by one-dimensional eigenspaces of K_0: $\quad K_0\psi_n = \lambda_n\psi_n$.

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Then for each ψ_n there are neighbouring eigenvectors ψ_{n-1}, ψ_{n+1} such that

$$K_1\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}, \tag{7}$$

$$\begin{aligned} K_2\psi_n = (q^{\frac12}\lambda_{n+1} - q^{-\frac12}\lambda_n)a_n\psi_{n+1} + (q^{\frac12} - q^{-\frac12})\lambda_n b_n\psi_n \\ + (q^{\frac12}\lambda_{n-1} - q^{-\frac12}\lambda_n)c_n\psi_{n-1}, \end{aligned} \tag{8}$$

and $\lambda_{n+1} + \lambda_{n-1} = (q+q^{-1})\lambda_n$,
and raising and lowering relations by elimination from (7), (8).

Suppose that $AW(3)$ acts on a vector space spanned by one-dimensional eigenspaces of K_0: $\quad K_0\psi_n = \lambda_n\psi_n$.

Then for each ψ_n there are neighbouring eigenvectors ψ_{n-1}, ψ_{n+1} such that

$$K_1\psi_n = a_n\psi_{n+1} + b_n\psi_n + c_n\psi_{n-1}, \tag{7}$$

$$\begin{aligned} K_2\psi_n &= (q^{\frac{1}{2}}\lambda_{n+1} - q^{-\frac{1}{2}}\lambda_n)a_n\psi_{n+1} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\lambda_n b_n\psi_n \\ &\qquad + (q^{\frac{1}{2}}\lambda_{n-1} - q^{-\frac{1}{2}}\lambda_n)c_n\psi_{n-1}, \end{aligned} \tag{8}$$

and $\lambda_{n+1} + \lambda_{n-1} = (q + q^{-1})\lambda_n$,
and raising and lowering relations by elimination from (7), (8).

In the Askey-Wilson realization of $AW(3)$ (7) is three-term recurrence relation and (8) is q-structure relation.

string equation

If we take Askey-Wilson realization of $AW(3)$ with $K_1 = X$ but $K_0 = \text{const.}\, D$ instead of with special constant term added, and $K_2 := [K_0, K_1]_q$, then quadratic term occurs in right-hand side of second relation:

$[K_1, K_2]_q$ is linear combination of D, X, X^2 and 1.

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In this form $q = 1$ limit of $AW(3)$ possible with realization by Jacobi, etc.. Then string equation:

$[L, X] = 1, X, 1 - X^2$ for resp. Hermite, Laguerre, Jacobi
(Adler & van Moerbeke).
In Hermite case related to matrix models in quantum gravity
(Witten, 1991).

Matrix models

$$V(x) := \sum_{j=1}^{r} t_j x^j .$$

$$Z := \int_{\mathcal{M}_n} e^{-\mathrm{tr}\, V(M)}\, dM = \mathrm{const.} \int_{\mathbb{R}^n} \prod_{i<j} (x_i - x_j)^2 \prod_i e^{-V(x_i)}\, dx_1 \ldots dx_n .$$

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Suppose $\{p_n\}$ OP's on $L^2(d\mu)$ and L skew symmetric operator on $L^2(d\mu)$ with $Lp_n = a_n p_{n+1} - c_n p_{n-1}$ and $[L, X] = \pi(X)$ (π polynomial).
Let $p_n^{(t)}$ OP's on $L^2(e^{-V(x)}\, d\mu(x))$. Then
$L^{(t)} := L - \frac{1}{2} \sum_{j \geq 1} j t_j x^{j-1}$ skew-symmetric with respect to $e^{-V(x)}\, d\mu(x)$ and $L^{(t)} p_n^{(t)} \in \mathrm{Span}\{\mathrm{p}^{(t)}_{n-r+1}, \ldots, \mathrm{p}^{(t)}_{n+r-1}\}$ and $[L^{(t)}, X] = \pi(X)$ (string equation).

Macdonald polynomials

Macdonald polynomial $P_\lambda(x;q,t)$, root system A_{n-1},
$x=(x_1,\ldots,x_n)$, $\quad \lambda=(\lambda_1,\ldots,\lambda_n)$ partition,
P_λ symmetric polynomial in x, homogeneous of degree $|\lambda|$.

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Eigenfunction of q-difference operators D_r ($r=0,1,\ldots,n$):

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$$D_r = t^{\frac{1}{2}r(r-1)} \sum_{|I|=r} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,i},$$

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Elementary cases: $D_0 f=f$, $(D_n f)(x)=t^{\frac{1}{2}n(n-1)}f(qx)$.

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Pieri formula:

$$e_r(x)\tilde{P}_\lambda(x) = \sum_{\substack{|\theta|=r,\ 0\le\theta_j\le 1\\ \lambda+\theta \text{ partition}}} t^{\theta_2+2\theta_3+\ldots+(n-1)\theta_n} \times \prod_{1\le i<j\le n} \frac{1-q^{\lambda_i-\lambda_j} t^{j-i+\theta_i-\theta_j}}{1-q^{\lambda_i-\lambda_j} t^{j-i}} \tilde{P}_{\lambda+\theta}(x).$$

Structure relations

"Casimir" Pieri formula (Pieri for $r = 1$):

$$(x_1 + \cdots + x_n)\tilde{P}_\lambda(x) = \sum_{\substack{k=1,\ldots,n \\ \lambda+\varepsilon_k \text{ partition}}} A_{\lambda,k}\, \tilde{P}_{\lambda+\varepsilon_k}(x). \tag{9}$$

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$$[D_r, X_1 + \cdots + X_n]\tilde{P}_\lambda$$
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These n equations (9), (10), if linearly independent, will yield by elimination raising relations. $\tilde{P}_\lambda \to \tilde{P}_{\lambda+\varepsilon_k}$.
Can be done explicitly analogous to work in Jack case by García Fuertes, Lorente & Perelomov (2001).

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$$= \sum_{\substack{k=1,\ldots,n\\ \lambda+\varepsilon_k \text{ partition}}} A_{\lambda,k} \prod_{i=1}^{n}(1 + wt^{n-i}q^{\lambda_i+\delta_{i,k}})\tilde{P}_{\lambda+\varepsilon_k}$$

$(w := -t^{j-n}q^{-\lambda_j})$

$$= A_{\lambda,j}(1-q)\prod_{i\neq j}(1 - q^{\lambda_i-\lambda_j}t^{j-i})\,\tilde{P}_{\lambda+\varepsilon_j} .$$

Explicit raising relations, continued

Final form of raising relations:

$$\sum_{r=1}^{n-1}(-t^{j-n}q^{-\lambda_j})^r\,[D_r, X_1+\cdots+X_n]\,\tilde P_\lambda$$
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Similar results with case $e_{n-1}(X)\tilde{P}_\lambda$ of Pieri relations.
Use $P_{\lambda_1+1,\ldots,\lambda_n+1}(x)=x_1x_2\ldots x_n\,P_\lambda(x)$. Then

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Hence further structure relations and further explicit lowering relations $\tilde{P}_\lambda\to\tilde{P}_{\lambda-\varepsilon_k}$.

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Hence further structure relations and further explicit lowering relations $\tilde P_\lambda\to\tilde P_{\lambda-\varepsilon_k}$.
More generally, explicit lowering and raising relations $\tilde P_\lambda\to\tilde P_{\lambda\pm\varepsilon_I}$, $\quad\varepsilon_I:=\sum_{i\in I}\varepsilon_i$.

Lapointe-Vinet / Kirillov-Noumi lowering/raising operators

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Compare with Lapointe & Vinet (LMP, 1997; Adv. Math, 1997) and A. N. Kirillov & Noumi (1998, 1999):
Explicit raising and lowering operators independent of λ, acting on integral form J_λ of Macdonald polynomials:

$$K_m\, J_{\lambda_1,\lambda_2,\ldots,\lambda_m,0,\ldots,0} = J_{\lambda_1+1,\ldots,\lambda_m+1,0,\ldots,0} \quad (m = 1, 2, \ldots, n),$$

$$M_m\, J_{\lambda_1,\lambda_2,\ldots,\lambda_m,0,\ldots,0} = c_{\lambda,m} J_{\lambda_1-1,\ldots,\lambda_m-1,0,\ldots,0} \quad (m = 1, 2, \ldots, n).$$

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These operators are given in several forms. The form of the operators which leads to a proof of part of the Macdonald conjectures for the double Kostka coefficients, does not seem to be expressible in terms of structure operators (except for $m = 1$).

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However, Lapointe & Vinet (LMP, 1997) have the operators K_m in a form which fits into the present scheme.

Lapointe-Vinet creation operators

Observe:

$$D(w)\Big(e_m(X)\tilde{P}_{\lambda_1,\ldots,\lambda_m,0,\ldots,0}\Big) = \sum_{\substack{|\theta|=m,\, 0\le\theta_j\le 1\\ \lambda+\theta \text{ partition}}} A_{\lambda,\theta}\, D(w)\,\tilde{P}_{\lambda+\theta}.$$

There is a term with $\theta = (1^m)$ and in all other terms $\theta_{m+1} = 1$, and

$$D(w)\,\tilde{P}_{\lambda+\theta} = \Big(\prod_{i=1}^{n}(1 + wt^{n-i}q^{\lambda_i+\theta_i}\Big)\tilde{P}_{\lambda+\theta},$$

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$$D(-q^{-1}t^{m-n+1})\Big(e_m(X)\tilde{P}_{\lambda_1,\ldots,\lambda_m,0,\ldots,0}\Big) = \text{const.}\, P_{\lambda_1+1,\ldots,\lambda_m+1,0,\ldots,0}\,.$$

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The operator $D(-q^{-1}t^{m-n+1})\circ e_m(X)$ only depends on m: it is the same operator for all λ of length $\le m$.