The Incidence Algebra of a Composition Poset

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
www.math.msu.edu/~sagan

June 18, 2006

Compositions

Rational generating functions

Commuting variables

The zeta and Möbius functions

Comments and open problems

Outline

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Question: Is this an isolated incident or part of a larger picture?

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Given $u \leq w$ there is a unique *rightmost embedding, I,* such that $I \geq I'$ componentwise for all embeddings I'. The embedding above is rightmost.

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$$
\begin{array}{c}
1 \\
| \\
\epsilon
\end{array}
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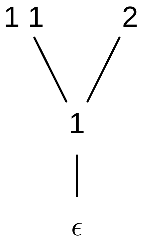

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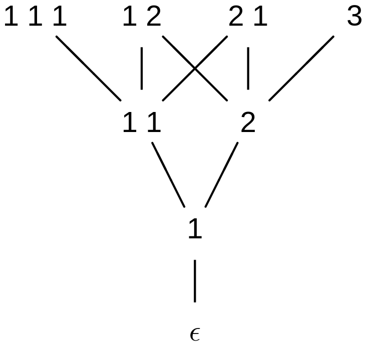

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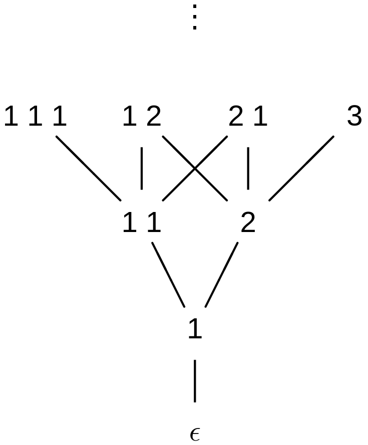

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Convention: If $S \subseteq A$, then we also let S stand for $\sum_{s\in S} s$.

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Now if $u = \bar{k}_1 \ldots \bar{k}_r$ then

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$$Z(u) = [\bar{n}]^* z(\bar{k}_1) \cdots z(\bar{k}_r) \quad \text{with} \quad z(\bar{k}) = [\bar{k}, \bar{n}][\,\overline{k-1}\,]^*.$$

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Corollary (B & S)

If $u \in [\bar{n}]^*$ has $t(u) = (t_1, \ldots, t_n)$ then

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since $t(\epsilon) = (0, 0, \ldots)$.

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Outline

Compositions

Rational generating functions

Commuting variables

The zeta and Möbius functions

Comments and open problems

The *incidence algebra* of poset P over the rationals $\mathbb{Q}$ is

$$I(P) = \{\phi : P \times P \to \mathbb{Q} \ : \ \phi(u,w) = 0 \text{ if } u \not\leq w\}.$$

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Question: What is μ in composition order on $\mathbb{P}^*$? We first discuss μ in subword order on A^*.

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A *run* of k's in $w = k_1 \ldots k_t$ is a maximal interval $[r, s]$ with $k_r = k_{r+1} = \ldots = k_s$. An embedding η_u in w is *normal* if, for each k and every run $[r, s]$ of k's in w,

$$(r, s] \subseteq \mathrm{Supp}\,\eta_u.$$

Theorem (Björner)

In A^*: $\mu(u, w) = (-1)^{\#w - \#u}(\#\text{ of normal } \eta_u \text{ in } w)$. ■

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So $\mu(u, w) = (-1)^{8-4}\, 2 \ = \ 2$.

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So $\mu(u, w) = (-1)^2 + (-1)^0 = 2$.

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3. (Björner & S) using formal power series in noncommuting variables.

Outline

1. Is there a bijective proof that the norm generating function for compositions only depends on type? That is, given $u, u' \in \mathbb{P}^*$ with $t(u) = t(u')$, find a norm-preserving bijection

$$\{w \ : \ w \geq u\} \leftrightarrow \{w \ : \ w \geq u'\}.$$

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2. Björner and Reutenauer gave generating functions for the powers ζ^m for $m \geq 1$ in subword order on A^*. Björner and S were only able to do this for composition order on $[2]^*$, and the proof involved hypergeometric series identities. What can be said for $[n]^*$?

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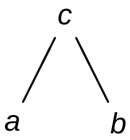
$\Lambda =$ 

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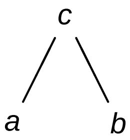
$\Lambda =$ 

Conjecture (Sagan & V)

For all $i \leq j$, the value $\mu(a^i, c^j)$ is the coefficient of x^{j-i} in $T_{i+j}(x)$, the Tchebyshev polynomial of the first kind.

THANKS FOR LISTENING!