Introduction to Symmetric Functions
Chapter 2

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Abstract. A development of the symmetric functions using the plethystic notation.

CHAPTER 3

The algebra structure of the ring of symmetric functions

Consider the polynomial ring in the variables p_i for $i \geq 1$, $\Lambda = \mathbb{Q}[p_1, p_2, p_3, \ldots]$. We will define the degree of a variable p_k to be k and so the degree of a monomial $p_{k_1} p_{k_2} \cdots p_{k_\ell}$ is simply $k_1 + k_2 + \cdots + k_\ell$. Λ is the ring of symmetric functions.

This is a very abstract way to begin, but at the end of this chapter we will draw a connection between this algebra and the space of class functions of the symmetric group. From this perspective Λ can be seen as an infinite dimensional graded vector space where the symmetric functions of degree m are a finite dimensional subspace.

The elements p_i are refered to as the power generators and since we are considering them as the variables in a commutative polynomial ring the space is spanned by the monomials in these variables. To specify a basis of this space we may assume that the variables are listed in weakly decreasing order. That is, if we denote Λ_m by the symmetric functions of degree m, then the set $\{p_\lambda : \lambda \vdash m\}$ forms a basis for Λ_m where $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$. The set $\bigoplus_{m \geq 0} \{p_\lambda\}_{\lambda \vdash m}$ is called the power basis. Note that the degree of a monomial p_λ is given by $|\lambda|$.

Λ has a natural 'un-multiplication' operation called a coproduct. In the sense if the product represents a way of putting elements in the algebra together, the coproduct represents ways of pulling elements apart. This can be a very interesting operation, especially when the multiplication and comultiplication interact.

Formally, we define the multiplication function on this algebra $\mu : \Lambda \otimes \Lambda \longrightarrow \Lambda$ as

$$\mu(p_\lambda \otimes p_\mu) = p_\lambda p_\mu = p_{\lambda_1} \cdots p_{\ell(\lambda)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}} = p_{\lambda \uplus \mu}. \tag{3.1}$$

The comultiplication will be denoted as $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$ is given by $\Delta(p_k) = 1 \otimes p_k + p_k \otimes 1$ (this is the property that the power generators are primitive in this algebra). We impose that it is a ring homomorphism, that is for $f, g \in \Lambda$, $\Delta(fg) = \Delta(f)\Delta(g)$ and and $c, d \in \mathbb{Q}$, $\Delta(cf + dg) = c\Delta(f) + d\Delta(g)$. Hence for an arbitrary basis element we have

Proposition 3.1.

$$\Delta(p_\lambda) = \sum_{\mu \uplus \nu = \lambda} \prod_{i=1}^{\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu \otimes p_\nu = \sum_{\mu \uplus \nu = \lambda} \frac{z_\lambda}{z_\mu z_\nu} p_\mu \otimes p_\nu \tag{3.2}$$

By rearranging the coefficients of (3.2) we can see a natural basis to consider is p_λ/z_λ because it arises naturally in this formula in the sense that

$$\Delta\left(\frac{p_\lambda}{z_\lambda}\right) = \sum_{\mu\uplus\nu=\lambda} \frac{p_\mu}{z_\mu}\otimes\frac{p_\nu}{z_\nu}. \tag{3.3}$$

PROOF. First we note that the coefficients in the two formulations of equation (3.2) are equal since

$$\frac{z_\lambda}{z_\mu z_\nu} = \prod_{i=1}^{\lambda_1} \frac{i^{m_i(\lambda)} m_i(\lambda)!}{i^{m_i(\mu)} m_i(\mu)! i^{m_i(\nu)} m_i(\nu)!} = \prod_{i=1}^{\lambda_1} \frac{m_i(\lambda)!}{m_i(\mu)! m_i(\nu)!} = \prod_{i=1}^{\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)}.$$

We show this proposition by induction on the number of parts of λ. We have a base case since the formula clearly works if λ has only one part by definition.

Let λ be a partition of n and denote $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$. It follows that

$$\Delta(p_\lambda) = \Delta(p_{\lambda_1})\Delta(p_{\overline{\lambda}}) = (p_{\lambda_1}\otimes 1 + 1\otimes p_{\lambda_1})\left(\sum_{\mu\uplus\nu=\overline{\lambda}} \prod_{i=1}^{\lambda_2} \binom{m_i(\overline{\lambda})}{m_i(\mu)} p_\mu\otimes p_\nu\right). \tag{3.4}$$

If $\lambda_1 \neq \lambda_2$ there is nothing to prove since $\binom{m_i(\overline{\lambda})}{m_i(\mu)} = \binom{m_i(\lambda)}{m_i(\mu)}$ for all $1 \leq i \leq \lambda_2$ and the expansion of the right hand side is exactly as stated in the proposition since $\binom{m_{\lambda_1}(\lambda)}{m_{\lambda_1}(\mu)} = 1$.

If $\lambda_1 = \lambda_2$ then $m_{\lambda_1}(\overline{\lambda}) = m_{\lambda_1}(\lambda) - 1$ and $m_i(\overline{\lambda}) = m_i(\lambda)$ for $1 \leq i < \lambda_1$, hence we see

$$\begin{aligned}
\Delta(p_{\lambda_1})\Delta(p_{\overline{\lambda}}) &= (p_{\lambda_1}\otimes 1 + 1\otimes p_{\lambda_1})\left(\sum_{j=0}^{m_{\lambda_1}(\overline{\lambda})} \sum_{\substack{\mu\uplus\nu=\overline{\lambda}\\ m_{\lambda_1}(\mu)=j}} \binom{m_{\lambda_1}(\lambda)-1}{j} \prod_{1\leq i<\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu\otimes p_\nu\right) \\
&= \sum_{j=0}^{m_{\lambda_1}(\lambda)} \sum_{\substack{\mu\uplus\nu=\overline{\lambda}\\ m_{\lambda_1}(\mu)=j}} \left(\binom{m_{\lambda_1}(\lambda)-1}{j} + \binom{m_{\lambda_1}(\lambda)-1}{j-1}\right) \prod_{1\leq i<\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu\otimes p_\nu \\
&= \sum_{\mu\uplus\nu=\overline{\lambda}} \binom{m_{\lambda_1}(\lambda)}{m_{\lambda_1}(\mu)} \prod_{1\leq i<\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu\otimes p_\nu.
\end{aligned}$$

And so it follows by induction that this formula holds for all partitions λ. $\square$

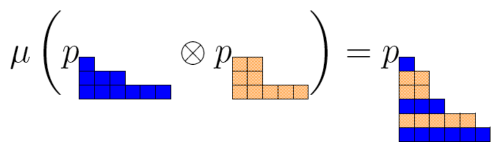
The p_k have the property that $\Delta(p_k) = p_k\otimes 1 + 1\otimes p_k$ and hence are called the primitive elements of this algebra. We remark that there is a map S called the antipode with the property

$$\mu\circ(id\otimes S)\circ\Delta(f) = 0 \tag{3.5}$$

for all $f\in\Lambda$ such that f has 0 constant term. We set $S(p_\lambda) = (-1)^{\ell(\lambda)}p_\lambda$ and extend this map linearly and it is easy to check that $\mu\circ(id\otimes S)\otimes\Delta(p_k) = \mu(p_k\otimes 1 - 1\otimes p_k) = 0$ and similarly for p_λ. This implies that $\mu \circ (id \otimes S) \circ \Delta(f)$ is equal to the constant term of f for all $f \in \Lambda$.

Therefore, so far our algebra of symmetric functions is very simple, but we should develop some intuitive ideas on how to picture what this algebra is. Now if $f \in \Lambda$, then f is some polynomial in variables p_i. Since we are using partitions to index our basis we often just write p_λ when we talk about the basis elements when we consider Λ as a vector space over $\mathbb{Q}$. The indexing set of partitions can be represented by their Young diagrams so when we take the product of p_λ and p_μ, $\mu(p_\lambda \otimes p_\mu)$ represents shuffling the Young digrams together.

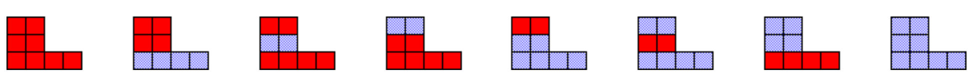
Example 14. Consider for instance $\lambda = (6,3,1)$ and $\mu = (5,2,2)$. This can be represented by the picture



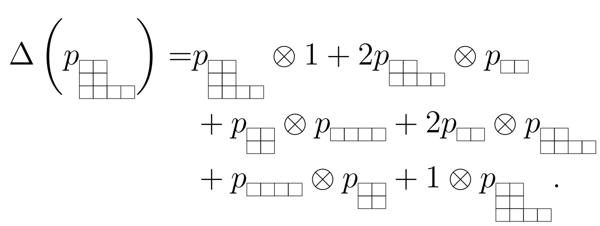
This diagram is simply representing the equation $p_{(6,3,1)}p_{(5,2,2)} = p_{(6,5,3,2,2,1)}$.

We should also develop a combinatorial picture of what happens when Δ acts on a term p_λ. Because the p_k are primitive elements, there will be $2^{\ell(\lambda)}$ terms in the expansion of $\Delta(p_\lambda)$.

Example 15. We compute the action of Δ on $p_{(5,2,2)}$ and do this by computing the number of ways of coloring the rows of the partition $(5,2,2)$ using two colors so that the whole row has the same color. This is represented by the following picture.



The blue rows will be in the left tensor and red will be in the right tensor (although the colors are symmetric) so we have determined



Here we are splitting the partition up into pieces such that their union is the original partition in all possible ways. Notice that the sum of the coefficients in this expression is $8 = 2^{\ell(5,2,2)}$. This picture will help us gain some intuition as we develop this algebra more completely.

Remark 1. The reader is encouraged to try to develop some sort of a picture each time a formula appears in this presentation since the formulas are difficult to appreciate unless some meaning is assigned to the symbols we are working with.

There are generally considered to be 6 'standard' bases of the symmetric functions since these bases are fundamental in the development of tools to describe the calculus of symmetric functions. After the power symmetric basis we will introduce the homogeneous basis and the elementary basis as these are defined as products of generators. We will save the definition of the Schur basis and the monomial basis for a later section. [1]

For $n > 0$, we set

$$h_n = \sum_{\lambda \vdash n} p_\lambda / z_\lambda, \tag{3.6}$$

these are the homogeneous generators. We also define

$$e_n = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} p_\lambda / z_\lambda \tag{3.7}$$

which are the elementary generators. For a partition λ we set $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\ell(\lambda)}$ and $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\ell(\lambda)}$. Clearly we have the triangularity relations that $h_\lambda = p_\lambda / \prod_{i=1}^{\ell(\lambda)} \lambda_i +$ terms containing p_μ with μ smaller than λ in lexicographic order (and a similar relation with e_λ). This implies that $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$ are bases for the symmetric functions and that $\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots] = \mathbb{Q}[e_1, e_2, e_3, \ldots]$. Also set as a convention $p_0 = h_0 = e_0 = 1$ and $p_{-n} = h_{-n} = e_{-n} = 0$ for $n > 0$, so that formulas which require us to refer to these elements make sense.

There are several ways of picturing what the elements h_n and e_n represent. In some sense, $n!h_n$ is the generating function of all permutations of the symmetric group Sym_n with weight 1 for each element which we can see in the following formula.

$$n!h_n = \sum_{\sigma \in Sym_n} p_{\lambda(\sigma)} = \sum_{\lambda \vdash n} \cdot \frac{n!}{z_\lambda} p_\lambda. \tag{3.8}$$

At the same time $n!e_n$ is a signed generating function with weight equal to $(-1)^{n-\ell(\lambda)}$ if the permutation has cycle type λ.

$$n!e_n = \sum_{\sigma \in Sym_n} \epsilon(\sigma) p_{\lambda(\sigma)}. \tag{3.9}$$

We will see when we introduce the Schur functions that these formulas are a special case of one where the elements of Λ are generating functions for the irreducible characters of the symmetric group and h_n is representing the trivial character and e_n is representing the sign character.

[1] There is another basis which is typically called the forgotten basis which completes the analogy, 'the homogeneous basis is to the monomial basis as the elementary basis is to the (um, I forget) basis.' There are few direct formulas for the forgotten basis except for those which are analogous to those for the monomial basis and hence remains somewhat underdeveloped in our account.

Set $P(t) = \sum_{r\geq 1} p_r t^r/r$ as a generating function for the power generators and set $H(t) = exp(P(t))$. Notice by the following calculation we have

$$\begin{aligned} H(t) &= exp\left(\sum_{r\geq 1} p_r t^r/r\right) = \prod_{r\geq 1} exp(p_r t^r/r) \\ &= \prod_{r\geq 1}\sum_{n\geq 0}\frac{p_r^n}{r^n n!}t^{nr} \\ &= \sum_{k\geq 0}\sum_{\lambda\vdash k}\frac{p_\lambda}{z_\lambda}t^k = \sum_{k\geq 0} h_k t^k. \end{aligned} \tag{3.10}$$

Similarly we may easily show that $E(t) = exp(-P(t)) = \sum_{n\geq 0}(-1)^n e_n t^n$. Simply by definition of these generating functions we have the relation

$$H(t)E(t) = exp(P(t))exp(-P(t)) = 1 \tag{3.11}$$

We can also consider the product of these generating functions explicitly and take the coefficient of t^n. On the right hand side the coefficient is 0 as long as $n > 0$ and the coefficient on the left hand side shows that

$$\sum_{k=0}^{n}(-1)^k h_{n-k}e_k = 0. \tag{3.12}$$

Define the ring homomorphism on Λ that sends $\omega(p_k) = (-1)^{k-1}p_k$. Clearly, ω is an involution and is related to the antipode map on Λ by $\omega(p_\lambda) = (-1)^{|\lambda|}S(p_\lambda)$. By going back to formulas (3.6) and (3.7) for h_k and e_k in terms of p_λ we see that ω relates the $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$ bases by $\omega(h_\lambda) = e_\lambda$.

By exploiting the generating functions $P(t), H(t)$ and $E(t)$ further we can extract other algebraic relations between the elements of this ring. For instance, notice that $P(t) = log(H(t))$ and hence $P'(t) = H'(t)/H(t)$. Therefore by taking the coefficient of t^{n-1} in $P'(t)H(t) = H'(t)$ we see that

$$nh_n = \sum_{k=1}^{n} h_{n-k}p_k. \tag{3.13}$$

By an application of ω on each side of this equation we also see that

$$ne_n = \sum_{k=1}^{n}(-1)^{k-1}e_{n-k}p_k \tag{3.14}$$

Equations (3.12), (3.13) and (3.14) give us a simple recursive method to express any of the algebraic generators of this space in terms of any other because the term containing h_n, e_n or p_n can be isolated to provide algebraic relations. These recursive definitions will be exactly the method that we use when we develop computer functions in Maple to change between bases.

EXAMPLE 16. If we wish to expand h_3 in the elementary basis we note that $h_3 = h_2e_1 - h_1e_2 + e_3$, $h_2 = h_1e_1 - e_2$ and $h_1 = e_1$. Combining these we find that $h_3 = e_1^3 - 2e_2e_1 + e_3$.

We may also use these equations derive simple results using calculations by induction. For instance, (3.13) may be used to derive by induction by induction the following action of the coproduct Δ, acting on the symmetric function h_k.

PROPOSITION 3.2.

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k} \tag{3.15}$$

PROOF. Assume by induction that we know that equation (3.15) is true for all $k < n$. Then we know that

$$\begin{aligned}
\Delta(nh_n) &= \sum_{r=1}^{n} \Delta(p_r h_{n-r}) \\
&= \sum_{r=1}^{n} \left((p_r \otimes 1) \sum_{k=0}^{n-r} h_{n-r-k} \otimes h_k + (1 \otimes p_r) \sum_{k=0}^{n-r} h_k \otimes h_{n-r-k} \right) \\
&= \sum_{r=1}^{n} \left(\sum_{k=0}^{n-r} p_r h_{n-r-k} \otimes h_k + \sum_{k=0}^{n-r} h_k \otimes p_r h_{n-r-k} \right) \\
&= \sum_{k=0}^{n-1} \sum_{r=1}^{n-k} p_r h_{n-r-k} \otimes h_k + \sum_{k=0}^{n-1} \sum_{r=1}^{n-k} h_k \otimes p_r h_{n-r-k} \\
&= \sum_{k=0}^{n-1} (n-k) h_{n-k} \otimes h_k + \sum_{k=0}^{n-1} h_k \otimes ((n-k) h_{n-k}) \\
&= nh_n \otimes 1 + n \sum_{k=1}^{n-1} h_k \otimes h_{n-k} + 1 \otimes nh_n \\
&= n \sum_{k=0}^{n} h_k \otimes h_{n-k}
\end{aligned}$$

□

This last result gives us an interesting combinatorial way of looking at the action of Δ on the functions h_λ. On a single h_n, Δ acts by summing over all possible ways of breaking up a block of size n into two pieces whose sum is n. Therefore when Δ acts on an h_λ, we can use this idea to come up with a combinatorial interpretation for the coefficient of $h_\mu \otimes h_\nu$ in $\Delta(h_\lambda)$.

Think of the rows of μ as red blocks with labels $1, 2, \ldots, \ell(\mu)$ whose horizontal lengths are $\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$ and the rows of ν are represented by blue blocks with horizontal lengths $\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}$. When Δ acts on h_λ it splits each of the rows of λ into two parts (with some of the parts possibly empty) and so we can interpret the coefficient $\Delta(h_\lambda)\Big|_{h_\mu \otimes h_\nu}$ as the number of ways of taking at most one red block and at most one blue block placing it next to each other to get rows of size $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$.

EXAMPLE 17. We wish to compute $\Delta(h_{(4,3,3)})\Big|_{h_{(2,2,1)}\otimes h_{(2,2,1)}}$, then we break up the rows of the partition $(4,3,3)$, each one into a red part and a blue part (possibly empty) such that red pieces sorted are a partition $(2,2,1)$ and the blue pieces sorted are a partition $(2,2,1)$. The can be done in exactly two ways,

Equation (3.14) can also be used to derive the following determinantal formula for p_n in terms of e_k.

$$p_n = \begin{vmatrix} ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \\ (n-1)e_{n-1} & e_{n-2} & e_{n-3} & \cdots & 1 \\ (n-2)e_{n-2} & e_{n-3} & e_{n-4} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ e_1 & 1 & 0 & \cdots & 0 \end{vmatrix} \tag{3.16}$$

This follows directly from equation (3.14) by expanding the determinant about the first row of the equation. It follows that the determinant satisifes the same recurrence as the p_k elements do in equation (3.14). Similarly, we also have

$$(-1)^{n-1}p_n = \begin{vmatrix} nh_n & h_{n-1} & h_{n-2} & \cdots & h_1 \\ (n-1)h_{n-1} & h_{n-2} & h_{n-3} & \cdots & 1 \\ (n-2)h_{n-2} & h_{n-3} & h_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_1 & 1 & 0 & \cdots & 0 \end{vmatrix} \tag{3.17}$$

which follows most easily by an application of the involution ω or by observing the same recurrence with equation (3.13).

There is another product defined on symmetric functions known as the 'Kronecker' or 'inner tensor' product. We will denote this product by $*$. It is defined on the power sum basis by

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu}\frac{p_\lambda}{z_\lambda}. \tag{3.18}$$

This product is associative and preserves the degree of the symmetric function, that is it maps $\Lambda_n \otimes \Lambda_n \to \Lambda_n$. We also know that the product is commutative $f*g = g*f$ since clearly this holds on the power basis. We will go into more detail of this product as we introduce more of the bases of the symmetric functions. The Kronecker product very naturally arises in the algebra of class functions and we will show that it is also nicely encoded in our notation.

There is an assocated coproduct with $*$ that defines a bialgebra structure on the symmetric functions. Define the corresponding coproduct as $\Delta' : \Lambda \to \Lambda \otimes \Lambda$ will be defined on the power basis

$$\Delta'(p_\lambda) = p_\lambda \otimes p_\lambda. \tag{3.19}$$

PROPOSITION 3.3. *The vector space Λ endowed with the product $\mu(f \otimes g) = fg$ and coproduct Δ' forms a bialgebra.*

PROOF. $\Delta'(p_\mu p_\lambda) = (p_\mu p_\lambda) \otimes (p_\mu p_\lambda)$. Similarly we have that $\Delta'(p_\mu)\Delta'(p_\lambda) = (p_\mu \otimes p_\mu)(p_\lambda \otimes p_\lambda) = (p_\mu p_\lambda) \otimes (p_\mu p_\lambda) = \Delta'(p_\mu p_\lambda)$. Therefore we have shown that Δ' is an algebra homomorphism with respect to the product. $\square$

Note: With the map $\varepsilon'(p_k) = 1$ (and more generally $\varepsilon'(p_\lambda) = 1$) is a counit and satisfies $\mu \circ (id \otimes \varepsilon') \circ \Delta' = id$, however this product/coproduct pair fails to have an antipode and hence is not a Hopf algebra (see exercise 12).

Δ' is not an algebra homomorphism with respect to the Kronecker product since $\Delta'(p_\mu * p_\lambda) = \delta_{\lambda\mu} z_\lambda p_\lambda \otimes p_\lambda$ while $\Delta'(p_\mu) * \Delta'(p_\lambda) = \delta_{\lambda\mu} z_\lambda^2 p_\lambda \otimes p_\lambda$ and hence it is not a bialgebra.

In addition to the Hopf algebra and bialgebra structures on Λ, one should also think of Λ as a vector space over $\mathbb{Q}$ and so it is convenient to define a scalar product on this to serve as a tool for computation. Define

$$\left\langle p_\lambda, \frac{p_\mu}{z_\mu} \right\rangle = \delta_{\lambda\mu} \tag{3.20}$$

where we use the notation $\delta_{xy} = 0$ if $x \neq y$ and $\delta_{xx} = 1$. The remarkable property of this scalar product is that it interacts nicely with the products and coproducts on this space.

PROPOSITION 3.4. *The scalar product is positive definite. In addition, it satisfies the following useful properties.*

$$\langle f, g \rangle = \langle g, f \rangle \tag{3.21}$$

If we set $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_\otimes = \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle$, then the coproduct Δ is dual to multiplication,

$$\langle f \otimes g, \Delta(h) \rangle_\otimes = \langle fg, h \rangle . \tag{3.22}$$

The coproduct Δ' is dual to the product $*$,

$$\langle f \otimes g, \Delta'(h) \rangle_\otimes = \langle f * g, h \rangle . \tag{3.23}$$

The involution ω and antipode S are self-dual,

$$\langle \omega(f), \omega(g) \rangle = \langle S(f), S(g) \rangle = \langle f, g \rangle . \tag{3.24}$$

Moreover,

$$\langle f, g \rangle = \varepsilon'(f * g) \tag{3.25}$$

and

$$\langle f * g, h \rangle = \langle g, f * h \rangle \tag{3.26}$$

PROOF. It suffices to verify these identities for a basis and then the result must extend by linearity, and for this we choose the basis $\{p_\lambda\}_\lambda$. Note that the fact that the scalar product is symmetric follows since $\langle p_\lambda, p_\mu \rangle = \langle p_\mu, p_\lambda \rangle = \delta_{\lambda\mu} z_\lambda$. We show equation (3.22) by expanding the left hand side using equation (3.2) and comparing it to the right hand side of the equation.

$$\langle p_\lambda \otimes p_\mu, \Delta(p_\nu)\rangle_\otimes = \sum_\gamma \prod_{i\geq 1} \binom{m_i(\nu)}{m_i(\gamma)} \langle p_\lambda, p_\gamma\rangle \langle p_\mu, p_{\nu\setminus\gamma}\rangle \tag{3.27}$$

$$= \delta_{\nu,\lambda\uplus\mu} \prod_{i\geq 1} \binom{m_i(\nu)}{m_i(\lambda)} z_\lambda z_\mu \tag{3.28}$$

On the right hand side of this equation we have

$$\langle p_\lambda p_\mu, p_\nu\rangle = \delta_{\nu,\lambda\uplus\mu} z_\nu \tag{3.29}$$

and by referring back to the definition of z_ν it is easy to see that (3.28) and (3.29) are equal.

Similarly we may verify (3.23),

$$\begin{aligned} \langle p_\lambda \otimes p_\mu, \Delta(p_\nu)\rangle_\otimes &= \langle p_\lambda, p_\nu\rangle \langle p_\mu, p_\nu\rangle = \delta_{\lambda\mu}\delta_{\mu\nu} z_\nu^2 \\ &= \langle z_\mu \delta_{\lambda\mu} p_\mu, p_\nu\rangle = \langle p_\lambda * p_\mu, p_\nu\rangle \end{aligned} \tag{3.30}$$

$$\langle \omega(p_\lambda), \omega(p_\mu)\rangle = (-1)^{|\lambda|+|\mu|-\ell(\lambda)-\ell(\mu)} \langle p_\lambda, p_\mu\rangle = \tag{3.31}$$

$$\langle S(p_\lambda), S(p_\mu)\rangle = (-1)^{\ell(\lambda)+\ell(\mu)} \langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda = \langle p_\lambda, p_\mu\rangle \tag{3.32}$$

Equation (3.25) follows because we have set $p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda$ and hence $\varepsilon'(p_\lambda * p_\mu) = \delta_{\lambda\mu} z_\lambda \varepsilon'(p_\lambda) = \delta_{\lambda\mu} z_\lambda = \langle p_\lambda, p_\mu\rangle$. This implies our last identity as well since the product $*$ is associative and symmetric and

$$\langle f * g, h\rangle = \varepsilon'(f * (g * h)) = \varepsilon'(g * (f * h)) = \langle g, f * h\rangle \tag{3.33}$$

□

Now for any symmetric function homomorphism we can ask what the operation which is dual with respect to the scalar product. That is, for $\phi \in Hom(\Lambda, \Lambda)$ we ask what is the operator ϕ^* with the property

$$\langle \phi(f), g\rangle = \langle f, \phi^*(g)\rangle \tag{3.34}$$

Notice that we have already shown that many of the operators which we have considered so far (e.g. S, ω, the action Kronecker product by a symmetric function $f * \cdot$) are self dual. In the last proposition we also showed that the coproduct Δ is dual to the product operation m and the coproduct Δ' is dual to the Kronecker product.

This leads to a useful computational tool, the operation which is dual to multiplication of a symmetric function f, which we will denote by $f^\perp$. That is, $f^\perp$ is defined as the operator with the property

$$\langle f \cdot g, h\rangle = \langle g, f^\perp h\rangle .$$

Since multiplication by f is an operation which raises the degree of a symmetric function by the degree of f, $f^\perp$ is an operator which lowers the degree of a symmetric function by the degree of f.

We show the following useful properties of this operation.

PROPOSITION 3.5.

$$f^{\perp}(g) = \sum_{\lambda} \langle f p_\lambda, g \rangle \, p_\lambda / z_\lambda \tag{3.35}$$

$p_k^{\perp} = k\frac{\partial}{\partial p_k}$ and, in particular,

$$p_k^{\perp}(p_\lambda) = k m_k(\lambda) p_{\lambda \ominus (k)} \tag{3.36}$$

where $p_{\lambda\ominus(k)}$ is zero if λ does not have a part of size k. If $\Delta(f) = \sum_i a_i \otimes b_i$, then

$$f^{\perp}(gh) = \sum_i a_i^{\perp}(g) b_i^{\perp}(h) \tag{3.37}$$

PROOF. The first equation follows because the coefficient of p_λ in $f \in \Lambda$ is given by $\langle p_\lambda/z_\lambda, f\rangle$, therefore the expansion of f in the p_λ basis is simply $f = \sum_\lambda \langle p_\lambda/z_\lambda, f\rangle \, p_\lambda$. In the case that we expand $f^{\perp}(g)$ in the power basis, we have that

$$f^{\perp} g = \sum_{\lambda} \left\langle p_\lambda/z_\lambda, f^{\perp}(g) \right\rangle p_\lambda = \sum_{\lambda} \langle f p_\lambda, g\rangle \, p_\lambda/z_\lambda. \tag{3.38}$$

The coefficient of p_μ in $p_k^{\perp} p_\lambda$ is given by $\left\langle p_k^{\perp} p_\lambda, p_\mu/z_\mu \right\rangle = \langle p_\lambda, p_k p_\mu/z_\mu \rangle$ which is equal to 0 unless $\mu \uplus (k) = \lambda$. If $\mu = \lambda \ominus (k)$, then the scalar product evaluates to $z_\lambda/z_\mu = k m_k(\lambda)$ and otherwise the result is 0. It follows that $p_k^{\perp} = k\frac{\partial}{\partial p_k}$ since the action of these operators is the same on the monomial p_λ.

The fact that $p_k^{\perp}(fg) = p_k^{\perp}(f)g + f p_k^{\perp}(g)$ follows from the product rule for derivatives since we can interpret $p_k^{\perp}$ as a differential operator. Since $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$ we have shown that equation (3.37) holds for any p_k. We know that $\Delta(p_\lambda) = \Delta(p_{\lambda_1})\Delta(p_{\lambda_2})\cdots\Delta(p_{\lambda_k})$ while $p_\lambda^{\perp} = p_{\lambda_1}^{\perp} p_{\lambda_2}^{\perp} \cdots p_{\lambda_k}^{\perp}$, therefore (3.37) must hold for any p_λ. It follows by extending this result linearly that it also holds for any symmetric function f. ☐

PROPOSITION 3.6. *For $k \geq 0$, the action of the operators $p_k^{\perp}$, $e_k^{\perp}$ and $h_k^{\perp}$ for $k \geq 0$ on the symmetric functions p_n, e_n and h_n is given by the following table.*

$$\begin{array}{cccc} & h_n & e_n & p_n \\ h_k^{\perp} & h_{n-k} & (\delta_{k0} + \delta_{k1}) e_{n-k} & \delta_{kn} + \delta_{k0} p_n \\ e_k^{\perp} & (\delta_{k0} + \delta_{k1}) h_{n-k} & e_{n-k} & (-1)^{k-1}\delta_{kn} + \delta_{k0} p_n \\ p_k^{\perp} & h_{n-k} & (-1)^{k-1} e_{n-k} & n\delta_{nk} + \delta_{k0} p_n \end{array} \tag{3.39}$$

PROOF. $e_k^{\perp}(e_n)$, $e_k^{\perp}(h_n)$, $e_k^{\perp}(p_n)$ and $p_k^{\perp}(e_n)$ can all be calculated from the action of the operator $h_k^{\perp}$ or $p_k^{\perp}$ since we have that $\omega(f^{\perp} g) = (\omega(f))^{\perp}(\omega(g))$.

$p_k^{\perp}(\frac{p_\lambda}{z_\lambda})$ is $\frac{p_{\lambda\ominus(k)}}{z_{\lambda\ominus(k)}}$ if λ contains a part of size k and 0 if λ does not contain a part of size k. Therefore if $p_k^{\perp}$ acts on $h_n = \sum_{\lambda \vdash n} p_\lambda/z_\lambda$ the result will be $h_{n-k} = \sum_{\lambda \vdash n-k} p_\lambda / z_\lambda$. This justifies the last line of the table.

If $k = 0$ then $h_k^\perp = 1$, but otherwise $h_k^\perp(p_n) = 0$ unless $k = n$ since only then will there be a partition which contains a part of size n so that $\sum_{\lambda \vdash k} p_\lambda^\perp(p_n)/z_\lambda$ is non-zero.

We will prove $h_k^\perp h_n$ using induction and formula (3.13).

$$h_k^\perp(h_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp p_i^\perp(h_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp(h_{n-i}) \tag{3.40}$$

since we have already calculated that $p_i^\perp(h_n) = h_{n-i}$. If we assume by induction that $h_{k-i}^\perp(h_{n-i}) = h_{n-k}$ for $1 \le i \le k$, then it follows that $h_k^\perp(h_n) = h_{n-k}$.

We can prove in a similar manner the formula for $h_k^\perp(e_n)$. We handle the $k = 0$ and $k = 1$ cases separately for there we already know $h_0^\perp(e_n) = e_n$ and $h_1^\perp(e_n) = p_1^\perp(e_n) = e_{n-1}$.

For $k > 1$, if we assume that $h_{k-i}^\perp(e_n)$ is known for all $i > 0$ and all n then we calculate that

$$h_k^\perp(e_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp p_i^\perp(e_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp(-1)^{i-1}e_{n-i} \tag{3.41}$$

Only two terms of this equation will survive, $i = k-1$ and $i = k$. Therefore,

$$h_k^\perp(e_n) = \frac{1}{k}((-1)^k e_{n-k} + (-1)^{k-1}e_{n-k}) = 0 \tag{3.42}$$

□

We can take this one step further to calculate explicitly the action of $h_k^\perp$, $e_k^\perp$ and $p_k^\perp$ on h_λ, e_λ, and p_λ. It is sufficient to give an expression for the expressions $p_k^\perp(h_\lambda), p_k^\perp(p_\lambda), h_k^\perp(h_\lambda), h_k^\perp(e_\lambda)$, and $h_k^\perp(p_\lambda)$ since the others can be found by applying ω to both sides of the equation. We will need to use the relation that $h_k^\perp(fg) = \sum_{i=0}^k h_i^\perp(f)h_{k-i}^\perp(g)$ and $p_k^\perp(fg) = p_k^\perp(f)g + fp_k^\perp(g)$. The real difficulty in this problem is in finding a nice way of elegantly expressing these quantities.

PROPOSITION 3.7.

$$p_k^\perp(h_\lambda) = \sum_{i=1}^{\ell(\lambda)} h_{\lambda \ominus (\lambda_i) \uplus (\lambda_i - k)} \tag{3.43}$$

$$p_k^\perp(p_\lambda) = m_k(\lambda) k p_{\lambda \ominus (k)} \tag{3.44}$$

$$h_k^\perp(h_\lambda) = \sum_{|\alpha|=k} \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i - \alpha_i} \tag{3.45}$$

where the sum over all sequences α such that $0 \le \alpha_i \le \lambda_i$.

$$h_k^\perp(e_\lambda) = \sum_{|\alpha|=k} \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i - \alpha_i} \tag{3.46}$$

where the sum over all sequences α *such that* $0 \leq \alpha_i \leq 1$.

$$h_k^\perp(p_\lambda) = \sum_{S \subset \{1,2,\ldots,\ell(\lambda)\}} \prod_{i \notin S} p_{\lambda_i} \tag{3.47}$$

where the sum is over all subsets S *of* $\{1, 2, \ldots, \ell(\lambda)\}$ *such that* $\sum_{i \in S} \lambda_i = k$.

Proof. Most of these expressions follow from the previous proposition and the action of $h_k^\perp$ and $p_k^\perp$ on a product. Notice that

$$p_k^\perp(h_\lambda) = \sum_{i=1}^{\ell(\lambda)} p_k^\perp(h_{\lambda_i}) h_{\lambda \ominus (\lambda_i)} = \sum_{i=1}^{\ell(\lambda)} h_{(\lambda_i - k)} h_{\lambda \ominus (\lambda_i)} \tag{3.48}$$

which is the same as equation (3.43).

Also we have

$$\begin{aligned} p_k^\perp(p_\lambda) &= \sum_{i=1}^{\ell(\lambda)} p_k^\perp(p_{\lambda_i}) p_{\lambda \ominus (\lambda_i)} \\ &= \sum_{i=1}^{\ell(\lambda)} \delta_{\lambda_i, k} k p_{(\lambda_i - k)} p_{\lambda \ominus (\lambda_i)} \end{aligned} \tag{3.49}$$

and because there is exactly one non-zero term for each part of size k in λ so this is equal to the expression in equation (3.44).

Similarly to compute $h_k^\perp(h_\lambda)$, we compute

$$h_k^\perp(h_\lambda) = \sum_{i=0}^{k} h_i^\perp(h_{\lambda_1}) h_{k-i}^\perp(h_{\lambda \ominus (\lambda_1)}) \tag{3.50}$$

We can assume by induction on the length of λ that we know the formula for $h_{k-i}^\perp(h_{\lambda \ominus (\lambda_1)})$ is given by equation (3.45) (the base case is known because $\ell(\lambda) = 1$ is given in the previous proposition). We have then that equation (3.50) is equal to

$$= \sum_{i=0}^{k} h_{\lambda_1 - i} \left(\sum_{|\tilde{\alpha}| = k-i} \prod_{i=2}^{\ell(\lambda)} h_{\lambda_i - \tilde{\alpha}_i} \right) \tag{3.51}$$

and this is equivalent to equation (3.45) since if $i > \lambda_1$ then $h_{\lambda_1 - i} = 0$ so the size of the first part of α is restricted by both k and the size of λ_1.

Similarly, we can show again by assuming (3.46) is true for all partitions of length less than $\ell(\lambda)$ by induction, then

$$h_k^\perp(e_\lambda) = \sum_{i=0}^{k} h_i^\perp(e_{\lambda_1}) \sum_{|\tilde{\alpha}|=k-i} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde{\alpha}_i}$$

$$= e_{\lambda_1} \sum_{|\tilde{\alpha}|=k} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde{\alpha}_i} + e_{\lambda_1-1} \sum_{|\tilde{\alpha}|=k-1} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde{\alpha}_i}. \tag{3.52}$$

This equation is then equivalent to (3.46) for an equation indexed by a partition equal to the length of the partition λ.

Now to calculate $h_k^\perp(p_\lambda)$ we again assume by induction that (3.47) holds for partitions of length less than $\ell(\lambda)$.

$$h_k^\perp(p_\lambda) = p_{\lambda_1} h_k^\perp(p_{\lambda\ominus(\lambda_1)}) + h_{k-\lambda_1}^\perp(p_{\lambda\ominus(\lambda_1)}) \tag{3.53}$$

which follows from the action of $h_k^\perp$ on p_n given in the previous proposition. In this equation, if $k-\lambda_1 < 0$ then the second term in this sum is equal to 0. Using the inductive assumption, (3.53) is equal to

$$= p_{\lambda_1} \sum_{S\subset\{2,\ldots,\ell(\lambda)\}} \prod_{\substack{i\notin S\\ i\neq 1}} p_{\lambda_i} + \sum_{T\subset\{2,\ldots,\ell(\lambda)\}} \prod_{\substack{i\notin T\\ i\neq 1}} p_{\lambda_i} \tag{3.54}$$

where the the first sum is over all subsets S such that $\sum_{\substack{i\notin S\\ i\neq 1}} \lambda_i = k$ and the second sum is over subsets T such that $\sum_{\substack{i\notin T\\ i\neq 1}} \lambda_i = k-\lambda_1$. This is equivalent to equation (3.47) for a partition of length equal to $\ell(\lambda)$. $\square$

There is a third operation of multiplication which we have not yet mentioned which is a type of composition of symmetric functions. We define $p_n[p_m] = p_{nm}$ and then extend this definition in a natural manner. That is we set,

$$p_n[p_\lambda] = \prod_{i=1}^{\ell(\lambda)} p_{n\lambda_i}. \tag{3.55}$$

For $c, d \in \mathbb{Q}$ and $f, g \in \Lambda$ this operation is linear by

$$p_n[cf + dg] = c\ p_n[f] + d\ p_n[g]. \tag{3.56}$$

In particular we have, $p_n[\sum_\lambda c_\lambda p_\lambda] = \sum_\lambda c_\lambda \prod_{i=1}^{\ell(\lambda)} p_{n\lambda_i}$. Then for $f \in \Lambda$ and for a partition λ we define

$$p_\lambda[f] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[f]. \tag{3.57}$$

Finally for $f = \sum_\lambda c_\lambda p_\lambda$ and $g \in \Lambda$, we define

$$f[g] = \sum_\lambda c_\lambda p_\lambda[g]. \tag{3.58}$$

This definition implies that for $c, d \in \mathbb{Q}$ and $f, g, h \in \Lambda$, $(c\ f + d\ g)[h] = c\ f[h] + d\ g[h]$ but in general $f[c\ g + d\ h] \neq c\ f[g] + d\ f[h]$ (note this will hold if $f = p_n$).

3.1. The class functions of the symmetric group

We have defined the algebra of 'symmetric functions' without much a hint as why we have chosen this as the name of the algebra since the elements of Λ are neither symmetric nor functions. One motivation for studying this algebra is that it is isomorphic to the space of class functions of the symmetric group.

Let us consider Φ_n the linear vector space over $\mathbb{Q}$ of the class functions of the symmetric group Sym_n. We know that Φ_n is a vector space spanned by the elements C_λ where λ is a partition of n and

$$C_\lambda(\pi) = \begin{cases} 1 & \text{if } \pi \text{ has cycle type } \lambda \\ 0 & \text{otherwise} \end{cases}. \tag{3.59}$$

We also know that Φ_n is spanned by the set of irreducible characters of Sym_n.

We will define the Frobenius map between the space of class functions Φ_n and the space of symmetric functions of degree n. That is we define $\mathcal{F} : \Phi_n \to \Lambda_n$ by the action on the basis C_λ as

$$\mathcal{F}(C_\lambda) = \frac{p_\lambda}{z_\lambda}. \tag{3.60}$$

This map is clearly an isomorphism since the sets $\{C_\lambda\}_{\lambda \vdash n}$ and $\{p_\lambda/z_\lambda\}_{\lambda \vdash n}$ are both bases of Φ_n and Λ_n respectively.

Let χ^{triv_n} represent the trivial character on the symmetric group Sym_n. This means that $\chi^{triv_n} = \sum_{\lambda \vdash n} C_\lambda$ and therefore we have

$$\mathcal{F}(\chi^{triv_n}) = \sum_{\lambda \vdash n} \mathcal{F}(C_\lambda) = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} = h_n.$$

As well we may denote the sign character on Sym_n by χ^{sgn_n}. Since the sign of a permutation with cycle type λ is $(-1)^{|\lambda| - \ell(\lambda)}$ we have that $\chi^{sgn_n} = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} C_\lambda$ and therefore

$$\mathcal{F}(\chi^{sgn_n}) = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} \mathcal{F}(C_\lambda) = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} \frac{p_\lambda}{z_\lambda} = e_n.$$

The homogeneous and elementary generators are natural elements to consider in this context since the trivial and sign characters are the two one dimensional characters of Sym_n.

The definition of the scalar product on the symmetric functions may have seemed somewhat arbitary when we introduced it for symmetric fucntions but it is actually motivated by the scalar product of class functions and the connection by the following proposition.

PROPOSITION 3.8. *If* $\chi, \psi \in \Phi_n$, *then*

$$\langle \mathcal{F}(\chi), \mathcal{F}(\psi) \rangle = \langle \chi, \psi \rangle \tag{3.61}$$

where on the left the scalar product is over symmetric functions with $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$ *and on the right it is the scalar product on the class functions defined as* $\langle \chi, \psi \rangle = \frac{1}{n!} \sum_{\sigma \in Sym_n} \chi(\sigma)\psi(\sigma^{-1})$.

PROOF. Because the map $\mathcal{F}$ is linear and the both the scalar products are bilinear it suffices to show that this result holds for a basis. That is, we need only show that

$$\langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \langle C_\lambda, C_\mu \rangle \tag{3.62}$$

since then we know that for $\chi = \sum_{\lambda \vdash n} c_\lambda C_\lambda$ and $\psi = \sum_{\mu \vdash n} d_\mu C_\mu$ and

$$\langle \mathcal{F}(\chi), \mathcal{F}(\psi) \rangle = \sum_{\lambda,\mu \vdash n} c_\lambda d_\mu \langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \sum_{\lambda,\mu \vdash n} c_\lambda d_\mu \langle C_\lambda, C_\mu \rangle = \langle \chi, \psi \rangle .$$

Now since $\mathcal{F}(C_\lambda) = p_\lambda / z_\lambda$ it is easy to establish (3.62) for a fixed λ and μ.

$$\langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \langle p_\lambda / z_\lambda, p_\mu / z_\mu \rangle = \delta_{\lambda\mu} / z_\lambda,$$

while at the same time

$$\langle C_\lambda, C_\mu \rangle = \frac{1}{n!} \sum_{\sigma \in Sym_n} C_\lambda(\sigma) C_\mu(\sigma^{-1}) = \delta_{\lambda\mu} / z_\lambda.$$

□

Remember that the irreducible characters of the symmetric group are an orthonormal basis of class functions. The images of the irreducible characters will be the fundamental basis for the symmetric functions and we will introduce this basis in a later chapter.

The space of symmetric functions Λ is an algebra since it is a vector space endowed with a multiplication operation which takes elements in $\Lambda_n \times \Lambda_m$ and sends them to Λ_{n+m}. The set of class functions is endowed with a similar multiplication operation. We set $\Phi = \bigoplus_{n \geq 0} \Phi_n$.

Recall that for $\chi \in \Phi_n$ and $\psi \in \Phi_m$, we have $\chi \otimes \psi$ is a class function of $Sym_m \times Sym_n$ defined by $\chi \otimes \psi(\pi, \sigma) = \chi(\pi)\psi(\sigma)$. To make this a character of Sym_{n+m} we consider the induced character $\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} \in \Phi_{n+m}$. This is our analogous operation in the space of class functions to the operation of multiplication in the symmetric functions. In fact, this operation is more than just analogous, it satisfies the following property:

PROPOSITION 3.9. *For* $\chi \in \Phi_n$ *and* $\psi \in \Phi_m$,

$$\mathcal{F}(\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \mathcal{F}(\chi)\mathcal{F}(\psi) \tag{3.63}$$

PROOF. The operation of inducing two class functions is linear in both the first and in the second position as we have for $\chi = \sum_{\lambda \vdash n} c_\lambda C_\lambda$ and $\psi = \sum_{\mu \vdash m} d_\mu C_\mu$ then

$$\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} = \sum_{\lambda \vdash n} \sum_{\mu \vdash m} c_\lambda d_\mu C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} . \tag{3.64}$$

Therefore we need only show that $\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \mathcal{F}(C_\lambda)\mathcal{F}(C_\mu)$. To do this we will expand $\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}})$ in the basis $\{p_\nu\}_{\nu \vdash n+m}$.

$$\begin{aligned}\left\langle C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}, C_\nu \right\rangle &= \left\langle C_\lambda \otimes C_\mu, C_\nu \downarrow_{Sym_n \times Sym_m}^{Sym_{n+m}} \right\rangle \\ &= \frac{1}{n!m!} \sum_{\sigma \in Sym_n, \tau \in Sym_m} C_\lambda(\sigma) C_\mu(\tau) C_\nu \downarrow_{Sym_n \times Sym_m}^{Sym_{n+m}} (\sigma, \tau).\end{aligned}$$

Every term in this last sum is equal to 0 unless $\lambda \uplus \mu = \nu$ and only then when σ is of cycle type λ and τ is of cycle type μ. Therefore the right hand side is equal to $\frac{\delta_{\lambda \uplus \mu, \nu}}{z_\lambda z_\mu}$ and hence

$$\left\langle \mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}), \mathcal{F}(C_\nu) \right\rangle = \frac{\delta_{\lambda \uplus \mu, \nu}}{z_\lambda z_\mu}$$

and hence

$$\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \frac{p_{\lambda \uplus \mu}}{z_\lambda z_\mu} = \frac{p_\lambda}{z_\lambda} \frac{p_\mu}{z_\mu} = \mathcal{F}(C_\lambda)\mathcal{F}(C_\mu).$$

$\square$

This last proposition gives us an interpretation for h_λ and e_λ since we have already noted that the image of the trivial and sign characters in the Frobenius map are h_n and e_n respectively. Define Sym_λ to be the subgroup of Sym_n isomorphic to $Sym_{\lambda_1} \times Sym_{\lambda_2} \times \cdots \times Sym_{\lambda_{\ell(\lambda)}}$ in the natural manner. Denote the trivial and sign characters on this subgroup as χ^{triv_λ} and χ^{sgn_λ} so that for all $\pi \in Sym_\lambda$, $\chi^{triv_\lambda}(\pi) = 1$ and $\chi^{sgn_\lambda}(\pi) = sgn(\pi)$ and more precisely $\chi^{triv_\lambda} = \chi^{triv_{\lambda_1}} \otimes \chi^{triv_{\lambda_2}} \otimes \cdots \otimes \chi^{triv_{\lambda_{\ell(\lambda)}}}$ and similarly $\chi^{sgn_\lambda} = \chi^{sgn_{\lambda_1}} \otimes \chi^{sgn_{\lambda_2}} \otimes \cdots \otimes \chi^{sgn_{\lambda_{\ell(\lambda)}}}$. From the previous proposition we have

$$h_\lambda = \mathcal{F}(\chi^{triv_\lambda} \uparrow_{Sym_\lambda}^{Sym_n})$$

and

$$e_\lambda = \mathcal{F}(\chi^{sgn_\lambda} \uparrow_{Sym_\lambda}^{Sym_n}).$$

We defined a second type of multiplication on symmetric functions which we called the inner or Kronecker product of symmetric functions. The definition this operation $*$ is given by $\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda}$. It arises naturally in the following sense.

Proposition 3.10. *For $\chi, \psi \in Phi_n$ and we define $\chi \cdot \psi$ as the class function $\chi \cdot \psi(g) := \chi(g)\psi(g)$. This is the inner tensor product of characters. We have*

$$\mathcal{F}(\chi \cdot \psi) = \mathcal{F}(\chi) * \mathcal{F}(\psi). \tag{3.65}$$

Proof. Again it suffices to verify this identity on a basis for the class functions because it will hold for any linear combination of the class functions as well. This is easy to verify for the class functions C_λ, since

$$C_\lambda \cdot C_\mu(\pi) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu \text{ and } C_\lambda(\pi) = 1 \end{cases} = \delta_{\lambda\mu} C_\lambda(\pi)$$

This means that $C_\lambda \cdot C_\mu = \delta_{\lambda\mu} C_\lambda$ and so we have

$$\begin{aligned}\mathcal{F}(C_\lambda \cdot C_\mu) &= \delta_{\lambda\mu}\mathcal{F}(C_\lambda)\\ &= \delta_{\lambda\mu}\frac{p_\lambda}{z_\lambda}\\ &= \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu}\\ &= \delta_{\lambda\mu}\mathcal{F}(C_\lambda) * \mathcal{F}(C_\mu)\end{aligned}$$

□

***** interpretation of ω here?

The coproduct operation also has an interpretation in the algebra of class functions, however we first need to extend our definition of the Frobenius map to the algebra of class functions on $Sym_k \times Sym_{n-k}$. Recall that we have for class functions $\chi \in \Phi_k$ and $\psi \in \Phi_{n-k}$ that the function $\chi \otimes \psi$ defined to be $\chi \otimes \psi(\pi, \sigma) := \chi(\pi)\psi(\sigma)$ for $\pi \in Sym_k$ and $\sigma \in Sym_{n-k}$ and $\chi \otimes \psi$ is a class function of $Sym_k \times Sym_{n-k}$. This is called the the outer tensor product of class functions. Note that for any basis of class functions of G, $\{C^{(i)}\}$, and of the class functions of H, $\{D^{(i)}\}$, then $\{C^{(i)} \otimes D^{(j)}\}$ is basis for the class functions of $G \times H$.

To extend the definition of the Frobenius map to include class functions of $Sym_k \times Sym_{n-k}$ which has as a basis $\{C_\lambda \otimes C_\mu\}_{\substack{\lambda \vdash k \\ \mu \vdash n-k}}$ we set

$$\mathcal{F}(C_\lambda \otimes C_\mu) := \frac{p_\lambda}{z_\lambda} \otimes \frac{p_\mu}{z_\mu} = \mathcal{F}(C_\lambda) \otimes \mathcal{F}(C_\mu) \tag{3.66}$$

and this definition is extended linearly. This implies that we have more generally, for $\chi \in \Phi_n$ and $\psi \in \Phi_m$,

$$\mathcal{F}(\chi \otimes \psi) = \mathcal{F}(\chi) \otimes \mathcal{F}(\psi). \tag{3.67}$$

Using this extension of notation we have the following interpretation of the coproduct operation on symmetric functions.

Proposition 3.11.

$$\Delta(\mathcal{F}(\chi)) = \sum_{k=0}^{n} \mathcal{F}(\chi \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}) \tag{3.68}$$

Proof. It suffices to show that this result again holds on a basis and the natural basis to consider is again C_λ for $\lambda \vdash n$. We have that

$$C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}} (\pi, \sigma) = \begin{cases} 1 & \text{if } C_\mu(\pi) = 1 \text{ and } C_\nu(\sigma) = 0 \text{ for} \\ & \mu \vdash k, \nu \vdash n-k \text{ with } \mu \uplus \nu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

In other words we see that

$$C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}} = \sum_{\substack{\mu \vdash k \\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} C_\mu \otimes C_\nu$$

This implies that

$$\mathcal{F}(C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}) = \sum_{\substack{\mu \vdash k \\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} \frac{1}{z_\mu z_\nu} p_\mu \otimes p_\nu.$$

As well we have

$$\begin{aligned}\Delta(z_\lambda \mathcal{F}(C_\lambda)) = \Delta(p_\lambda) &= \sum_{k=0}^{n} \sum_{\substack{\mu \vdash k \\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} \frac{z_\lambda}{z_\mu z_\nu} p_\mu \otimes p_\nu \\ &= \sum_{k=0}^{n} z_\lambda \mathcal{F}(C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}),\end{aligned}$$

and therefore the proposition holds for all class functions. □

3.2. Exercises

(1) (a) Expand $p_{(2,2)}$ in the elementary and homogeneous bases.
  (b) Expand $e_{(2,2)}$ in the power basis.
  (c) Expand $h_{(2,2)}$ in the homogeneous basis.

(2) Calculate the following scalar products
  (a) $\langle h_{(2,2,1)}, p_{(3,2)} \rangle$
  (b) $\langle h_{(3,2)}, p_{(3,2)} \rangle$
  (c) $\langle h_{(3,2)}, p_{(2,2,1)} \rangle$
  (d) $\langle h_{(3,2)}, h_{(4,1)} \rangle$
  (e) $\langle h_{(3,2)}, h_{(3,1,1)} \rangle$
  (f) $\langle h_{(3,2)}, h_{(2,2,1)} \rangle$

(3) Calculate the following inner products using the formulas given in this section. Assume that $|\lambda| = n$.
  (a) $\langle h_n, p_\lambda \rangle$
  (b) $\langle e_n, p_\lambda \rangle$
  (c) $\langle p_n, h_\lambda \rangle$
  (d) $\langle p_{1^n}, h_\lambda \rangle$
  (e) $\langle p_\lambda, h_\lambda \rangle$
  (f) $\langle h_n, h_n \rangle$
  (g) $\langle e_n, h_n \rangle$
  (h) $\langle h_n, h_\lambda \rangle$
  (i) $\langle e_n, h_\lambda \rangle$

(4) Show that $\Delta \circ \omega = (\omega \otimes \omega) \circ \Delta$ by showing that it holds true on the power basis. Use this to show that $\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$.

(5) Show that $(1 \otimes \omega) \circ \Delta' = (\omega \otimes 1) \circ \Delta' = \Delta' \circ \omega$.

(6) Show the following determinental formulas for h_n.

(a)

$$h_m = \begin{vmatrix} p_1 & -1 & 0 & \cdots & 0 \\ \frac{p_2}{2} & \frac{p_1}{2} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{p_{m-1}}{m-1} & \frac{p_{m-2}}{m-1} & \cdots & \frac{p_1}{m-1} & -1 \\ \frac{p_m}{m} & \frac{p_{m-1}}{m} & \frac{p_{m-2}}{m} & \cdots & \frac{p_1}{m} \end{vmatrix}$$

(b)

$$h_n = \begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_n \\ 1 & e_1 & e_2 & \cdots & e_{n-1} \\ 0 & 1 & e_1 & \cdots & e_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}$$

State and prove the corresponding determinental formulas for e_n in terms of p_n and h_n.

(7) Show that $\Delta(H(t)) = H(t) \otimes H(t)$ and $\Delta(E(t)) = E(t) \otimes E(t)$ and $\Delta(P(t)) = P(t) \otimes 1 + 1 \otimes P(t)$.
(8) Show that $p_k^\perp = k\frac{\partial}{\partial p_k}$
(9) Use the fact that $E(t)H(t) = 1$ to develop a formula for h_n in terms of the elementary basis.
(10) Use the relationship $P(t) = log(1 + \sum_{n\geq 1} h_n t^n)$ to derive a formula for p_n in terms of the homogeneous basis.
(11) Use the previous result to expand p_λ in the homogeneous basis.
(12) Prove that Λ endowed with the bialgebra with product μ and coproduct Δ' does not have a corresponding antipode and hence is not a Hopf algebra.
(13) Prove that for any $f \in \Lambda_n$, $h_n * f = f$ and $e_n * f = \omega f$.
(14) Prove that if $g \in \Lambda_n$ has the property that for all $f \in \Lambda_n$, $g * (g * f)) = f$ then $\langle g, p_\lambda \rangle = \pm 1$ for all λ partitions of n.
(15) Show that $\mathbb{Z}[p_1, p_2, p_3, \ldots] \subseteq \mathbb{Z}[e_1, e_2, e_3, \ldots]$ and that the converse of this statment (i.e. that $\mathbb{Z}[e_1, e_2, e_3, \ldots] \subseteq \mathbb{Z}[p_1, p_2, p_3, \ldots]$) is not true.
(16) Show that $\mathbb{Z}[h_1, h_2, \ldots, h_k] = \mathbb{Z}[e_1, e_2, \ldots, e_k]$.
(17) (a) Show that the linear span of the symmetric functions $\{f + \omega(f) : f \in \Lambda\}$ forms a subalgebra of the symmetric functions under the standard product.
 (b) Show that this algebra is not a bialgbra with the coproduct Δ.
 (c) Show that a linear basis for this space is given by the set $\{p_\lambda : |\lambda| - \ell(\lambda) \; mod \; 2 = 0\}$
 (d) Show that the space is closed under the Kronecker product of equation (3.18) and the coproduct Δ' of equation (3.19).
(18) Show that $\mathbb{Q}[p_1, p_3, p_5, \ldots]$ is a Hopf subalgebra of the symmetric functions and that it is also closed under the Kronecker product and coproduct Δ'. This subalgebra is sometimes known as the Q-function algebra.

3.3. Solutions to exercises

(1) (a)

$$p_2^2 = \left(-2\,e_2 + {e_1}^2\right)^2 = {e_1}^4 - 4\,e_2\,{e_1}^2 + 4\,{e_2}^2$$
$$= \left(-2\,h_2 + {h_2}^2\right)^2 = {h_1}^4 - 4\,h_2\,{h_1}^2 + 4\,{h_2}^2$$

(b)

$$e_2^2 = 1/4\,\left(-p_2 + {p_1}^2\right)^2 = 1/4\,{p_1}^4 - 1/2\,p_2\,{p_1}^2 + 1/4\,{p_2}^2$$

(c)

$$h_2^2 = \left(-e_2 + {e_1}^2\right)^2 = {e_1}^4 - 2\,e_2\,{e_1}^2 + {e_2}^2$$

(2) By direct calculation

$$h_{(3,2)} = 1/12\,{p_1}^5 + 1/3\,p_2\,{p_1}^3 + 1/6\,p_3\,{p_1}^2 + 1/4\,{p_2}^2 p_1 + 1/6\,p_3\,p_2$$
$$h_{(2,2,1)} = 1/4\,{p_1}^5 + 1/2\,p_2{p_1}^3 + 1/4\,{p_2}^2 p_1$$
$$h_{(4,1)} = 1/24\,{p_1}^5 + 1/4\,p_2{p_1}^3 + 1/3\,p_3{p_1}^2 + 1/8\,{p_2}^2 p_1 + 1/4\,p_4 p_1$$
$$h_{(3,1,1)} = 1/6\,{p_1}^5 + 1/2\,p_2{p_1}^3 + 1/3\,p_3{p_1}^2$$

(a) $\left\langle h_{(2,2,1)}, p_{(3,2)}\right\rangle = 0$
(b) $\left\langle h_{(3,2)}, p_{(3,2)}\right\rangle = 1$
(c) $\left\langle h_{(3,2)}, p_{(2,2,1)}\right\rangle = 2$
(d) $\left\langle h_{(3,2)}, h_{(4,1)}\right\rangle = 3$
(e) $\left\langle h_{(3,2)}, h_{(3,1,1)}\right\rangle = 4$
(f) $\left\langle h_{(3,2)}, h_{(2,2,1)}\right\rangle = 5$

(3) (a) $\left\langle \sum_{\mu\vdash n} p_\mu/z_\mu, p_\lambda \right\rangle = 1$
(b) $\left\langle \sum_{\mu\vdash n} (-1)^{|\mu|-\ell(\mu)} p_\mu/z_\mu, p_\lambda \right\rangle = (-1)^{|\lambda|-\ell(\lambda)}$
(c) $\langle p_n, h_\lambda\rangle = \delta_{\lambda,(n)}$
(d) $\langle p_{1^n}, h_\lambda\rangle = z_{1^n}/\prod_{i=1}^{\ell(\lambda)} z_{1^{\lambda_i}} = \binom{n}{\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)}}$
(e) $\langle p_\lambda, h_\lambda\rangle = \prod_i m_i(\lambda)!$
(f) $\langle h_n, h_n\rangle = \left\langle \sum_{\mu\vdash n} p_\mu/z_\mu, \sum_{\lambda\vdash n} p_\lambda/z_\lambda \right\rangle = \sum_{\lambda\vdash n} 1/z_\lambda$ which is equal to 1 since $n!/z_\lambda =$ the number of permuations with cycle type λ and $\sum_{\lambda\vdash n} n!/z_\lambda = n!$.
(g) $\langle e_n, h_n\rangle = \sum_{\lambda\vdash n}(-1)^{|\lambda|-\ell(\lambda)}/z_\lambda = 0$ if $n > 1$ and is equal to 1 if $n = 1$. This follows since $\sum_{\lambda\vdash n}(-1)^{|\lambda|-\ell(\lambda)} n!/z_\lambda =$ the number of permutations of even length $-$ the number of permutations of odd length. The numbers of these subsets of these permutations must be equal because composition with a permutation of length 1 is an involution which interchanges these sets.
(h) $\langle h_n, h_\lambda\rangle$

(4) For any partition $k > 0$, $\Delta\circ\omega(p_k) = (-1)^{k-1}\Delta(p_k) = (\omega\otimes\omega)\circ\Delta(p_k)$. Since ω and Δ are both ring homomorphisms, this formula holds on any $f \in \Lambda$. Using this identity, $\Delta(e_n) = \Delta\circ\omega(h_n) = (\omega\otimes\omega)\circ\Delta(h_n) = (\omega\otimes\omega)(\sum_{k=0}^n h_k\otimes h_{n-k}) = \sum_{k=0}^n e_k\otimes e_{n-k}$.

(5) Again we need only show that this property holds on a basis to conclude that it holds for all symmetric functions $\Delta'\circ\omega(p_\lambda) = \Delta'((-1)^{|\lambda|-\ell(\lambda)}p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)}(p_\lambda\otimes p_\lambda) = (\omega\otimes 1)(p_\lambda\otimes p_\lambda) = (\omega\otimes 1)\circ\Delta'(p_\lambda)$ Similarly we have $\Delta'\circ\omega(p_\lambda) = (1\otimes\omega)\circ\Delta'(p_\lambda)$.

(6) Let $M_n = [a_{i,j}]$ be the $n \times n$ matrix with $a_{i,j} = p_{n-i-j+2}$ if $n-i-j+2 > 0$ and $a_{i,j} = i-1$ if $n-i-j+2 = 0$ and $a_{i,j} = 0$ otherwise. Show $\frac{(-1)^n}{n!} det M_n$ satisfies the relation of equation (3.13) by expanding the determinant about the first row. Notice that the $(1,k)$ minor $M_n^{(1,k)}$ (the minor formed by deleting the 1^{st} row and k^{th} column of M_n) has determinant equal to $(-1)^{n-1}(n-1)!$ if $k=1$ and $(-1)^{n-k}(n-1)_k det M_{k-1}$ for $2 \leq k \leq n$. Therefore, $n\frac{(-1)^n}{n!}M_n = p_n p_n$

(7)

$$\begin{aligned} H(t) \otimes H(t) &= \left(\sum_{r\geq 0} h_r t^r\right) \otimes \left(\sum_{m \geq 0} h_m t^m\right) \\ &= \sum_{n \geq 0} t^n \sum_{k=0}^{n} h_{n-k} \otimes h_k \\ &= \sum_{n\geq 0} t^n \Delta(h_n) = \Delta(H(t)) \end{aligned}$$

Apply the result of problem 4 to show as well that $\Delta(E(t)) = E(t) \otimes E(t)$.

$$\begin{aligned} \Delta(P(t)) &= \sum_{r\geq 1} \Delta(\frac{p_r}{r})t^r = \sum_{r\geq 1}\left(\frac{p_r}{r} \otimes 1 + 1 \otimes \frac{p_r}{r}\right) t^r \\ &= P(t) \otimes 1 + 1 \otimes P(t) \end{aligned}$$

(8) $p_k^{\perp}(p_\lambda) = m_k(\lambda) k p_{\lambda \ominus (k)}$ from equation (3.44). Notice that $\frac{\partial}{\partial p_k}(p_\lambda) = m_k(\lambda) p_{\lambda\ominus(k)}$, so $p_k^{\perp}(p_\lambda) = k\frac{\partial}{\partial p_k}(p_\lambda)$ and $p_k^{\perp}(f) = k \frac{\partial}{\partial p_k}(f)$.

(9)

$$H(t) = 1 + \sum_{\ell \geq 1} (-1)^\ell \left(\sum_{k\geq 1} e_k (-t)^k\right)^\ell$$

Now the coefficient of t^n on both sides of this equation will be h_n on the left, and on the right a term appears for every partition with length ℓ and a coefficient equal to -1 raised to the size of the partition times a multinomial coefficient.

$$\begin{aligned} h_n &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda)\ m_2(\lambda)\ \cdots} e_\lambda \\ &= \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_{i\geq 1} m_i(\lambda)!} e_\lambda \end{aligned}$$

(10)

$$P(t) = \sum_{\ell\geq 1} \frac{(-1)^{\ell-1}}{\ell} \left(\sum_{m\geq 1} h_m t^m\right)^\ell$$

The coefficient of t^n on the left hand side of this equation is $\frac{p_n}{n}$ and on the right hand side for each partition there is a term with a multinomial coefficient which depends on the length and a factor of $(-1)^{\ell-1}$ over the length of the partition.

$$p_n = n \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)-1} \frac{(\ell(\lambda)-1)!}{\prod_{i\geq 1} m_i(\lambda)!} h_\lambda$$

(11) With the formula from the previous problem it is a matter of finding a good way of expressing the product p_λ. The coefficient of h_μ in p_λ will be positive if $\ell(\mu) - \ell(\lambda)$ even and negative otherwise. The coefficient of h_λ in p_μ is

$$(-1)^{\ell(\lambda)-\ell(\mu)} \sum_{\nu^{(1)} \uplus \nu^{(2)} \uplus \cdots \uplus \nu^{(\ell(\mu))} = \lambda} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i(\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}$$

where the sum is over all sequences of partitions with $\nu^{(i)} \vdash \lambda_i$.

(12) In order for the bialgebra structure to have a Hopf algebra structure it must hold that $m \circ (id \otimes S') \circ \Delta' = u \circ \varepsilon'$ where $m(f \otimes g) = f * g$. Act by this expression on p_1 and we must have that

$$p_1 S'(p_1) = 1$$

This cannot happen unless $S'(p_1) = 1/p_1$ which is not in our algebra.

(13) For $\lambda \vdash n$ we clearly have that $h_n * p_\lambda = \left(\sum_{\mu \vdash n} \frac{p_\mu}{z_\mu}\right) * p_\lambda = p_\lambda$ and $e_n * p_\lambda = \left(\sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \frac{p_\mu}{z_\mu}\right) * p_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda = \omega(p_\lambda)$. Therefore by linearity it holds that $h_n * f = f$ and $e_n * f = \omega(f)$.

(14) Note that $g = \sum_\mu c_\mu p_\mu$. Therefore, $g * p_\lambda = z_\lambda c_\lambda p_\lambda$ and $g * (g * p_\lambda) = z_\lambda^2 c_\lambda^2 p_\lambda = p_\lambda$, and so we know that $z_\lambda^2 c_\lambda^2 = 1$ or $z_\lambda c_\lambda = \pm 1$. We also have $\langle g, p_\lambda \rangle = c_\lambda z_\lambda = \pm 1$.

(15) Since each $p_k \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$ from equation (3.16) or problem number 10, we know that $p_\lambda \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$. It follows that each $f = \sum_\lambda c_\lambda p_\lambda$ with each $c_\lambda \in \mathbb{Z}$, then $f \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$. Many counterexamples to the converse exists (e.g. $e_2 = p_2/2 + p_1^2/2$).

(16) From equation (3.12) or problem number 6b or 9 we know that $h_k \in \mathbb{Z}[e_1, e_2, \ldots, e_k]$ and with an application of ω on these equations we know equally that $e_k \in \mathbb{Z}[h_1, h_2, \ldots, h_k]$.

(17) (a) $\{f + \omega(f) : f \in \Lambda\}$ are the set of functions which are invariant under the involution ω. That property is clearly invariant under products since ω is a ring homomorphism.
(b) This is not invariant under the coproduct Δ, since for instance $\Delta(e_{(2,2)} + h_{(2,2)})$ in the degree $(2,2)$ tensor is not invariant under ω.
(c) Note that $p_\lambda + \omega(p_\lambda) = 2p_\lambda$ if $|\lambda| + \ell(\lambda)$ is even and it is equal to 0 if $|\lambda| + \ell(\lambda)$ is odd. Since p_λ is a linear basis of Λ, $\{p_\lambda : |\lambda| - \ell(\lambda) \; mod \; 2\}$ is a linear basis for $\{f + \omega(f) : f \in \Lambda\}$.
(d) If λ has $|\lambda| - \ell(\lambda)$ even, then $p_\lambda * p_\mu = 0$ or is a multiple of p_λ and hence the basis elements are closed. Δ also sends the basis elements p_λ to $p_\lambda \otimes p_\lambda$.

(18) The fact that this ring is a Hopf subalgebra and closed under the Kronecker product and coproduct is true for any subalgebra generated by a subcollection of the the p_k and this need only be checked by verifying these operations are closed on the p_λ basis where the parts of λ are taken from the p_k which generate the subalgebra.

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