Introduction to Symmetric Functions

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Abstract. A development of the symmetric functions using the plethystic notation.

CHAPTER 1

The Symmetric Group

1.1. Representations of permutations

(1) two-line notation
(2) one-line notation
(3) cycle notation
(4) generators and relations (Coxeter-Moore relations)
(5) set of points
(6) compact diagrams

(1) inverse
(2) cycle structure
(3) conjugacy classes and z_λ
(4) length of a permutation
(5) longest permutation
(6) descent set
(7) ascent set
(8) major/comajor index
(9) charge
(10) Eulerian
(11) Euler-Mahonian
(12) determinants

1.2. Classes of permutations

(1) even/odd
(2) involutions
(3) 321-avoiding
(4) more pattern avoiding/ counting pattern occurences
(5) Grassmannian
(6) rotationally invariant

CHAPTER 2

Partitions

Start with n unlabeled objects and partition them (break them into subsets) into non-empty subsets. Since the objects are indistinguishable from each other, a partition is specified by the sizes of the non-empty subsets and we can list these sizes in weakly decreasing order. An object of this type is called a partition of n and we will represent this as a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with λ_i representing the number of objects in the i^{th} subset and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ where k is the number of non-empty subsets of the partition. We denote 'λ is a partition of n' by $\lambda \vdash n$. k is known as the length of the partition and we will denote it by $\ell(\lambda) := k$. By defintion, we have $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} = n$ and we will adopt the convention that for any $i > \ell(\lambda)$, $\lambda_i = 0$ so that $\sum_{i\geq 1} \lambda_i = n$.

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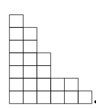
For each partition we will associate a subset of the points in the first quadrant of the cartesian lattice, $D(\lambda) = \{(i,j) : 0 \leq i < \lambda_{j+1}, j \geq 0\}$. The cardinality of this set is $|D(\lambda)| = n$ and we will represent $D(\lambda)$ by drawing a diagram in the coordinate plane with a square for each $(i,j) \in D(\lambda)$ with the bottom left hand corner of the square at the coordinate (i,j). A set of cells S corresponds to a partition if and only if $(i,j) \in S$ implies that $(i',j') \in S$ for all $0 \leq i' \leq i$ and $0 \leq j' \leq j$.

Example 1. Let $\lambda = (6,5,3,3,2,2,1)$ which is a partition of 22 and $D(\lambda)$ is represented by the following diagram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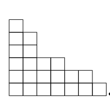
Let $n_i(\lambda)$ be the number of $j \leq \ell(\lambda)$ such that $\lambda_j \geq i$ and set $\lambda' = (n_1(\lambda), n_2(\lambda), \ldots, n_{\lambda_1}(\lambda))$. Since $n_i(\lambda) \geq n_{i+1}(\lambda)$, λ' is also a partition and we refer to it as the conjugate partition to λ.

Exercise 1. Show that $(i,j) \in D(\lambda)$ if and only if $(j,i) \in D(\lambda')$ and conclude the diagram associated to $D(\lambda')$ is exactly the diagram for $D(\lambda)$ flipped about the line $x = y$.

Example 2. As in the previous example $\lambda = (6,5,3,3,2,2,1)$, then we can easily calculate each of the $n_i(\lambda)$ and determine $\lambda' = (7,6,4,2,2,1)$. $D(\lambda')$ is represented by the diagram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From this point forward we will identify λ and $D(\lambda)$ in abuse of notation. This will appear in our formulas as expressions like $\lambda \subseteq \mu$ (or the language λ is contained in μ) to mean $D(\lambda) \subseteq D(\mu)$ or $s \in \lambda$ in place of $s \in D(\lambda)$. We shall also try to develop notation which is consistent with this identification, for instance the size of the partition λ will be $|\lambda|$ which represents $|D(\lambda)|$.

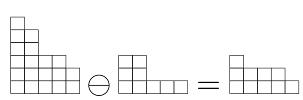
The union of two partitions $\lambda \cup \mu$ is another abuse of notation which will represent the partition whose diagram contains $D(\lambda) \cup D(\mu)$. Intuitively $\lambda \cup \mu$ is the smallest partition which contains both the partition λ and the partition μ. An equivalent definition will be the weakly increasing sequence $(max(\lambda_1, \mu_1), max(\lambda_2, \mu_2), \ldots, max(\lambda_m, \mu_m))$ with $m = max(\ell(\lambda), \ell(\mu))$. Similarly $\lambda \cap \mu$ will be the partition which is contained in both the partition λ and the partition μ (that is, $D(\lambda \cap \mu) = D(\lambda) \cap D(\mu)$) so that $\lambda \cap \mu = (min(\lambda_1, \mu_1), min(\lambda_2, \mu_2), \ldots, min(\lambda_m, \mu_m))$.

We will sometime need to consider the partition $\lambda \uplus \mu$ which will be the sequence

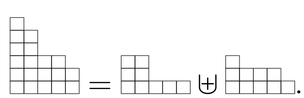
$$(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$$

rearranged so that the entries are in decreasing order. The complementary operation to $\uplus$ will be the $\lambda \ominus \mu$ where if there is a subset $S = \{s_1 < s_2 < \ldots < s_{\ell(\lambda)}\}$ such that $(\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_{\ell(\lambda)}}) = \mu$ then $\lambda \ominus \mu$ is the ordered sequence of parts $(\lambda_i : i \notin S)$. These operations are complementary in the sense that $\lambda \uplus \mu = \nu$ if and only if $\lambda = \nu \ominus \mu$ (and by symmetry $\mu = \nu \ominus \lambda$). We will have that $|\lambda \uplus \mu| = |\lambda| + |\mu|$ and $\ell(\lambda \uplus \mu) = \ell(\lambda) + \ell(\mu)$.

Example 3. Let $\lambda = (5, 5, 4, 2, 2, 1)$ and $\mu = (5, 2, 2)$ then $\lambda \ominus \mu = (5, 4, 1)$ and $\lambda = \mu \uplus (\lambda \ominus \mu)$. In pictures this is represented as



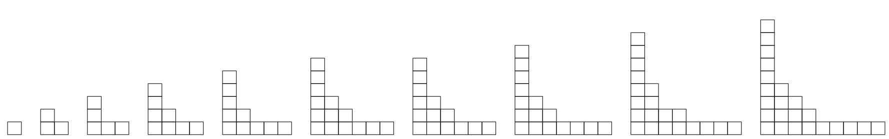
and



Exercise 2. Prove:

$$\left| \bigcup_{\mu \vdash n} \mu \right| = \sum_{k=1}^{n} d(k)$$

where $d(k)$ is the number of divisors of the number k. For example, we have



are the first nine partitions $\bigcup_{\mu \vdash n} \mu$ and $\sum_{k=1}^{n} d(k)$ takes on the values $1, 3, 5, 8, 10, 14, \ldots$.

Exercise 3. Show that $(\lambda \uplus \mu)' = (\lambda'_1 + \mu'_1, \lambda'_2 + \mu'_2, \ldots, \lambda'_r + \mu'_r)$ with $r = max(\ell(\lambda), \ell(\mu))$.

Let $m_i(\lambda) = n_i(\lambda) - n_{i-1}(\lambda)$ (with the convention that $n_0(\lambda) = 0$) so that $m_i(\lambda)$ represents the number of parts of size λ which are exactly equal to i. Another way of representing the partition λ will be by specifying the number of parts of each size i (that is we indicate the values of $m_i(\lambda)$ for each i). This will be done with the notation $\lambda = (k^{m_k(\lambda)}, (k-1)^{m_{k-1}(\lambda)}, \ldots, 2^{m_2(\lambda)}, 1^{m_1(\lambda)})$ for the value $k = \lambda_1$.

There are several statistics which are associated to a given partition. If λ is a partition then we define $z_\lambda := 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! 3^{m_3(\lambda)} m_3(\lambda)! \cdots$. This statistic is of interest because $|\lambda|!/z_\lambda$ is the number of permutations of $|\lambda|$ that have cycle type λ.

We also will denote $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$ which is something of a measure of how tall a partition is since a single row partition will have $n((k)) = 0$ and a single columned partition will have $n((1^k)) = \binom{k}{2}$. This statistic often appears naturally when considering weighted sums of partitions. For instance, the generating function for the number of partitions of size n is $\prod_{i \geq 1} \frac{1}{1-x^i}$ and to look for a q-analog of this expression we might consider the generating function

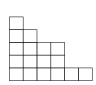
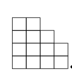
$$\prod_{i \geq 1} \frac{1}{1 - qx^i} = \sum_{n \geq 0} \sum_{\lambda \vdash n} q^{n(\lambda)} x^{|\lambda|}. \tag{2.1}$$

If $\mu \subseteq \lambda$, then the symbol λ/μ will represent the collection of cells $D(\lambda) - D(\mu)$ (where the operation $-$ means the difference of sets). λ/μ is called a skew partition. Unless otherwise specified when a skew partition λ/μ is indicated in a formula, it is automatically assumed that $\mu \subseteq \lambda$.

If λ/μ has at most one cell per column then we will say that it is a horizontal strip and indicate this by the notation $\lambda/\mu \in \mathcal{H}$. This condition can also be expressed by the inequalities $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots$.

If λ/μ has at most one cell per column then it will be called a vertical strip (and indicated by $\lambda/\mu \in \mathcal{V}$). $\lambda/\mu \in \mathcal{V}$ if and only if $\lambda_i \geq \mu_i \geq \lambda_i - 1$ for all $i \geq 1$. Furthermore if $\lambda/\mu \in \mathcal{H}$ (respectively $\lambda/\mu \in \mathcal{V}$) and $|\lambda/\mu| = k$ then we will use the notation $\lambda/\mu \in \mathcal{H}_k$ (resp. $\lambda/\mu \in \mathcal{V}_k$).

Example 4. Let $\lambda = (6,4,4,2,1)$ and $\mu = (4,4,3,2)$ are represented by the following diagrams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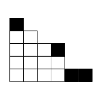
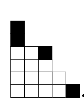
Then the skew diagram λ/μ is a horizontal strip and it is represented by the cells which are colored in the diagram below

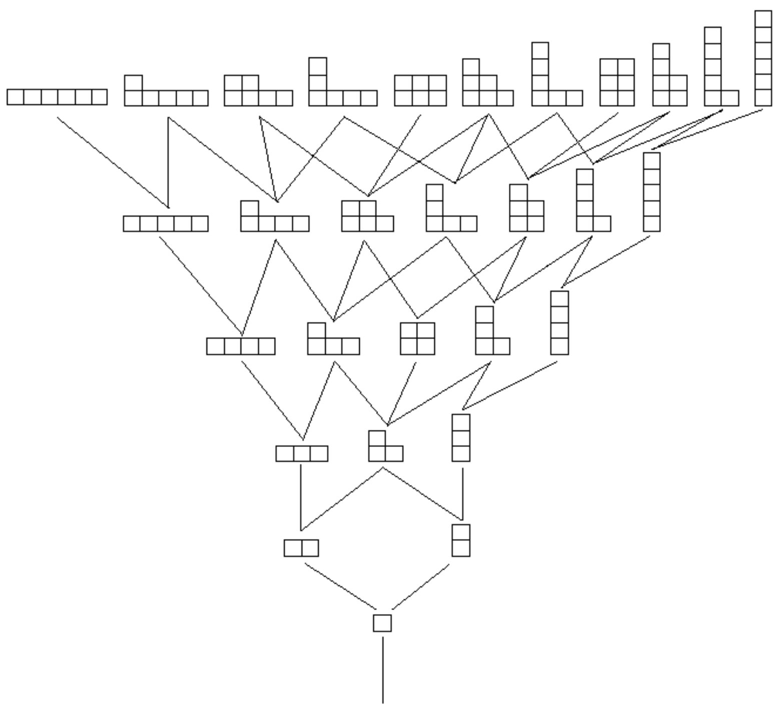


$\lambda' = (5,4,3,3,1,1)$ and $\mu' = (4,4,3,2)$ and λ'/μ' is a vertical strip and the diagram consists of the cells which are colored in the diagram


.

There are several ways of ordering the set of partitions. The first is to consider one partition μ is smaller than another λ if $D(\mu) \subseteq D(\lambda)$. We will denote this as an abuse of notation by $\mu \subseteq \lambda$. Note that $\mu \subseteq \lambda$ if and only if $\lambda_i \leq \mu_i$ for all i. This is a partial order on the set of partitions in that it possible that both $\mu \not\subseteq \lambda$ and $\lambda \not\subseteq \mu$.

Example 5. We list the partitions which are size less than or equal to 6 and place a line between two partitions if the smaller is less than the larger in containment order. This partial order defines a lattice structure on the set of partitions.

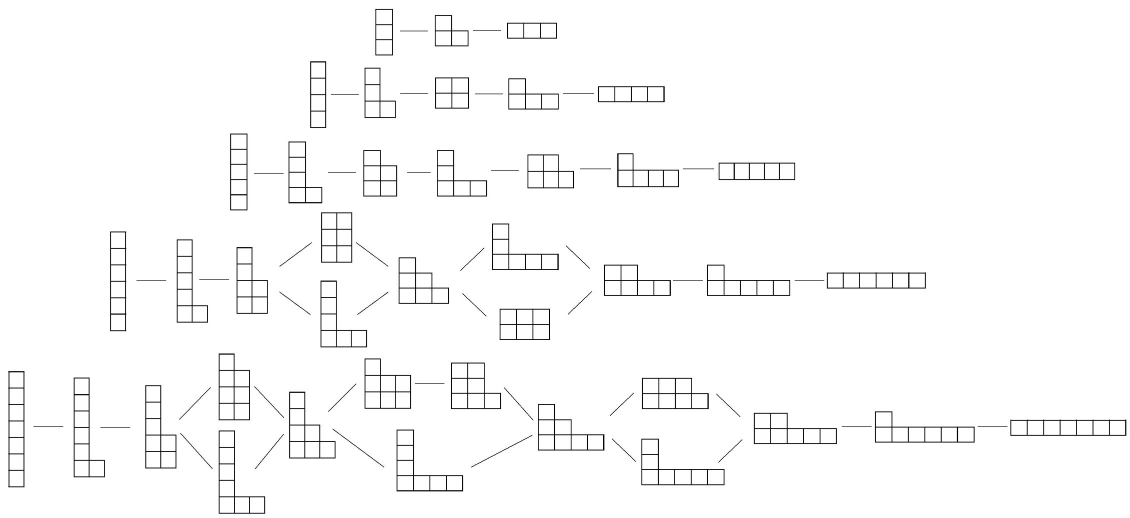


Exercise 4. Prove that $|\{\mu : \mu \subseteq \lambda\}| = |\{i : m_i(\lambda) > 0\}| = |\{\mu : \mu \supseteq \lambda\}| - 1$

Another way of ordering the partitions is to compare the entries lexicographically, that is if λ, μ are partitions then $\lambda <_{lex} \mu$ if either $\lambda_1 < \mu_1$ or $\lambda_1 = \mu_1$ and $(\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)}) <_{lex} (\mu_2, \mu_3, \ldots, \mu_{\ell(\mu)})$. This is a total order on partitions but it is possible that $|\lambda| > |\mu|$ and $\lambda <_{lex} \mu$. Although this is a perfectly natural order on the set of partitions it does not seem to be the one that arises most naturally in the algebra structures that we will study. This and similar orders (such as degree lex or reverse lex) are ones that we might use for studying ideals in a polynomial ring. For the time being this structure is not as important as the third order which we will define.

The last order we will consider is also a partial order on partitions which arises naturally from the algebra structure that we will be looking at. This order will be denoted by $<$ and we will say that $\lambda \leq \mu$ if and only if $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ for all $i \geq 1$. This partial order is a total order on partitions of size strictly less than 6. At $n = 6$ both the pair $(4, 1, 1)$ and $(3, 3)$ and the pair $(3, 1, 1, 1)$ and $(2, 2, 2)$ are not comparable with respect to this order.

EXAMPLE 6. Below we show the poset of partitions of size $3, 4, 5, 6$ and 7. Notice that for partitions of $3, 4$ and 5 this is a linear order but for the partitions of size 6 and 7 it is not since for example $(2, 2, 2)$ and $(3, 1, 1, 1)$ are not comparable in this order.



An intuitive definition of the statement $\mu \leq \lambda$ is that the diagram for μ is narrower and taller than λ. We also clearly have for $\mu \leq \lambda$ then $\mu \leq_{lex} \lambda$. The converse of this statement is not true.

EXERCISE 5. Prove that $\lambda \leq \mu$ if an only if $\mu' \leq \lambda'$.

For any n, let $p(n)$ represent the number of partitions of n. $\frac{1}{1-q^k} = \sum_{r\geq 0} q^{rk}$ is a generating function for the partitions which only contain parts of size k (that is, there is a rectangular partition at of size n consisting of parts of size only k if and only if $n = rk$ for some integer r). Since in a partition the parts may be chosen independently, the generating function for the sequence $p(n)$ is the product of the generating functions $\frac{1}{1-q^i}$ for all i, that is

$$\sum_{n\geq 0} q^n p(n) = \prod_{i\geq 1} \frac{1}{1-q^i}. \tag{2.2}$$

We can use this generating function formula to give the first few values of $p(n)$.

$$\sum_{n\geq 0} q^n p(n) = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + 15\,q^7$$
$$+ 22\,q^8 + 30\,q^9 + 42\,q^{10} + 56\,q^{11} + 77\,q^{12} + \cdots$$

Despite the fact that we can use the generating function to calculate the number of partitions, it does not really seem to be a very satisfying formula because in order to use it we potentially need to consider expanding the product of n terms in order to calculate the value of $p(n)$.

A remarkable fact was noticed by Leonhard Euler in 17** that we will exploit to find a recurrence on the number of partitions of n. He calculated the first several terms of the product $\prod_{i\geq 1}(1-q^i)$ and noticed that the coefficient was always equal to ± 1 or 0. In fact he was quite easily able to guess a formula for this denominator and several years later was able to prove it. We will take advantage of its existence to derive a method of calculating the number of partitions of size n.

Proposition 2.1. *(Euler's pentagonal number theorem)*

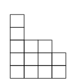
$$\prod_{i\geq 1} 1-q^i = 1 + \sum_{m\geq 1}(-1)^m \left(q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}}\right) \tag{2.3}$$

There is a clever proof of this proposition that comes from one of the first American mathematicians F. Franklin [?]. The proof uses a technique which is fairly ubiquitous in algebraic combinatorics, to show that terms in a sum cancel associate a combinatorial object to each term in the sum and then show that they cancel by producing a map which sends an element with positive weight to a term with negative weight.

There are several other accounts of this proof: [?], [?], [?], [?].

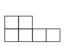
We will need to talk about partitions as a combinatorial object. λ is a partition if it is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{\ell(\lambda)}$ where we use the notation $\ell(\lambda)$ to represent the number of parts of λ. The symbol $|\lambda|$ will represent the size of the partition so that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}$. λ is a strict partition if in addition $\lambda_1 > \lambda_2 > \cdots \lambda_{\ell(\lambda)}$.

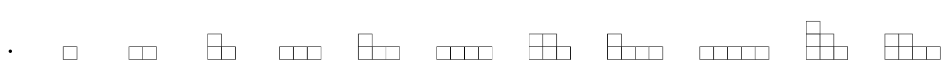
There is a way of graphically representing a partition with rows of boxes. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is represented by a row of λ_1 boxes below a row of λ_2 boxes below a row of λ_3 boxes etc. Each of these rows of cells will be left justified. For example the partition $(4, 4, 3, 1, 1)$ is represented by the following diagram:



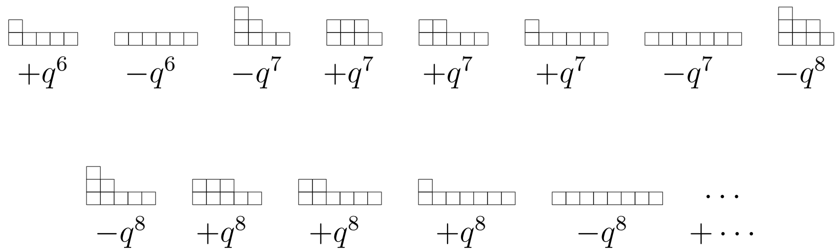
Example 7. We note that the left hand side of this equation is the generating function for all strict partitions (partitions where all parts are distinct) weighted with $(-1)^{\ell(\lambda)}q^{|\lambda|}$. That is,

$$\prod_{i\geq 1} 1-q^i = \sum_{\lambda\ strict} (-1)^{\ell(\lambda)} q^{|\lambda|} \tag{2.4}$$

This follows by observing that to determine the coefficient of q^n by expansion of the product on the left we have a contribution of $(-1)^k q^{\lambda_1+\lambda_2+\cdots+\lambda_k}$ for every sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i > \lambda_{i+1}$ for $1 \leq i < k$. Below we expand the terms of this generating function through degree 8. For example, a term of the form $(-q^4)(-q^2)$ is represented by the picture  and we record the weight of $+q^6$ just below the picture.



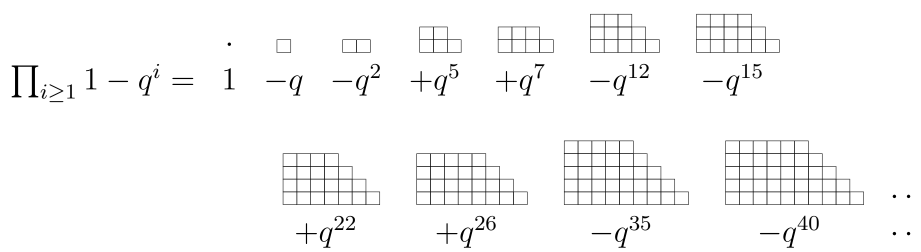
$1 \quad -q \quad -q^2 \quad +q^3 \quad -q^3 \quad +q^4 \quad -q^4 \quad +q^5 \quad +q^5 \quad -q^5 \quad -q^6 \quad +q^6$



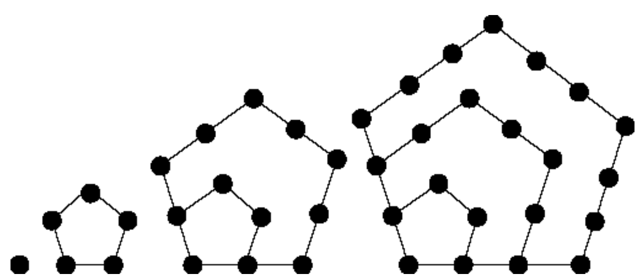
Now we notice that all of the terms cancel except for the ones stated in the theorem, that is we have

$$\prod_{i\geq 1} 1 - q^i = 1 - q - q^2 + q^5 + q^7 + \cdots$$

In fact, we will show that one way of looking at this expression is to observe terms which survive are those that correspond to the following pictures:



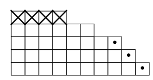
From the image in this example one might think that the theorem would be better named the *trapazoidal* number theorem. There is a reason that the numbers $m(3m-1)/2$ are referred to as pentagonal numbers and if $m \to -m$ then the pentagonal number is transformed to $\to -m(-3m-1)/2 = m(3m+1)/2$. Observe the picture below how a sequence of pentagons have exactly $m(3m-1)/2$ points in them (and this continues for $m > 4$).



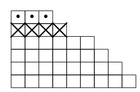
Proof. To show that this proposition holds we show that there is an involution ϕ on the strict partitions λ of n such that $\phi(\lambda)$ is also a partition of n and the length of $\phi(\lambda)$ will have length either one smaller or one larger than that of λ. This means that if the weight of a strict partition is $(-1)^{\ell(\lambda)}q^{|\lambda|}$ then the weight of $\phi(\lambda)$ is $-(-1)^{\ell(\lambda)}q^{|\lambda|}$ and so this term corresponding to $\phi(\lambda)$ will cancel with the term corresponding to λ. This involution will fail to 'work' for the partitions of the form $(2m-1, 2m-2, \ldots, m)$ which are of size $2m^2 - \frac{(m+1)m}{2} = \frac{m(3m-1)}{2}$ and $(2m, 2m-1, \ldots, m+1)$ which are of size $2m^2 - \frac{(m-1)m}{2} = \frac{m(3m+1)}{2}$.

For a strict partition λ we will let r equal to the smallest part of λ ($r = \lambda_{\ell(\lambda)}$) and let s equal the number of parts which are consecutive at the beginning of the partition. In other words s is the largest integer such that $(\lambda_1, \lambda_2, \ldots, \lambda_s) = (\lambda_1, \lambda_1 - 1, \ldots, \lambda_1 - s + 1)$.

If $s \neq \ell(\lambda)$ and $r > s$ then we will let $\phi(\lambda)$ equal the partition $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_{\ell(\lambda)}, s)$. That is, if the diagram for the partition looks something like the following where there is an $\times$ in each of the cells corresponding to r and a dot in the cells corresponding to s

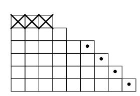


then $\phi(\lambda)$ will be the partition with the diagonal of s cells filled with a dot moved to the top row of the partition.

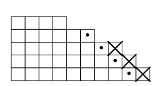


$\phi(\lambda)$ is still a strict partition and it has the property that the longest string of consecutive parts at the beginning of the partition is greater than or equal to s.

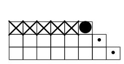
If $s \neq \ell(\lambda)$ and $r \leq s$ then we will let $\phi(\lambda)$ equal to the partition $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{\ell(\lambda)})$. For example, if our diagram is similar to the one below with the cells marked with an $\times$ representing the row of size r and those marked with the $\cdot$ represent the cells which correspond to the s consecutive parts at the beginning of the partition.



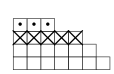
The partition corresponding to $\phi(\lambda)$ is then represented by the following picture.



Notice that it is also possible that $s = \ell(\lambda)$ and we consider this case separately because we need that r is at least 2 more cells than s does before we can move the s cells to the top row. In this case if $r > s + 1$ then we will remove the s cells along the diagonal and turn them into the shortest row so that $\phi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, s)$. For example we have the picture on the left will be transformed to the one on the right.

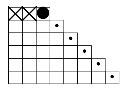


λ

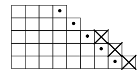


$\phi(\lambda)$

If $s = \ell(\lambda)$ and $r < s$ then it is still possible to move the shortest row of λ to the first r rows. We will set $\phi(\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_r, \ldots, \lambda_{\ell(\lambda)-1})$, this corresponds to the case when we have a partition of the form of the one below.



λ



$\phi(\lambda)$

If we describe what is happening to the diagram the map ϕ does one of two things, either it removes the smallest row of $r = \lambda_{\ell(\lambda)}$ cells of the partition and places one cell more in each of the first r rows (in the case that $r < s$ or $r = s$ and $s < \ell(\lambda)$) or it removes one cell from each of the first s rows and adds a row of size s to the top of the diagram (in the case that $r > s + 1$ or $r = s + 1$ and $s < \ell(\lambda)$).

Observe that if the weight of λ is $(-1)^{\ell(\lambda)}$ then since $\phi(\lambda)$ has the same number of cells and either one more or one less row than λ then the weight of $\phi(\lambda)$ is the negative of the weight of λ.

Also observe for each of the 4 cases we have considered, $\phi(\phi(\lambda))$ is just λ. This implies we can say that in the expansion of the expression $\sum_{\lambda\ strict}(-1)^{\ell(\lambda)}q^{|\lambda|}$, the term corresponding to the partition λ will cancel with the term corresponding to the partition $\phi(\lambda)$ because the then $\phi(\lambda)$ will also cancel with $\phi(\phi(\lambda)) = \lambda$.

There are two cases that we have not considered. These terms do not cancel. One is that $r = s$ and $s = \ell(\lambda)$ and so we have a partition of the form $(2m-1, 2m-2, \ldots, m)$ and the other is that $r = s+1$ and $s = \ell(\lambda)$ and this is a partition of the form $(2m, 2m-1, \ldots, m+1)$. $\square$

We encourage the reader to take a pencil and draw an arrow between the diagrams of the strict partitions given in the example above to show that the involution works as expected.

What this implies is that we can derive a recurrence which will allow us to calculate the number of partitions of n. Notice if we multiply the formula of Euler and the generating function for the number of partitions together we get 1.

$$1 = \prod_{i\geq 1} \frac{1}{1-q^i} \prod_{i\geq 1}(1-q^i) = \left(\sum_{n\geq 0} p(n)q^n\right)\left(1 + \sum_{m\geq 1}(-1)^m \left(q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}}\right)\right) \tag{2.5}$$

Now take the coefficient of q^n in both sides of this equation. For $n > 0$ we see that

$$0 = p(n) + \sum_{m\geq 1}(-1)^m \left(p\left(n - m(3m-1)/2\right) + p\left(n - m(3m+1)/2\right)\right) \tag{2.6}$$

where we are assuming here that $p(k) = 0$ if k is a negative number. By isolating $p(n)$ by itself we have the following recurrence.

Corollary 2.2. *$p(n) = 0$ for $n < 0$, $p(0) = 1$ and for $n > 0$ we have*

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \cdots$$

EXAMPLE 8.

$$
\begin{aligned}
p(1) &= p(0) = 1\\
p(2) &= p(1) + p(0) = 2\\
p(3) &= p(2) + p(1) = 3\\
p(4) &= p(3) + p(2) = 5\\
p(5) &= p(4) + p(3) - p(0) = 7\\
p(6) &= p(5) + p(4) - p(1) = 11\\
p(7) &= p(6) + p(5) - p(2) - p(0) = 15\\
p(8) &= p(7) + p(6) - p(3) - p(1) = 22\\
p(9) &= p(8) + p(7) - p(4) - p(2) = 30\\
p(10) &= p(9) + p(8) - p(5) - p(3) = 42\\
p(11) &= p(10) + p(9) - p(6) - p(4) = 56\\
p(12) &= p(11) + p(10) - p(7) - p(5) + p(0) = 77
\end{aligned}
$$

and this agrees with the generating function formula which we computed in an earlier example. This formulation is easier for implementing in an algorithm than the generating function formula.

It will be useful to consider the generating function for all partitions which fit inside of an $n \times k$ rectangle. This generating function $C_q(n,k) = \sum_{\lambda \subseteq (n^k)} q^{|\lambda|}$ must be finite and hence is a polynomial in q.

Every partition which fits inside of this $n \times k$ rectangle will have the property that either λ has a part of size n or it does not. If λ has a part of size n then $\lambda - (n)$ is a partition which fits inside of an $n \times (k-1)$ rectangle. In terms of the generating functions, this translates to the following recursion:

$$
C_q(n,k) = \sum_{\substack{\lambda \subseteq (n^k)\\ \lambda_1 = n}} q^{|\lambda|} + \sum_{\substack{\lambda \subseteq (n^k)\\ \lambda_1 \neq n}} q^{|\lambda|} = q^n C_q(n,k-1) + C_q(n-1,k). \tag{2.7}
$$

Since the generating function for the partitions which fit inside of an $n \times k$ rectangle is the same as the generating function which fit inside of a $k \times n$ rectangle, it follows that $C_q(n,k) = C_q(k,n)$ and hence using equation (2.7) we also have

$$
C_q(n,k) = q^k C_q(n-1,k) + C_q(n,k-1). \tag{2.8}
$$

Now set $[n]_q = \frac{1-q^n}{1-q} = \sum_{i=1}^n q^{i-1}$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q = \frac{1}{(1-q)^n} \prod_{i=1}^n (1-q^i)$, and

$$
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{\prod_{i=1}^n (1-q^i)}{\prod_{i=1}^k (1-q^i) \prod_{j=1}^{n-k} (1-q^j)}. \tag{2.9}
$$

The symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is called the q-binomial coefficient or Gaussian polynomial. It is not obvious from this definition that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is even an integer, in fact it seems to be no more than a rational function in q, but like the binomial coefficient the denominator always cancels with the numerator. This fact is not transparent until we make the following identification with the generating functions $C_q(n,k)$ which are clearly polynomials in q. What is clear however is that if $q = 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$ because we can easily check $[n]_1 = n$ and $[n]_1! = n!$. The following statement makes it clear however that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with non-negative integer coefficients.

EXERCISE 6. (credit D. Stanton) If n and k are relatively prime then

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

is a polynomial in q with positive integer coefficients. Is there a set of combinatorial objects for which this is a generating function?

PROPOSITION 2.3. *For $n, k \geq 0$,*

$$C_q(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

PROOF. We will establish that the q-binomial coefficients satisfy the same recurrence as the polynomials $C_q(n,k)$ and that they agree for the obvious bases cases and hence are the equal. For $n, k \geq 0$,

$$\begin{aligned}
\begin{bmatrix} n+k \\ k \end{bmatrix}_q &= \frac{\prod_{i=1}^{n+k}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\
&= \frac{(1-q^n+q^n-q^{n+k})\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\
&= \frac{\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n-1}(1-q^j)} + q^n \frac{\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k-1}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\
&= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q
\end{aligned}$$

Now since the generating function $C_q(n,k)$ also satisfies $C_q(n,k) = C_q(n-1,k) + q^n C_q(n,k-1)$ and $C_q(n,0) = C_q(0,k) = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} k \\ k \end{bmatrix}_q = 1$ then we have that $C_q(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$ for all $n, k \geq 0$. □

EXAMPLE 9. Calculate the polynomial

$$\begin{aligned}
\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q &= \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}{(1-q)(1-q^2)(1-q)(1-q^2)(1-q^3)} \\
&= (1+q^2)(1+q+q^2+q^3+q^4) \\
&= 1+q+2q^2+2q^3+2q^4+q^5+q^6
\end{aligned}$$

This polynomial is counting the number of partitions which fit inside of a 2×3 rectangle. We draw the 2×3 rectangle below 10 times and fill it with each of the partitions which fit in this rectangle.

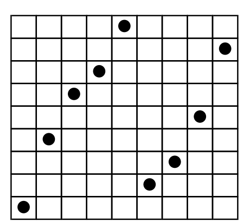
2.0.1. Grassmannian permutations and partitions. Recall that a permutation $\sigma \in Sym_n$ is called Grassmannian if it has at exactly one descent. If the descent is a position k then the sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ is an increasing sequence of integers. In fact, this sequence completely determines the permutation since $\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_n$ is also an increasing sequence and consists of the remaining integers. If $\{\sigma_1, \sigma_2, \ldots, \sigma_k\} = \{1, 2, \ldots k\}$ then there cannot be a descent at position k because the next smallest value that can appear is $k+1$. If the first k values are not just the values 1 through k and they are in increasing order and the last $n-k$ values are in increasing order then σ is a Grassmannian permutation.

So every subset of size k of the integers $\{1, 2, \ldots, n\}$ except for the set $\{1, 2, \ldots, k\}$ determines a Grassmannian permutation with a descent at position k therefore there are $\binom{n}{k} - 1$ Grassmannian permutations in Sym_n with a descent at position k and

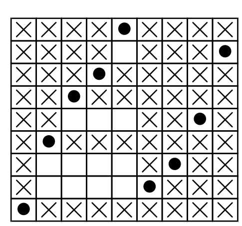
$$\sum_{k=1}^{n-1} \left(\binom{n}{k} - 1 \right) = \left(\sum_{k=0}^{n} \binom{n}{k} \right) - n = 2^{n-1} - n \tag{2.10}$$

Grassmannian permutations in Sym_n with a descent anywhere.

To every σ in Sym_n we can associate an $n \times n$ subset of cells with a mark in the (i, σ_i) cell for each $1 \leq i \leq n$ (the bottom left hand corner of this diagram will have the coordinate $(1,1)$). For example, consider the Grassmannian permutation 146792358 which will be associated to the diagram



Now for every point in this grid, place an $\times$ in every cell that lies directly north or directly east of a $\bullet$.



Notice that the cells which do not have either a $\bullet$ or a $\times$ appear as empty rows and these will be decreasing in size. If the permutation is Grassmannian then the sequence of rows which are not filled can be viewed in a natural way as a partition. If the descent in the permutation is at position k then this partition will have width less than or equal to k and height less than or equal to $n-k$ since the rows $\sigma_1, \sigma_2, \ldots, \sigma_k$ will all be filled.

2.1. Tableaux

A tableau in the most general sense we will consider will be a function from a subset S of points in $\mathbb{Z} \times \mathbb{Z}$ to a set of labels M. If $S = D(\lambda)$ (or $D(\lambda/\mu)$) then λ (respectively λ/μ) is referred to as the shape of the tableau T. $\lambda(T)$ will sometimes be used to denote the domain of the tableau T and in particular when the domain of this function is a partition $D(\lambda)$ or $D(\lambda/\mu)$ it will represent λ or λ/μ.

A tableau can then be represented with a diagram where $\lambda(T)$ is drawn as the subset of cells on the $\mathbb{Z} \times \mathbb{Z}$ grid just as we drew the diagram for a partition in the previous section, and then the value of T_s is recorded in the cell $s \in \lambda(T)$.

EXAMPLE 10. Let T be a tableau with $\lambda(T)$ equal to $D(\lambda)$ where $\lambda = (2, 1)$. In addition, say that $T_{(0,0)} = 1$, $T_{(1,0)} = 3$ and $T_{(0,1)} = 2$. The diagram for tableau T will be

$$\begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}\,.$$

Tableaux are a convenient means of recording information about partitions and it will be important for us to examine certain classes of tableaux. For the moment we will concentrate on tableaux with $\lambda(T)$ is the diagram for a partition or a skew-partition with labels of the tableaux (the range of the tableaux) limited to positive integers.

We will define the content of a tableau to be the sequence which records the number of cells labeled with each value i. Let $n_i(T) = \#\{s : T_s = i\}$ and define $\mu(T) = (n_1(T), n_2(T), n_3(T) \ldots)$. As long as T is a finite tableau (we will never consider the infinite case) the sequence $\mu(T)$ will have all but a finite number of entries non-zero and largest label which appears in the tableau T will be denoted $\ell(\mu(T))$.

We will call a tableau *column strict* if $T_{(i,j)} \leq T_{(i+1,j)}$ for all pairs $(i, j), (i + 1, j) \in \lambda(T)$ (the entries are weakly increasing in the rows of the partition) and $T_{(i,j)} < T_{(i,j+1)}$ whenever $(i, j), (i, j + 1) \in \lambda(T)$ (the entries are strictly increasing in the columns of the partition).

A column strict tableau can be seen as a sequence of partitions, if we restrict our attention to the set of cells $\{s : T_s \leq k\}$ (we shall denote T restricted to this domain by $T\big|_{1\ldots k}$) then the shape will always be a partition. Since $T\big|_{1\ldots k-1}$ and $T\big|_{1\ldots k}$ have the shape of partitions and there can be at most one cell labeled by a k in each column from our restrictions on the labeling of cells, then $\lambda(T\big|_{1\ldots k})/\lambda(T\big|_{1\ldots k-1}) \in \mathcal{H}$.

Stated in another way, a column strict tableau T can be thought of as a sequence of partitions. Set $\lambda^{(k)} := \lambda(T\big|_{1\ldots k})$, then

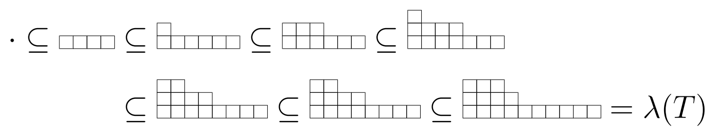
$$\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(n)} \subseteq \lambda(T) \tag{2.11}$$

with the property that $\lambda^{(k)}/\lambda^{(k-1)} \in \mathcal{H}$.

Example 11. Let

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 4&5&7\\ \cline{1-3} 2&3&3&4\\ \cline{1-4} 1&1&1&1&2&2&4&5&7&7\\ \hline\end{array}.$$

The content of the tableau T is $\mu(T) = (4, 3, 2, 3, 2, 0, 3)$. Notice that the tableau T satisfies the condition of being column strict. One can think of T as an object which conveniently records a sequence of partitions and the condition of being column strict records the fact that each partition differs from the next by a horizontal strip.



Reading word of T. Content of T. $n_i(T)$ number of occurrences of the

2.1.1. Crystal Operators. If the number of cells labeled $i + 1$ is greater than the number of cells labeled with an i then an operation that we would like to be able to perform is to change one of the labels of $i + 1$ to an i in such a way that the operation is reversible. We will define these operations as crystal operators because they appear in crystal basis theory although they are also useful in the theory of symmetric functions.

Fix an integer i and let $w = R(T)$ be the reading word of the tableau T. Consider the sequence of symbols ')' and '(' formed by reading w from left to right and recording a close parenthesis ')' for each i that occurs in w and an open parenthesis for each $i+1$ that occurs in w.

We will let $\varepsilon_i(T)$ be the number of open parentheses that are not matched with a close parenthesis that lies to the right and let $\phi_i(T)$ be the number of close parentheses that do not match with an open parenthesis to the left.

Restated in another way, $\phi_i(T) = max\{n_i(w_{1...k}) - n_{i+1}(w_{1...k}) : 1 \leq k \leq \ell(w)\}$ and $\varepsilon_i(T) = max\{n_{i+1}(w_{k...\ell(w)}) - n_i(w_{k...\ell(w)}) : 1 \leq k \leq \ell(w)\}$ where $n_i(w)$ is the number of occurrences of i in the word $w = w_1 w_2 \cdots w_{\ell(w)}$ and $w_{a...b}$ is the subword of w equal to $w_a w_{a+1} \cdots w_b$.

Example 12.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 5&6&6\\ \cline{1-3} 3&4&5&7&7&8\\ \cline{1-6} 2&2&3&4&4&5&5&9\\ \hline 1&1&1&1&2&2&4&8\\ \hline\end{array}$$

For this tableaux $R(T) = w = 566345778223445591112248$. We list below the value of i and the sequence of parentheses that is relevant to the calculation of $\phi_i(T)$ and $\varepsilon_i(T)$.

$i = 1$	(()))((	$\phi_1(T) = 2$	$\varepsilon_1(T) = 2$
$i = 2$	())())	$\phi_2(T) = 2$	$\varepsilon_1(T) = 0$
$i = 3$	)()(((	$\phi_3(T) = 1$	$\varepsilon_1(T) = 3$
$i = 4$	()()(()	$\phi_4(T) = 0$	$\varepsilon_1(T) = 1$
$i = 5$	)(()))	$\phi_5(T) = 2$	$\varepsilon_1(T) = 0$
$i = 6$	))((	$\phi_6(T) = 2$	$\varepsilon_1(T) = 2$
$i = 7$	))((	$\phi_7(T) = 2$	$\varepsilon_1(T) = 2$
$i = 8$	)()	$\phi_8(T) = 1$	$\varepsilon_1(T) = 0$

We will define the crystal operator f_i to act on tableaux such that $\phi_i(T) > 0$ (and if $\phi_i(T) \leq 0$ then f_iT will be undefined or as it is sometimes stated, $f_iT = 0$). Let $s \in \lambda(T)$ be the cell such that $T_s = i$ and s corresponds to the rightmost close parenthesis ')' which is unmatched in the parenthesis sequence. f_iT is the tableau such that $(f_iT)_s = i + 1$ and $(f_iT)_c = T_c$ for $c \in \lambda(T)$ and $c \neq s$.

The crystal operator e_i is the inverse of this operation. It is defined on tableau such that $\varepsilon_i(T) > 0$. If $s \in \lambda(T)$ is the cell of T such that $T_s = i+1$ and it corresponds to the leftmost open parenthesis '(' which is unmatched in the parenthesis sequence then $(e_iT)_s = i$ and e_iT has all of the other entries exactly the same as in T (that is, $(e_iT)_c = T_c$ for $c \in \lambda(T)$ and $c \neq s$).

Example 13. Let

$$T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&5 \\ \hline 2&3&3&4 \\ \hline 1&1&1&3 \\ \hline\end{array}$$

then we have

$$f_1T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&5 \\ \hline 2&3&3&4 \\ \hline 1&1&2&3 \\ \hline\end{array} \qquad e_2T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&5 \\ \hline 2&2&3&4 \\ \hline 1&1&1&3 \\ \hline\end{array}$$

$$f_3T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&5 \\ \hline 2&3&4&4 \\ \hline 1&1&1&3 \\ \hline\end{array} \qquad e_4T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&4 \\ \hline 2&3&3&4 \\ \hline 1&1&1&3 \\ \hline\end{array}$$

$$f_4T = \begin{array}{|c|c|c|c|}\cline{1-3} 3&4&5 \\ \hline 2&3&3&5 \\ \hline 1&1&1&3 \\ \hline\end{array}$$

while each of f_2T, e_1T and e_3T are undefined.

From these definitions we observe the following properties of these operators:

Proposition 2.4. *If $\phi_i(T) > 0$ then $\phi_i(f_iT) = \phi_i(T) - 1$ and $\varepsilon_i(f_iT) = \varepsilon_i(T) + 1$, and in addition*

$$e_if_iT = T. \tag{2.12}$$

Similarly, if $\varepsilon_i(T) > 0$ *then* $\phi_i(e_iT) = \phi_i(T) + 1$ *and* $\varepsilon_i(e_iT) = \varepsilon_i(T) - 1$*, and in addition*

$$f_ie_iT = T. \tag{2.13}$$

Proof. Notice that the operation f_i has the effect of changing the corresponding parenthesis sequence of T by changing an unmatched close parenthesis to an unmatched open parenthesis so that $\phi_i(f_iT)$ will be one smaller than $\phi(T)$ and $\varepsilon_i(f_iT)$ will be one larger than $\phi(T)$. In fact, since the rightmost unmatched close parenthesis is changed to the leftmost unmatched open parenthesis, e_i will will have the inverse effect when it acts on f_iT.

Similar statements about the operator e_i justify the second part of this proposition. □

The crystal operators can be used to define a symmetric group action on the content of tableaux. Let $s_iT = f_i^{\phi_i(T)-\varepsilon(T)}T$ where we have set $f_i^{-k}T = e_i^kT$ for $k > 0$.

The operators s_i are the generators for a symmetric group action on the content of the tableau T, since if $\mu(T) = (a_1, a_2, \ldots, a_n)$ then $\mu(s_iT) = (a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_n)$. We must justify however that the operators s_i which satisfy the Coxeter relations and therefore define a symmetric group action.

Proposition 2.5. *For* $1 \leq i \leq max\{T_s : s \in \lambda(T)\}$,

$$s_i^2T = T,$$
$$s_is_jT = s_js_iT \quad for \quad |i-j| > 1$$
$$s_is_{i+1}s_iT = s_{i+1}s_is_{i+1}T$$

Proof. From the previous proposition we see that $\phi_i(s_iT) = \phi_i(f_i^{\phi_i(T)-\varepsilon_i(T)}T) = \phi_i(T) - (\phi_i(T) - \varepsilon_i(T)) = \varepsilon_i(T)$ and similarly, $\varepsilon(s_iT) = \varepsilon_i(T) + \phi_i(T) - \varepsilon_i(T) = \phi_i(T)$. Therefore to compute

$$\begin{aligned} s_i^2T = s_i(s_iT) &= f_i^{\phi_i(s_iT)-\varepsilon_i(s_iT)}s_iT \\ &= f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = T \end{aligned}$$

There are two cases to show why the last expression is T. If $\phi_i(T) - \varepsilon_i(T) > 0$, then $f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = e_i^{\phi_i(T)-\varepsilon_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = T$. If we have $\phi_i(T) - \varepsilon_i(T) < 0$, then $f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = f_i^{\phi_i(T)-\varepsilon_i(T)}e_i^{\phi_i(T)-\varepsilon_i(T)}T = T$.

Now assume that $|i-j| > 1$. The action of s_i on T changes some of the labels with i to $i+1$ (or the reverse). Since the entries labeled with an i and $i+1$ are the same in both T and s_jT, s_i has the same effect on both of these tableaux. Similarly, the entries of j and $j+1$ are the same in both T and s_iT and so s_j has the effect on both these tableaux. For this reason $s_is_jT = s_js_iT$. □

A column strict tableau T will be called *standard* if it is a bijection from the the set $\lambda(T)$ to the labels $\{1, 2, \ldots, |\lambda(T)|\}$.

Theorem 2.6. *The number of standard tableaux of shape* λ *with* $\lambda \vdash n$ *is equal to*

$$\frac{n!}{\prod_{s\in\lambda} h_\lambda(s)} \tag{2.14}$$

where $h_\lambda(s)$ *is hook length of the cell* s *in the partition* λ *(that is,* $h_\lambda(i,j) = \lambda_i - i + 1 + \lambda'_j - j$*).*

CHAPTER 3

The algebra structure of the ring of symmetric functions

Consider the polynomial ring in the variables p_i for $i \geq 1$, $\Lambda = \mathbb{Q}[p_1, p_2, p_3, \ldots]$. We will define the degree of a variable p_k to be k and so the degree of a monomial $p_{k_1} p_{k_2} \cdots p_{k_\ell}$ is simply $k_1 + k_2 + \cdots + k_\ell$. Λ is the ring of symmetric functions.

This is a very abstract way to begin, but at the end of this chapter we will draw a connection between this algebra and the space of class functions of the symmetric group. From this perspective Λ can be seen as an infinite dimensional graded vector space where the symmetric functions of degree m are a finite dimensional subspace.

The elements p_i are refered to as the power generators and since we are considering them as the variables in a commutative polynomial ring the space is spanned by the monomials in these variables. To specify a basis of this space we may assume that the variables are listed in weakly decreasing order. That is, if we denote Λ_m by the symmetric functions of degree m, then the set $\{p_\lambda : \lambda \vdash m\}$ forms a basis for Λ_m where $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$. The set $\bigoplus_{m \geq 0} \{p_\lambda\}_{\lambda \vdash m}$ is called the power basis. Note that the degree of a monomial p_λ is given by $|\lambda|$.

Λ has a natural 'un-multiplication' operation called a coproduct. In the sense if the product represents a way of putting elements in the algebra together, the coproduct represents ways of pulling elements apart. This can be a very interesting operation, especially when the multiplication and comultiplication interact.

Formally, we define the multiplication function on this algebra $\mu : \Lambda \otimes \Lambda \longrightarrow \Lambda$ as

$$\mu(p_\lambda \otimes p_\mu) = p_\lambda p_\mu = p_{\lambda_1} \cdots p_{\ell(\lambda)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}} = p_{\lambda \uplus \mu}. \tag{3.1}$$

The comultiplication will be denoted as $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$ is given by $\Delta(p_k) = 1 \otimes p_k + p_k \otimes 1$ (this is the property that the power generators are primitive in this algebra). We impose that it is a ring homomorphism, that is for $f, g \in \Lambda$, $\Delta(fg) = \Delta(f)\Delta(g)$ and and $c, d \in \mathbb{Q}$, $\Delta(cf + dg) = c\Delta(f) + d\Delta(g)$. Hence for an arbitrary basis element we have

Proposition 3.1.

$$\Delta(p_\lambda) = \sum_{\mu \uplus \nu = \lambda} \prod_{i=1}^{\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu \otimes p_\nu = \sum_{\mu \uplus \nu = \lambda} \frac{z_\lambda}{z_\mu z_\nu} p_\mu \otimes p_\nu \tag{3.2}$$

By rearranging the coefficients of (3.2) we can see a natural basis to consider is p_λ/z_λ because it arises naturally in this formula in the sense that

$$\Delta\left(\frac{p_\lambda}{z_\lambda}\right) = \sum_{\mu \uplus \nu = \lambda} \frac{p_\mu}{z_\mu} \otimes \frac{p_\nu}{z_\nu}. \tag{3.3}$$

Proof. First we note that the coefficients in the two formulations of equation (3.2) are equal since

$$\frac{z_\lambda}{z_\mu z_\nu} = \prod_{i=1}^{\lambda_1} \frac{i^{m_i(\lambda)} m_i(\lambda)!}{i^{m_i(\mu)} m_i(\mu)! i^{m_i(\nu)} m_i(\nu)!} = \prod_{i=1}^{\lambda_1} \frac{m_i(\lambda)!}{m_i(\mu)! m_i(\nu)!} = \prod_{i=1}^{\lambda_1} \binom{m_i(\lambda)}{m_i(\mu)}.$$

We show this proposition by induction on the number of parts of λ. We have a base case since the formula clearly works if λ has only one part by definition.

Let λ be a partition of n and denote $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$. It follows that

$$\Delta(p_\lambda) = \Delta(p_{\lambda_1})\Delta(p_{\overline{\lambda}}) = (p_{\lambda_1} \otimes 1 + 1 \otimes p_{\lambda_1}) \left(\sum_{\mu \uplus \nu = \overline{\lambda}} \prod_{i=1}^{\lambda_2} \binom{m_i(\overline{\lambda})}{m_i(\mu)} p_\mu \otimes p_\nu \right). \tag{3.4}$$

If $\lambda_1 \neq \lambda_2$ there is nothing to prove since $\binom{m_i(\overline{\lambda})}{m_i(\mu)} = \binom{m_i(\lambda)}{m_i(\mu)}$ for all $1 \leq i \leq \lambda_2$ and the expansion of the right hand side is exactly as stated in the proposition since $\binom{m_{\lambda_1}(\lambda)}{m_{\lambda_1}(\mu)} = 1$.

If $\lambda_1 = \lambda_2$ then $m_{\lambda_1}(\overline{\lambda}) = m_{\lambda_1}(\lambda) - 1$ and $m_i(\overline{\lambda}) = m_i(\lambda)$ for $1 \leq i < \lambda_1$, hence we see

$$\begin{aligned}
\Delta(p_{\lambda_1})\Delta(p_{\overline{\lambda}}) &= (p_{\lambda_1} \otimes 1 + 1 \otimes p_{\lambda_1}) \left(\sum_{j=0}^{m_{\lambda_1}(\overline{\lambda})} \sum_{\substack{\mu \uplus \nu = \overline{\lambda} \\ m_{\lambda_1}(\mu) = j}} \binom{m_{\lambda_1}(\lambda) - 1}{j} \prod_{1 \leq i < \lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu \otimes p_\nu \right) \\
&= \sum_{j=0}^{m_{\lambda_1}(\lambda)} \sum_{\substack{\mu \uplus \nu = \overline{\lambda} \\ m_{\lambda_1}(\mu) = j}} \left(\binom{m_{\lambda_1}(\lambda) - 1}{j} + \binom{m_{\lambda_1}(\lambda) - 1}{j-1} \right) \prod_{1 \leq i < \lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu \otimes p_\nu \\
&= \sum_{\mu \uplus \nu = \overline{\lambda}} \binom{m_{\lambda_1}(\lambda)}{m_{\lambda_1}(\mu)} \prod_{1 \leq i < \lambda_1} \binom{m_i(\lambda)}{m_i(\mu)} p_\mu \otimes p_\nu.
\end{aligned}$$

And so it follows by induction that this formula holds for all partitions λ. □

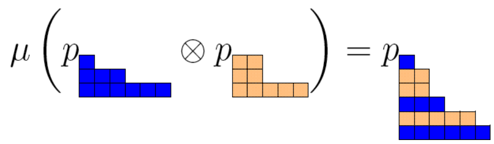
The p_k have the property that $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$ and hence are called the primitive elements of this algebra. We remark that there is a map S called the antipode with the property

$$\mu \circ (id \otimes S) \circ \Delta(f) = 0 \tag{3.5}$$

for all $f \in \Lambda$ such that f has 0 constant term. We set $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda$ and extend this map linearly and it is easy to check that $\mu \circ (id \otimes S) \otimes \Delta(p_k) = \mu(p_k \otimes 1 - 1 \otimes p_k) = 0$ and similarly for p_λ. This implies that $\mu \circ (id \otimes S) \circ \Delta(f)$ is equal to the constant term of f for all $f \in \Lambda$.

Therefore, so far our algebra of symmetric functions is very simple, but we should develop some intuitive ideas on how to picture what this algebra is. Now if $f \in \Lambda$, then f is some polynomial in variables p_i. Since we are using partitions to index our basis we often just write p_λ when we talk about the basis elements when we consider Λ as a vector space over $\mathbb{Q}$. The indexing set of partitions can be represented by their Young diagrams so when we take the product of p_λ and p_μ, $\mu(p_\lambda \otimes p_\mu)$ represents shuffling the Young digrams together.

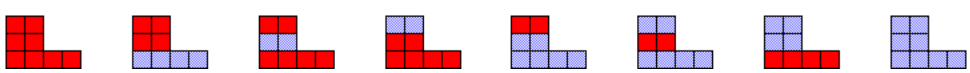
Example 14. Consider for instance $\lambda = (6, 3, 1)$ and $\mu = (5, 2, 2)$. This can be represented by the picture



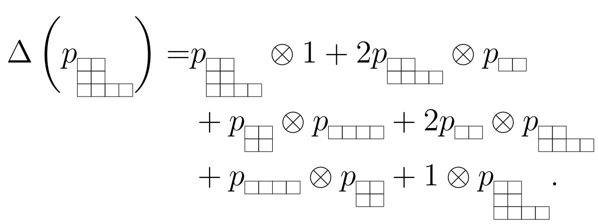
This diagram is simply representing the equation $p_{(6,3,1)}p_{(5,2,2)} = p_{(6,5,3,2,2,1)}$.

We should also develop a combinatorial picture of what happens when Δ acts on a term p_λ. Because the p_k are primitive elements, there will be $2^{\ell(\lambda)}$ terms in the expansion of $\Delta(p_\lambda)$.

Example 15. We compute the action of Δ on $p_{(5,2,2)}$ and do this by computing the number of ways of coloring the rows of the partition $(5, 2, 2)$ using two colors so that the whole row has the same color. This is represented by the following picture.



The blue rows will be in the left tensor and red will be in the right tensor (although the colors are symmetric) so we have determined



Here we are splitting the partition up into pieces such that their union is the original partition in all possible ways. Notice that the sum of the coefficients in this expression is $8 = 2^{\ell(5,2,2)}$. This picture will help us gain some intuition as we develop this algebra more completely.

Remark 1. The reader is encouraged to try to develop some sort of a picture each time a formula appears in this presentation since the formulas are difficult to appreciate unless some meaning is assigned to the symbols we are working with.

There are generally considered to be 6 'standard' bases of the symmetric functions since these bases are fundamental in the development of tools to describe the calculus of symmetric functions. After the power symmetric basis we will introduce the homogeneous basis and the elementary basis as these are defined as products of generators. We will save the definition of the Schur basis and the monomial basis for a later section. [1]

For $n > 0$, we set

$$h_n = \sum_{\lambda \vdash n} p_\lambda / z_\lambda, \tag{3.6}$$

these are the homogeneous generators. We also define

$$e_n = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} p_\lambda / z_\lambda \tag{3.7}$$

which are the elementary generators. For a partition λ we set $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\ell(\lambda)}$ and $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\ell(\lambda)}$. Clearly we have the triangularity relations that $h_\lambda = p_\lambda / \prod_{i=1}^{\ell(\lambda)} \lambda_i +$ terms containing p_μ with μ smaller than λ in lexicographic order (and a similar relation with e_λ). This implies that $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$ are bases for the symmetric functions and that $\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots] = \mathbb{Q}[e_1, e_2, e_3, \ldots]$. Also set as a convention $p_0 = h_0 = e_0 = 1$ and $p_{-n} = h_{-n} = e_{-n} = 0$ for $n > 0$, so that formulas which require us to refer to these elements make sense.

There are several ways of picturing what the elements h_n and e_n represent. In some sense, $n!h_n$ is the generating function of all permutations of the symmetric group Sym_n with weight 1 for each element which we can see in the following formula.

$$n!h_n = \sum_{\sigma \in Sym_n} p_{\lambda(\sigma)} = \sum_{\lambda \vdash n} \cdot \frac{n!}{z_\lambda} p_\lambda. \tag{3.8}$$

At the same time $n!e_n$ is a signed generating function with weight equal to $(-1)^{n-\ell(\lambda)}$ if the permutation has cycle type λ.

$$n!e_n = \sum_{\sigma \in Sym_n} \epsilon(\sigma) p_{\lambda(\sigma)}. \tag{3.9}$$

We will see when we introduce the Schur functions that these formulas are a special case of one where the elements of Λ are generating functions for the irreducible characters of the symmetric group and h_n is representing the trivial character and e_n is representing the sign character.

[1]There is another basis which is typically called the forgotten basis which completes the analogy, 'the homogeneous basis is to the monomial basis as the elementary basis is to the (um, I forget) basis.' There are few direct formulas for the forgotten basis except for those which are analogous to those for the monomial basis and hence remains somewhat underdeveloped in our account.

Set $P(t) = \sum_{r\geq 1} p_r t^r/r$ as a generating function for the power generators and set $H(t) = exp(P(t))$. Notice by the following calculation we have

$$
\begin{aligned}
H(t) &= exp\left(\sum_{r\geq 1} p_r t^r/r\right) = \prod_{r\geq 1} exp(p_r t^r/r) \\
&= \prod_{r\geq 1}\sum_{n\geq 0} \frac{p_r^n}{r^n n!} t^{nr} \qquad (3.10)\\
&= \sum_{k\geq 0}\sum_{\lambda\vdash k} \frac{p_\lambda}{z_\lambda} t^k = \sum_{k\geq 0} h_k t^k.
\end{aligned}
$$

Similarly we may easily show that $E(t) = exp(-P(t)) = \sum_{n\geq 0}(-1)^n e_n t^n$. Simply by definition of these generating functions we have the relation

$$H(t)E(t) = exp(P(t))exp(-P(t)) = 1 \qquad (3.11)$$

We can also consider the product of these generating functions explicitly and take the coefficient of t^n. On the right hand side the coefficient is 0 as long as $n > 0$ and the coefficient on the left hand side shows that

$$\sum_{k=0}^{n} (-1)^k h_{n-k} e_k = 0. \qquad (3.12)$$

Define the ring homomorphism on Λ that sends $\omega(p_k) = (-1)^{k-1}p_k$. Clearly, ω is an involution and is related to the antipode map on Λ by $\omega(p_\lambda) = (-1)^{|\lambda|}S(p_\lambda)$. By going back to formulas (3.6) and (3.7) for h_k and e_k in terms of p_λ we see that ω relates the $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$ bases by $\omega(h_\lambda) = e_\lambda$.

By exploiting the generating functions $P(t), H(t)$ and $E(t)$ further we can extract other algebraic relations between the elements of this ring. For instance, notice that $P(t) = log(H(t))$ and hence $P'(t) = H'(t)/H(t)$. Therefore by taking the coefficient of t^{n-1} in $P'(t)H(t) = H'(t)$ we see that

$$nh_n = \sum_{k=1}^{n} h_{n-k} p_k. \qquad (3.13)$$

By an application of ω on each side of this equation we also see that

$$ne_n = \sum_{k=1}^{n} (-1)^{k-1} e_{n-k} p_k \qquad (3.14)$$

Equations (3.12), (3.13) and (3.14) give us a simple recursive method to express any of the algebraic generators of this space in terms of any other because the term containing h_n, e_n or p_n can be isolated to provide algebraic relations. These recursive definitions will be exactly the method that we use when we develop computer functions in Maple to change between bases.

EXAMPLE 16. If we wish to expand h_3 in the elementary basis we note that $h_3 = h_2 e_1 - h_1 e_2 + e_3$, $h_2 = h_1 e_1 - e_2$ and $h_1 = e_1$. Combining these we find that $h_3 = e_1^3 - 2e_2 e_1 + e_3$.

We may also use these equations derive simple results using calculations by induction. For instance, (3.13) may be used to derive by induction by induction the following action of the coproduct Δ, acting on the symmetric function h_k.

PROPOSITION 3.2.

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k} \tag{3.15}$$

PROOF. Assume by induction that we know that equation (3.15) is true for all $k < n$. Then we know that

$$\begin{aligned}
\Delta(nh_n) &= \sum_{r=1}^{n} \Delta(p_r h_{n-r}) \\
&= \sum_{r=1}^{n} \left((p_r \otimes 1) \sum_{k=0}^{n-r} h_{n-r-k} \otimes h_k + (1 \otimes p_r) \sum_{k=0}^{n-r} h_k \otimes h_{n-r-k} \right) \\
&= \sum_{r=1}^{n} \left(\sum_{k=0}^{n-r} p_r h_{n-r-k} \otimes h_k + \sum_{k=0}^{n-r} h_k \otimes p_r h_{n-r-k} \right) \\
&= \sum_{k=0}^{n-1} \sum_{r=1}^{n-k} p_r h_{n-r-k} \otimes h_k + \sum_{k=0}^{n-1} \sum_{r=1}^{n-k} h_k \otimes p_r h_{n-r-k} \\
&= \sum_{k=0}^{n-1} (n-k) h_{n-k} \otimes h_k + \sum_{k=0}^{n-1} h_k \otimes ((n-k) h_{n-k}) \\
&= nh_n \otimes 1 + n \sum_{k=1}^{n-1} h_k \otimes h_{n-k} + 1 \otimes nh_n \\
&= n \sum_{k=0}^{n} h_k \otimes h_{n-k}
\end{aligned}$$

□

This last result gives us an interesting combinatorial way of looking at the action of Δ on the functions h_λ. On a single h_n, Δ acts by summing over all possible ways of breaking up a block of size n into two pieces whose sum is n. Therefore when Δ acts on an h_λ, we can use this idea to come up with a combinatorial interpretation for the coefficient of $h_\mu \otimes h_\nu$ in $\Delta(h_\lambda)$.

Think of the rows of μ as red blocks with labels $1, 2, \ldots, \ell(\mu)$ whose horizontal lengths are $\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$ and the rows of ν are represented by blue blocks with horizontal lengths $\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}$. When Δ acts on h_λ it splits each of the rows of λ into two parts (with some of the parts possibly empty) and so we can interpret the coefficient $\Delta(h_\lambda)\Big|_{h_\mu \otimes h_\nu}$ as the number of ways of taking at most one red block and at most one blue block placing it next to each other to get rows of size $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$.

EXAMPLE 17. We wish to compute $\Delta(h_{(4,3,3)})\Big|_{h_{(2,2,1)}\otimes h_{(2,2,1)}}$, then we break up the rows of the partition $(4,3,3)$, each one into a red part and a blue part (possibly empty) such that red pieces sorted are a partition $(2,2,1)$ and the blue pieces sorted are a partition $(2,2,1)$. The can be done in exactly two ways,

Equation (3.14) can also be used to derive the following determinantal formula for p_n in terms of e_k.

$$p_n = \begin{vmatrix} ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \\ (n-1)e_{n-1} & e_{n-2} & e_{n-3} & \cdots & 1 \\ (n-2)e_{n-2} & e_{n-3} & e_{n-4} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ e_1 & 1 & 0 & \cdots & 0 \end{vmatrix} \tag{3.16}$$

This follows directly from equation (3.14) by expanding the determinant about the first row of the equation. It follows that the determinant satisifes the same recurrence as the p_k elements do in equation (3.14). Similarly, we also have

$$(-1)^{n-1}p_n = \begin{vmatrix} nh_n & h_{n-1} & h_{n-2} & \cdots & h_1 \\ (n-1)h_{n-1} & h_{n-2} & h_{n-3} & \cdots & 1 \\ (n-2)h_{n-2} & h_{n-3} & h_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_1 & 1 & 0 & \cdots & 0 \end{vmatrix} \tag{3.17}$$

which follows most easily by an application of the involution ω or by observing the same recurrence with equation (3.13).

There is another product defined on symmetric functions known as the 'Kronecker' or 'inner tensor' product. We will denote this product by $*$. It is defined on the power sum basis by

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu}\frac{p_\lambda}{z_\lambda}. \tag{3.18}$$

This product is associative and preserves the degree of the symmetric function, that is it maps $\Lambda_n \otimes \Lambda_n \to \Lambda_n$. We also know that the product is commutative $f*g = g*f$ since clearly this holds on the power basis. We will go into more detail of this product as we introduce more of the bases of the symmetric functions. The Kronecker product very naturally arises in the algebra of class functions and we will show that it is also nicely encoded in our notation.

There is an assocated coproduct with $*$ that defines a bialgebra structure on the symmetric functions. Define the corresponding coproduct as $\Delta' : \Lambda \to \Lambda \otimes \Lambda$ will be defined on the power basis

$$\Delta'(p_\lambda) = p_\lambda \otimes p_\lambda. \tag{3.19}$$

Proposition 3.3. *The vector space Λ endowed with the product $\mu(f \otimes g) = fg$ and coproduct Δ' forms a bialgebra.*

Proof. $\Delta'(p_\mu p_\lambda) = (p_\mu p_\lambda) \otimes (p_\mu p_\lambda)$. Similarly we have that $\Delta'(p_\mu)\Delta'(p_\lambda) = (p_\mu \otimes p_\mu)(p_\lambda \otimes p_\lambda) = (p_\mu p_\lambda) \otimes (p_\mu p_\lambda) = \Delta'(p_\mu p_\lambda)$. Therefore we have shown that Δ' is an algebra homomorphism with respect to the product. □

Note: With the map $\varepsilon'(p_k) = 1$ (and more generally $\varepsilon'(p_\lambda) = 1$) is a counit and satisfies $\mu \circ (id \otimes \varepsilon') \circ \Delta' = id$, however this product/coproduct pair fails to have an antipode and hence is not a Hopf algebra (see exercise 12).

Δ' is not an algebra homomorphism with respect to the Kronecker product since $\Delta'(p_\mu * p_\lambda) = \delta_{\lambda\mu} z_\lambda p_\lambda \otimes p_\lambda$ while $\Delta'(p_\mu) * \Delta'(p_\lambda) = \delta_{\lambda\mu} z_\lambda^2 p_\lambda \otimes p_\lambda$ and hence it is not a bialgebra.

In addition to the Hopf algebra and bialgebra structures on Λ, one should also think of Λ as a vector space over $\mathbb{Q}$ and so it is convenient to define a scalar product on this to serve as a tool for computation. Define

$$\left\langle p_\lambda, \frac{p_\mu}{z_\mu} \right\rangle = \delta_{\lambda\mu} \tag{3.20}$$

where we use the notation $\delta_{xy} = 0$ if $x \neq y$ and $\delta_{xx} = 1$. The remarkable property of this scalar product is that it interacts nicely with the products and coproducts on this space.

Proposition 3.4. *The scalar product is positive definite. In addition, it satisfies the following useful properties.*

$$\langle f, g \rangle = \langle g, f \rangle \tag{3.21}$$

If we set $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_\otimes = \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle$, then the coproduct Δ is dual to multiplication,

$$\langle f \otimes g, \Delta(h) \rangle_\otimes = \langle fg, h \rangle . \tag{3.22}$$

The coproduct Δ' is dual to the product $*$,

$$\langle f \otimes g, \Delta'(h) \rangle_\otimes = \langle f * g, h \rangle . \tag{3.23}$$

The involution ω and antipode S are self-dual,

$$\langle \omega(f), \omega(g) \rangle = \langle S(f), S(g) \rangle = \langle f, g \rangle . \tag{3.24}$$

Moreover,

$$\langle f, g \rangle = \varepsilon'(f * g) \tag{3.25}$$

and

$$\langle f * g, h \rangle = \langle g, f * h \rangle \tag{3.26}$$

Proof. It suffices to verify these identities for a basis and then the result must extend by linearity, and for this we choose the basis $\{p_\lambda\}_\lambda$. Note that the fact that the scalar product is symmetric follows since $\langle p_\lambda, p_\mu \rangle = \langle p_\mu, p_\lambda \rangle = \delta_{\lambda\mu} z_\lambda$. We show equation (3.22) by expanding the left hand side using equation (3.2) and comparing it to the right hand side of the equation.

$$\langle p_\lambda \otimes p_\mu, \Delta(p_\nu)\rangle_\otimes = \sum_\gamma \prod_{i\geq 1} \binom{m_i(\nu)}{m_i(\gamma)} \langle p_\lambda, p_\gamma\rangle \langle p_\mu, p_{\nu\setminus\gamma}\rangle \tag{3.27}$$

$$= \delta_{\nu,\lambda\uplus\mu} \prod_{i\geq 1} \binom{m_i(\nu)}{m_i(\lambda)} z_\lambda z_\mu \tag{3.28}$$

On the right hand side of this equation we have

$$\langle p_\lambda p_\mu, p_\nu\rangle = \delta_{\nu,\lambda\uplus\mu} z_\nu \tag{3.29}$$

and by referring back to the definition of z_ν it is easy to see that (3.28) and (3.29) are equal.

Similarly we may verify (3.23),

$$\begin{aligned}\langle p_\lambda \otimes p_\mu, \Delta(p_\nu)\rangle_\otimes &= \langle p_\lambda, p_\nu\rangle \langle p_\mu, p_\nu\rangle = \delta_{\lambda\mu}\delta_{\mu\nu} z_\nu^2 \\ &= \langle z_\mu \delta_{\lambda\mu} p_\mu, p_\nu\rangle = \langle p_\lambda * p_\mu, p_\nu\rangle\end{aligned} \tag{3.30}$$

$$\langle \omega(p_\lambda), \omega(p_\mu)\rangle = (-1)^{|\lambda|+|\mu|-\ell(\lambda)-\ell(\mu)} \langle p_\lambda, p_\mu\rangle = \tag{3.31}$$

$$\langle S(p_\lambda), S(p_\mu)\rangle = (-1)^{\ell(\lambda)+\ell(\mu)} \langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda = \langle p_\lambda, p_\mu\rangle \tag{3.32}$$

Equation (3.25) follows because we have set $p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda$ and hence $\varepsilon'(p_\lambda * p_\mu) = \delta_{\lambda\mu} z_\lambda \varepsilon'(p_\lambda) = \delta_{\lambda\mu} z_\lambda = \langle p_\lambda, p_\mu\rangle$. This implies our last identity as well since the product $*$ is associative and symmetric and

$$\langle f * g, h\rangle = \varepsilon'(f * (g * h)) = \varepsilon'(g * (f * h)) = \langle g, f * h\rangle \tag{3.33}$$

□

Now for any symmetric function homomorphism we can ask what the operation which is dual with respect to the scalar product. That is, for $\phi \in Hom(\Lambda, \Lambda)$ we ask what is the operator ϕ^* with the property

$$\langle \phi(f), g\rangle = \langle f, \phi^*(g)\rangle \tag{3.34}$$

Notice that we have already shown that many of the operators which we have considered so far (e.g. S, ω, the action Kronecker product by a symmetric function $f * \cdot$) are self dual. In the last proposition we also showed that the coproduct Δ is dual to the product operation m and the coproduct Δ' is dual to the Kronecker product.

This leads to a useful computational tool, the operation which is dual to multiplication of a symmetric function f, which we will denote by $f^\perp$. That is, $f^\perp$ is defined as the operator with the property

$$\langle f \cdot g, h\rangle = \langle g, f^\perp h\rangle .$$

Since multiplication by f is an operation which raises the degree of a symmetric function by the degree of f, $f^\perp$ is an operator which lowers the degree of a symmetric function by the degree of f.

We show the following useful properties of this operation.

Proposition 3.5.

$$f^{\perp}(g) = \sum_{\lambda} \langle f p_\lambda, g\rangle\, p_\lambda/z_\lambda \tag{3.35}$$

$p_k^{\perp} = k\frac{\partial}{\partial p_k}$ and, in particular,

$$p_k^{\perp}(p_\lambda) = k m_k(\lambda) p_{\lambda\ominus(k)} \tag{3.36}$$

where $p_{\lambda\ominus(k)}$ is zero if λ does not have a part of size k. If $\Delta(f) = \sum_i a_i \otimes b_i$, then

$$f^{\perp}(gh) = \sum_i a_i^{\perp}(g) b_i^{\perp}(h) \tag{3.37}$$

Proof. The first equation follows because the coefficient of p_λ in $f \in \Lambda$ is given by $\langle p_\lambda/z_\lambda, f\rangle$, therefore the expansion of f in the p_λ basis is simply $f = \sum_\lambda \langle p_\lambda/z_\lambda, f\rangle\, p_\lambda$. In the case that we expand $f^{\perp}(g)$ in the power basis, we have that

$$f^{\perp} g = \sum_{\lambda} \left\langle p_\lambda/z_\lambda, f^{\perp}(g)\right\rangle p_\lambda = \sum_{\lambda} \langle f p_\lambda, g\rangle\, p_\lambda/z_\lambda. \tag{3.38}$$

The coefficient of p_μ in $p_k^{\perp} p_\lambda$ is given by $\left\langle p_k^{\perp} p_\lambda, p_\mu/z_\mu\right\rangle = \langle p_\lambda, p_k p_\mu/z_\mu\rangle$ which is equal to 0 unless $\mu \uplus (k) = \lambda$. If $\mu = \lambda \ominus (k)$, then the scalar product evaluates to $z_\lambda/z_\mu = k m_k(\lambda)$ and otherwise the result is 0. It follows that $p_k^{\perp} = k\frac{\partial}{\partial p_k}$ since the action of these operators is the same on the monomial p_λ.

The fact that $p_k^{\perp}(fg) = p_k^{\perp}(f)g + f p_k^{\perp}(g)$ follows from the product rule for derivatives since we can interpret $p_k^{\perp}$ as a differential operator. Since $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$ we have shown that equation (3.37) holds for any p_k. We know that $\Delta(p_\lambda) = \Delta(p_{\lambda_1})\Delta(p_{\lambda_2})\cdots\Delta(p_{\lambda_k})$ while $p_\lambda^{\perp} = p_{\lambda_1}^{\perp} p_{\lambda_2}^{\perp} \cdots p_{\lambda_k}^{\perp}$, therefore (3.37) must hold for any p_λ. It follows by extending this result linearly that it also holds for any symmetric function f. □

Proposition 3.6. *For $k \geq 0$, the action of the operators $p_k^{\perp}$, $e_k^{\perp}$ and $h_k^{\perp}$ for $k \geq 0$ on the symmetric functions p_n, e_n and h_n is given by the following table.*

(3.39)

	h_n	e_n	p_n
$h_k^{\perp}$	h_{n-k}	$(\delta_{k0} + \delta_{k1})e_{n-k}$	$\delta_{kn} + \delta_{k0} p_n$
$e_k^{\perp}$	$(\delta_{k0} + \delta_{k1})h_{n-k}$	e_{n-k}	$(-1)^{k-1}\delta_{kn} + \delta_{k0} p_n$
$p_k^{\perp}$	h_{n-k}	$(-1)^{k-1} e_{n-k}$	$n\delta_{nk} + \delta_{k0} p_n$

Proof. $e_k^{\perp}(e_n)$, $e_k^{\perp}(h_n)$, $e_k^{\perp}(p_n)$ and $p_k^{\perp}(e_n)$ can all be calculated from the action of the operator $h_k^{\perp}$ or $p_k^{\perp}$ since we have that $\omega(f^{\perp} g) = (\omega(f))^{\perp}(\omega(g))$.

$p_k^{\perp}(\frac{p_\lambda}{z_\lambda})$ is $\frac{p_{\lambda\ominus(k)}}{z_{\lambda\ominus(k)}}$ if λ contains a part of size k and 0 if λ does not contain a part of size k. Therefore if $p_k^{\perp}$ acts on $h_n = \sum_{\lambda\vdash n} p_\lambda/z_\lambda$ the result will be $h_{n-k} = \sum_{\lambda\vdash n-k} p_\lambda/z_\lambda$. This justifies the last line of the table.

If $k = 0$ then $h_k^\perp = 1$, but otherwise $h_k^\perp(p_n) = 0$ unless $k = n$ since only then will there be a partition which contains a part of size n so that $\sum_{\lambda \vdash k} p_\lambda^\perp(p_n)/z_\lambda$ is non-zero.

We will prove $h_k^\perp h_n$ using induction and formula (3.13).

$$h_k^\perp(h_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp p_i^\perp(h_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp(h_{n-i}) \tag{3.40}$$

since we have already calculated that $p_i^\perp(h_n) = h_{n-i}$. If we assume by induction that $h_{k-i}^\perp(h_{n-i}) = h_{n-k}$ for $1 \leq i \leq k$, then it follows that $h_k^\perp(h_n) = h_{n-k}$.

We can prove in a similar manner the formula for $h_k^\perp(e_n)$. We handle the $k = 0$ and $k = 1$ cases separately for there we already know $h_0^\perp(e_n) = e_n$ and $h_1^\perp(e_n) = p_1^\perp(e_n) = e_{n-1}$.

For $k > 1$, if we assume that $h_{k-i}^\perp(e_n)$ is known for all $i > 0$ and all n then we calculate that

$$h_k^\perp(e_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp p_i^\perp(e_n) = \frac{1}{k}\sum_{i=1}^{k} h_{k-i}^\perp(-1)^{i-1}e_{n-i} \tag{3.41}$$

Only two terms of this equation will survive, $i = k - 1$ and $i = k$. Therefore,

$$h_k^\perp(e_n) = \frac{1}{k}((-1)^k e_{n-k} + (-1)^{k-1}e_{n-k}) = 0 \tag{3.42}$$

□

We can take this one step further to calculate explicitly the action of $h_k^\perp$, $e_k^\perp$ and $p_k^\perp$ on h_λ, e_λ, and p_λ. It is sufficient to give an expression for the expressions $p_k^\perp(h_\lambda), p_k^\perp(p_\lambda), h_k^\perp(h_\lambda), h_k^\perp(e_\lambda)$, and $h_k^\perp(p_\lambda)$ since the others can be found by applying ω to both sides of the equation. We will need to use the relation that $h_k^\perp(fg) = \sum_{i=0}^{k} h_i^\perp(f)h_{k-i}^\perp(g)$ and $p_k^\perp(fg) = p_k^\perp(f)g + fp_k^\perp(g)$. The real difficulty in this problem is in finding a nice way of elegantly expressing these quantities.

PROPOSITION 3.7.

$$p_k^\perp(h_\lambda) = \sum_{i=1}^{\ell(\lambda)} h_{\lambda \ominus (\lambda_i) \uplus (\lambda_i - k)} \tag{3.43}$$

$$p_k^\perp(p_\lambda) = m_k(\lambda) k p_{\lambda \ominus (k)} \tag{3.44}$$

$$h_k^\perp(h_\lambda) = \sum_{|\alpha|=k} \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i - \alpha_i} \tag{3.45}$$

where the sum over all sequences α *such that* $0 \leq \alpha_i \leq \lambda_i$.

$$h_k^\perp(e_\lambda) = \sum_{|\alpha|=k} \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i - \alpha_i} \tag{3.46}$$

where the sum over all sequences α *such that* $0 \le \alpha_i \le 1$.

$$h_k^\perp(p_\lambda) = \sum_{S \subset \{1,2,\ldots,\ell(\lambda)\}} \prod_{i \notin S} p_{\lambda_i} \tag{3.47}$$

where the sum is over all subsets S *of* $\{1,2,\ldots,\ell(\lambda)\}$ *such that* $\sum_{i\in S} \lambda_i = k$.

Proof. Most of these expressions follow from the previous proposition and the action of $h_k^\perp$ and $p_k^\perp$ on a product. Notice that

$$p_k^\perp(h_\lambda) = \sum_{i=1}^{\ell(\lambda)} p_k^\perp(h_{\lambda_i}) h_{\lambda\ominus(\lambda_i)} = \sum_{i=1}^{\ell(\lambda)} h_{(\lambda_i-k)} h_{\lambda\ominus(\lambda_i)} \tag{3.48}$$

which is the same as equation (3.43).

Also we have

$$\begin{aligned} p_k^\perp(p_\lambda) &= \sum_{i=1}^{\ell(\lambda)} p_k^\perp(p_{\lambda_i}) p_{\lambda\ominus(\lambda_i)} \\ &= \sum_{i=1}^{\ell(\lambda)} \delta_{\lambda_i,k} k p_{(\lambda_i-k)} p_{\lambda\ominus(\lambda_i)} \end{aligned} \tag{3.49}$$

and because there is exactly one non-zero term for each part of size k in λ so this is equal to the expression in equation (3.44).

Similarly to compute $h_k^\perp(h_\lambda)$, we compute

$$h_k^\perp(h_\lambda) = \sum_{i=0}^{k} h_i^\perp(h_{\lambda_1}) h_{k-i}^\perp(h_{\lambda\ominus(\lambda_1)}) \tag{3.50}$$

We can assume by induction on the length of λ that we know the formula for $h_{k-i}^\perp(h_{\lambda\ominus(\lambda_1)})$ is given by equation (3.45) (the base case is known because $\ell(\lambda) = 1$ is given in the previous proposition). We have then that equation (3.50) is equal to

$$= \sum_{i=0}^{k} h_{\lambda_1-i} \left(\sum_{|\tilde\alpha|=k-i} \prod_{i=2}^{\ell(\lambda)} h_{\lambda_i-\tilde\alpha_i} \right) \tag{3.51}$$

and this is equivalent to equation (3.45) since if $i > \lambda_1$ then $h_{\lambda_1-i} = 0$ so the size of the first part of α is restricted by both k and the size of λ_1.

Similarly, we can show again by assuming (3.46) is true for all partitions of length less than $\ell(\lambda)$ by induction, then

$$h_k^\perp(e_\lambda) = \sum_{i=0}^{k} h_i^\perp(e_{\lambda_1}) \sum_{|\tilde\alpha|=k-i} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde\alpha_i}$$

$$= e_{\lambda_1} \sum_{|\tilde\alpha|=k} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde\alpha_i} + e_{\lambda_1-1} \sum_{|\tilde\alpha|=k-1} \prod_{i=2}^{\ell(\lambda)} e_{\lambda_i-\tilde\alpha_i}. \tag{3.52}$$

This equation is then equivalent to (3.46) for an equation indexed by a partition equal to the length of the partition λ.

Now to calculate $h_k^\perp(p_\lambda)$ we again assume by induction that (3.47) holds for partitions of length less than $\ell(\lambda)$.

$$h_k^\perp(p_\lambda) = p_{\lambda_1} h_k^\perp(p_{\lambda\ominus(\lambda_1)}) + h_{k-\lambda_1}^\perp(p_{\lambda\ominus(\lambda_1)}) \tag{3.53}$$

which follows from the action of $h_k^\perp$ on p_n given in the previous proposition. In this equation, if $k-\lambda_1<0$ then the second term in this sum is equal to 0. Using the inductive assumption, (3.53) is equal to

$$= p_{\lambda_1} \sum_{S\subset\{2,\ldots,\ell(\lambda)\}} \prod_{\substack{i\notin S\\ i\neq 1}} p_{\lambda_i} + \sum_{T\subset\{2,\ldots,\ell(\lambda)\}} \prod_{\substack{i\notin T\\ i\neq 1}} p_{\lambda_i} \tag{3.54}$$

where the the first sum is over all subsets S such that $\sum_{\substack{i\notin S\\ i\neq 1}} \lambda_i = k$ and the second sum is over subsets T such that $\sum_{\substack{i\notin T\\ i\neq 1}} \lambda_i = k-\lambda_1$. This is equivalent to equation (3.47) for a partition of length equal to $\ell(\lambda)$. □

There is a third operation of multiplication which we have not yet mentioned which is a type of composition of symmetric functions. We define $p_n[p_m]=p_{nm}$ and then extend this definition in a natural manner. That is we set,

$$p_n[p_\lambda] = \prod_{i=1}^{\ell(\lambda)} p_{n\lambda_i}. \tag{3.55}$$

For $c,d\in\mathbb{Q}$ and $f,g\in\Lambda$ this operation is linear by

$$p_n[cf+dg] = c\; p_n[f] + d\; p_n[g]. \tag{3.56}$$

In particular we have, $p_n[\sum_\lambda c_\lambda p_\lambda] = \sum_\lambda c_\lambda \prod_{i=1}^{\ell(\lambda)} p_{n\lambda_i}$. Then for $f\in\Lambda$ and for a partition λ we define

$$p_\lambda[f] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[f]. \tag{3.57}$$

Finally for $f=\sum_\lambda c_\lambda p_\lambda$ and $g\in\Lambda$, we define

$$f[g] = \sum_\lambda c_\lambda p_\lambda[g]. \tag{3.58}$$

This definition implies that for $c, d \in \mathbb{Q}$ and $f, g, h \in \Lambda$, $(c\ f + d\ g)[h] = c\ f[h] + d\ g[h]$ but in general $f[c\ g + d\ h] \neq c\ f[g] + d\ f[h]$ (note this will hold if $f = p_n$).

3.1. The class functions of the symmetric group

We have defined the algebra of 'symmetric functions' without much a hint as why we have chosen this as the name of the algebra since the elements of Λ are neither symmetric nor functions. One motivation for studying this algebra is that it is isomorphic to the space of class functions of the symmetric group.

Let us consider Φ_n the linear vector space over $\mathbb{Q}$ of the class functions of the symmetric group Sym_n. We know that Φ_n is a vector space spanned by the elements C_λ where λ is a partition of n and

$$C_\lambda(\pi) = \begin{cases} 1 & \text{if } \pi \text{ has cycle type } \lambda \\ 0 & \text{otherwise} \end{cases} . \tag{3.59}$$

We also know that Φ_n is spanned by the set of irreducible characters of Sym_n.

We will define the Frobenius map between the space of class functions Φ_n and the space of symmetric functions of degree n. That is we define $\mathcal{F} : \Phi_n \to \Lambda_n$ by the action on the basis C_λ as

$$\mathcal{F}(C_\lambda) = \frac{p_\lambda}{z_\lambda}. \tag{3.60}$$

This map is clearly an isomorphism since the sets $\{C_\lambda\}_{\lambda \vdash n}$ and $\{p_\lambda / z_\lambda\}_{\lambda \vdash n}$ are both bases of Φ_n and Λ_n respectively.

Let χ^{triv_n} represent the trivial character on the symmetric group Sym_n. This means that $\chi^{triv_n} = \sum_{\lambda \vdash n} C_\lambda$ and therefore we have

$$\mathcal{F}(\chi^{triv_n}) = \sum_{\lambda \vdash n} \mathcal{F}(C_\lambda) = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} = h_n.$$

As well we may denote the sign character on Sym_n by χ^{sgn_n}. Since the sign of a permutation with cycle type λ is $(-1)^{|\lambda| - \ell(\lambda)}$ we have that $\chi^{sgn_n} = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} C_\lambda$ and therefore

$$\mathcal{F}(\chi^{sgn_n}) = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} \mathcal{F}(C_\lambda) = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} \frac{p_\lambda}{z_\lambda} = e_n.$$

The homogeneous and elementary generators are natural elements to consider in this context since the trivial and sign characters are the two one dimensional characters of Sym_n.

The definition of the scalar product on the symmetric functions may have seemed somewhat arbitary when we introduced it for symmetric fucntions but it is actually motivated by the scalar product of class functions and the connection by the following proposition.

PROPOSITION 3.8. *If $\chi, \psi \in \Phi_n$, then*

$$\langle \mathcal{F}(\chi), \mathcal{F}(\psi) \rangle = \langle \chi, \psi \rangle \tag{3.61}$$

where on the left the scalar product is over symmetric functions with $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$ and on the right it is the scalar product on the class functions defined as $\langle \chi, \psi \rangle = \frac{1}{n!} \sum_{\sigma \in Sym_n} \chi(\sigma)\psi(\sigma^{-1})$.

PROOF. Because the map $\mathcal{F}$ is linear and the both the scalar products are bilinear it suffices to show that this result holds for a basis. That is, we need only show that

$$\langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \langle C_\lambda, C_\mu \rangle \tag{3.62}$$

since then we know that for $\chi = \sum_{\lambda \vdash n} c_\lambda C_\lambda$ and $\psi = \sum_{\mu \vdash n} d_\mu C_\mu$ and

$$\langle \mathcal{F}(\chi), \mathcal{F}(\psi) \rangle = \sum_{\lambda,\mu \vdash n} c_\lambda d_\mu \langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \sum_{\lambda,\mu \vdash n} c_\lambda d_\mu \langle C_\lambda, C_\mu \rangle = \langle \chi, \psi \rangle .$$

Now since $\mathcal{F}(C_\lambda) = p_\lambda / z_\lambda$ it is easy to establish (3.62) for a fixed λ and μ.

$$\langle \mathcal{F}(C_\lambda), \mathcal{F}(C_\mu) \rangle = \langle p_\lambda / z_\lambda, p_\mu / z_\mu \rangle = \delta_{\lambda\mu} / z_\lambda,$$

while at the same time

$$\langle C_\lambda, C_\mu \rangle = \frac{1}{n!} \sum_{\sigma \in Sym_n} C_\lambda(\sigma) C_\mu(\sigma^{-1}) = \delta_{\lambda\mu} / z_\lambda.$$

□

Remember that the irreducible characters of the symmetric group are an orthonormal basis of class functions. The images of the irreducible characters will be the fundamental basis for the symmetric functions and we will introduce this basis in a later chapter.

The space of symmetric functions Λ is an algebra since it is a vector space endowed with a multiplication operation which takes elements in $\Lambda_n \times \Lambda_m$ and sends them to Λ_{n+m}. The set of class functions is endowed with a similar multiplication operation. We set $\Phi = \bigoplus_{n \geq 0} \Phi_n$.

Recall that for $\chi \in \Phi_n$ and $\psi \in \Phi_m$, we have $\chi \otimes \psi$ is a class function of $Sym_m \times Sym_n$ defined by $\chi \otimes \psi(\pi, \sigma) = \chi(\pi)\psi(\sigma)$. To make this a character of Sym_{n+m} we consider the induced character $\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} \in \Phi_{n+m}$. This is our analogous operation in the space of class functions to the operation of multiplication in the symmetric functions. In fact, this operation is more than just analogous, it satisfies the following property:

PROPOSITION 3.9. *For $\chi \in \Phi_n$ and $\psi \in \Phi_m$,*

$$\mathcal{F}(\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \mathcal{F}(\chi)\mathcal{F}(\psi) \tag{3.63}$$

PROOF. The operation of inducing two class functions is linear in both the first and in the second position as we have for $\chi = \sum_{\lambda \vdash n} c_\lambda C_\lambda$ and $\psi = \sum_{\mu \vdash m} d_\mu C_\mu$ then

$$\chi \otimes \psi \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} = \sum_{\lambda \vdash n} \sum_{\mu \vdash m} c_\lambda d_\mu C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}} . \tag{3.64}$$

Therefore we need only show that $\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \mathcal{F}(C_\lambda)\mathcal{F}(C_\mu)$. To do this we will expand $\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}})$ in the basis $\{p_\nu\}_{\nu \vdash n+m}$.

$$\left\langle C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}, C_\nu \right\rangle = \left\langle C_\lambda \otimes C_\mu, C_\nu \downarrow_{Sym_n \times Sym_m}^{Sym_{n+m}} \right\rangle$$
$$= \frac{1}{n!m!} \sum_{\sigma \in Sym_n, \tau \in Sym_m} C_\lambda(\sigma) C_\mu(\tau) C_\nu \downarrow_{Sym_n \times Sym_m}^{Sym_{n+m}} (\sigma, \tau).$$

Every term in this last sum is equal to 0 unless $\lambda \uplus \mu = \nu$ and only then when σ is of cycle type λ and τ is of cycle type μ. Therefore the right hand side is equal to $\frac{\delta_{\lambda \uplus \mu, \nu}}{z_\lambda z_\mu}$ and hence

$$\left\langle \mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}), \mathcal{F}(C_\nu) \right\rangle = \frac{\delta_{\lambda \uplus \mu, \nu}}{z_\lambda z_\mu}$$

and hence

$$\mathcal{F}(C_\lambda \otimes C_\mu \uparrow_{Sym_n \times Sym_m}^{Sym_{n+m}}) = \frac{p_{\lambda \uplus \mu}}{z_\lambda z_\mu} = \frac{p_\lambda}{z_\lambda} \frac{p_\mu}{z_\mu} = \mathcal{F}(C_\lambda)\mathcal{F}(C_\mu).$$

□

This last proposition gives us an interpretation for h_λ and e_λ since we have already noted that the image of the trivial and sign characters in the Frobenius map are h_n and e_n respectively. Define Sym_λ to be the subgroup of Sym_n isomorphic to $Sym_{\lambda_1} \times Sym_{\lambda_2} \times \cdots \times Sym_{\lambda_{\ell(\lambda)}}$ in the natural manner. Denote the trivial and sign characters on this subgroup as χ^{triv_λ} and χ^{sgn_λ} so that for all $\pi \in Sym_\lambda$, $\chi^{triv_\lambda}(\pi) = 1$ and $\chi^{sgn_\lambda}(\pi) = sgn(\pi)$ and more precisely $\chi^{triv_\lambda} = \chi^{triv_{\lambda_1}} \otimes \chi^{triv_{\lambda_2}} \otimes \cdots \otimes \chi^{triv_{\lambda_{\ell(\lambda)}}}$ and similarly $\chi^{sgn_\lambda} = \chi^{sgn_{\lambda_1}} \otimes \chi^{sgn_{\lambda_2}} \otimes \cdots \otimes \chi^{sgn_{\lambda_{\ell(\lambda)}}}$. From the previous proposition we have

$$h_\lambda = \mathcal{F}(\chi^{triv_\lambda} \uparrow_{Sym_\lambda}^{Sym_n})$$

and

$$e_\lambda = \mathcal{F}(\chi^{sgn_\lambda} \uparrow_{Sym_\lambda}^{Sym_n}).$$

We defined a second type of multiplication on symmetric functions which we called the inner or Kronecker product of symmetric functions. The definition this operation $*$ is given by $\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda}$. It arises naturally in the following sense.

Proposition 3.10. *For $\chi, \psi \in Phi_n$ and we define $\chi \cdot \psi$ as the class function $\chi \cdot \psi(g) := \chi(g)\psi(g)$. This is the inner tensor product of characters. We have*

$$\mathcal{F}(\chi \cdot \psi) = \mathcal{F}(\chi) * \mathcal{F}(\psi). \tag{3.65}$$

Proof. Again it suffices to verify this identity on a basis for the class functions because it will hold for any linear combination of the class functions as well. This is easy to verify for the class functions C_λ, since

$$C_\lambda \cdot C_\mu(\pi) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu \text{ and } C_\lambda(\pi) = 1 \end{cases} = \delta_{\lambda\mu} C_\lambda(\pi)$$

This means that $C_\lambda \cdot C_\mu = \delta_{\lambda\mu} C_\lambda$ and so we have

$$\begin{aligned}\mathcal{F}(C_\lambda \cdot C_\mu) &= \delta_{\lambda\mu}\mathcal{F}(C_\lambda)\\ &= \delta_{\lambda\mu}\frac{p_\lambda}{z_\lambda}\\ &= \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu}\\ &= \delta_{\lambda\mu}\mathcal{F}(C_\lambda) * \mathcal{F}(C_\mu)\end{aligned}$$

□

***** interpretation of ω here?

The coproduct operation also has an interpretation in the algebra of class functions, however we first need to extend our definition of the Frobenius map to the algebra of class functions on $Sym_k \times Sym_{n-k}$. Recall that we have for class functions $\chi \in \Phi_k$ and $\psi \in \Phi_{n-k}$ that the function $\chi \otimes \psi$ defined to be $\chi \otimes \psi(\pi, \sigma) := \chi(\pi)\psi(\sigma)$ for $\pi \in Sym_k$ and $\sigma \in Sym_{n-k}$ and $\chi \otimes \psi$ is a class function of $Sym_k \times Sym_{n-k}$. This is called the the outer tensor product of class functions. Note that for any basis of class functions of G, $\{C^{(i)}\}$, and of the class functions of H, $\{D^{(i)}\}$, then $\{C^{(i)} \otimes D^{(j)}\}$ is basis for the class functions of $G \times H$.

To extend the definition of the Frobenius map to include class functions of $Sym_k \times Sym_{n-k}$ which has as a basis $\{C_\lambda \otimes C_\mu\}_{\substack{\lambda \vdash k\\ \mu \vdash n-k}}$ we set

$$\mathcal{F}(C_\lambda \otimes C_\mu) := \frac{p_\lambda}{z_\lambda} \otimes \frac{p_\mu}{z_\mu} = \mathcal{F}(C_\lambda) \otimes \mathcal{F}(C_\mu) \tag{3.66}$$

and this definition is extended linearly. This implies that we have more generally, for $\chi \in \Phi_n$ and $\psi \in \Phi_m$,

$$\mathcal{F}(\chi \otimes \psi) = \mathcal{F}(\chi) \otimes \mathcal{F}(\psi). \tag{3.67}$$

Using this extension of notation we have the following interpretation of the coproduct operation on symmetric functions.

PROPOSITION 3.11.

$$\Delta(\mathcal{F}(\chi)) = \sum_{k=0}^{n} \mathcal{F}(\chi \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}) \tag{3.68}$$

PROOF. It suffices to show that this result again holds on a basis and the natural basis to consider is again C_λ for $\lambda \vdash n$. We have that

$$C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}} (\pi, \sigma) = \begin{cases} 1 & \text{if } C_\mu(\pi) = 1 \text{ and } C_\nu(\sigma) = 0 \text{ for} \\ & \mu \vdash k, \nu \vdash n-k \text{ with } \mu \uplus \nu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

In other words we see that

$$C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}} = \sum_{\substack{\mu \vdash k\\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} C_\mu \otimes C_\nu$$

This implies that

$$\mathcal{F}(C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}) = \sum_{\substack{\mu \vdash k \\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} \frac{1}{z_\mu z_\nu} p_\mu \otimes p_\nu.$$

As well we have

$$\begin{aligned} \Delta(z_\lambda \mathcal{F}(C_\lambda)) = \Delta(p_\lambda) &= \sum_{k=0}^{n} \sum_{\substack{\mu \vdash k \\ \nu \vdash n-k}} \delta_{\mu \uplus \nu, \lambda} \frac{z_\lambda}{z_\mu z_\nu} p_\mu \otimes p_\nu \\ &= \sum_{k=0}^{n} z_\lambda \mathcal{F}(C_\lambda \downarrow^{Sym_n}_{Sym_k \times Sym_{n-k}}), \end{aligned}$$

and therefore the proposition holds for all class functions. □

3.2. Exercises

(1) (a) Expand $p_{(2,2)}$ in the elementary and homogeneous bases.
    (b) Expand $e_{(2,2)}$ in the power basis.
    (c) Expand $h_{(2,2)}$ in the homogeneous basis.
(2) Calculate the following scalar products
    (a) $\langle h_{(2,2,1)}, p_{(3,2)} \rangle$
    (b) $\langle h_{(3,2)}, p_{(3,2)} \rangle$
    (c) $\langle h_{(3,2)}, p_{(2,2,1)} \rangle$
    (d) $\langle h_{(3,2)}, h_{(4,1)} \rangle$
    (e) $\langle h_{(3,2)}, h_{(3,1,1)} \rangle$
    (f) $\langle h_{(3,2)}, h_{(2,2,1)} \rangle$
(3) Calculate the following inner products using the formulas given in this section. Assume that $|\lambda| = n$.
    (a) $\langle h_n, p_\lambda \rangle$
    (b) $\langle e_n, p_\lambda \rangle$
    (c) $\langle p_n, h_\lambda \rangle$
    (d) $\langle p_{1^n}, h_\lambda \rangle$
    (e) $\langle p_\lambda, h_\lambda \rangle$
    (f) $\langle h_n, h_n \rangle$
    (g) $\langle e_n, h_n \rangle$
    (h) $\langle h_n, h_\lambda \rangle$
    (i) $\langle e_n, h_\lambda \rangle$
(4) Show that $\Delta \circ \omega = (\omega \otimes \omega) \circ \Delta$ by showing that it holds true on the power basis. Use this to show that $\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$.
(5) Show that $(1 \otimes \omega) \circ \Delta' = (\omega \otimes 1) \circ \Delta' = \Delta' \circ \omega$.
(6) Show the following determinental formulas for h_n.

(a)

$$h_m = \begin{vmatrix} p_1 & -1 & 0 & \cdots & 0 \\ \frac{p_2}{2} & \frac{p_1}{2} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{p_{m-1}}{m-1} & \frac{p_{m-2}}{m-1} & \cdots & \frac{p_1}{m-1} & -1 \\ \frac{p_m}{m} & \frac{p_{m-1}}{m} & \frac{p_{m-2}}{m} & \cdots & \frac{p_1}{m} \end{vmatrix}$$

(b)

$$h_n = \begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_n \\ 1 & e_1 & e_2 & \cdots & e_{n-1} \\ 0 & 1 & e_1 & \cdots & e_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & e_1 \end{vmatrix}$$

State and prove the corresponding determinental formulas for e_n in terms of p_n and h_n.

(7) Show that $\Delta(H(t)) = H(t) \otimes H(t)$ and $\Delta(E(t)) = E(t) \otimes E(t)$ and $\Delta(P(t)) = P(t) \otimes 1 + 1 \otimes P(t)$.
(8) Show that $p_k^{\perp} = k\frac{\partial}{\partial p_k}$
(9) Use the fact that $E(t)H(t) = 1$ to develop a formula for h_n in terms of the elementary basis.
(10) Use the relationship $P(t) = log(1 + \sum_{n\geq 1} h_n t^n)$ to derive a formula for p_n in terms of the homogeneous basis.
(11) Use the previous result to expand p_λ in the homogeneous basis.
(12) Prove that Λ endowed with the bialgebra with product μ and coproduct Δ' does not have a corresponding antipode and hence is not a Hopf algebra.
(13) Prove that for any $f \in \Lambda_n$, $h_n * f = f$ and $e_n * f = \omega f$.
(14) Prove that if $g \in \Lambda_n$ has the property that for all $f \in \Lambda_n$, $g * (g * f)) = f$ then $\langle g, p_\lambda \rangle = \pm 1$ for all λ partitions of n.
(15) Show that $\mathbb{Z}[p_1, p_2, p_3, \ldots] \subseteq \mathbb{Z}[e_1, e_2, e_3, \ldots]$ and that the converse of this statment (i.e. that $\mathbb{Z}[e_1, e_2, e_3, \ldots] \subseteq \mathbb{Z}[p_1, p_2, p_3, \ldots]$) is not true.
(16) Show that $\mathbb{Z}[h_1, h_2, \ldots, h_k] = \mathbb{Z}[e_1, e_2, \ldots, e_k]$.
(17) (a) Show that the linear span of the symmetric functions $\{f + \omega(f) : f \in \Lambda\}$ forms a subalgebra of the symmetric functions under the standard product.
  (b) Show that this algebra is not a bialgbra with the coproduct Δ.
  (c) Show that a linear basis for this space is given by the set $\{p_\lambda : |\lambda| - \ell(\lambda) \; mod\, 2 = 0\}$
  (d) Show that the space is closed under the Kronecker product of equation (3.18) and the coproduct Δ' of equation (3.19).
(18) Show that $\mathbb{Q}[p_1, p_3, p_5, \ldots]$ is a Hopf subalgebra of the symmetric functions and that it is also closed under the Kronecker product and coproduct Δ'. This subalgebra is sometimes known as the Q-function algebra.

3.3. Solutions to exercises

(1) (a)

$$\begin{aligned} p_2^2 &= \left(-2\,e_2 + {e_1}^2\right)^2 = {e_1}^4 - 4\,e_2\,{e_1}^2 + 4\,{e_2}^2 \\ &= \left(-2\,h_2 + {h_2}^2\right)^2 = {h_1}^4 - 4\,h_2\,{h_1}^2 + 4\,{h_2}^2 \end{aligned}$$

(b)

$$e_2^2 = 1/4\,\left(-p_2 + {p_1}^2\right)^2 = 1/4\,{p_1}^4 - 1/2\,p_2\,{p_1}^2 + 1/4\,{p_2}^2$$

(c)

$$h_2^2 = \left(-e_2 + {e_1}^2\right)^2 = {e_1}^4 - 2\,e_2\,{e_1}^2 + {e_2}^2$$

(2) By direct calculation

$$\begin{aligned} h_{(3,2)} &= 1/12\,{p_1}^5 + 1/3\,p_2\,{p_1}^3 + 1/6\,p_3\,{p_1}^2 + 1/4\,{p_2}^2 p_1 + 1/6\,p_3\,p_2 \\ h_{(2,2,1)} &= 1/4\,{p_1}^5 + 1/2\,p_2{p_1}^3 + 1/4\,{p_2}^2 p_1 \\ h_{(4,1)} &= 1/24\,{p_1}^5 + 1/4\,p_2{p_1}^3 + 1/3\,p_3{p_1}^2 + 1/8\,{p_2}^2 p_1 + 1/4\,p_4 p_1 \\ h_{(3,1,1)} &= 1/6\,{p_1}^5 + 1/2\,p_2{p_1}^3 + 1/3\,p_3{p_1}^2 \end{aligned}$$

(a) $\left\langle h_{(2,2,1)}, p_{(3,2)} \right\rangle = 0$
(b) $\left\langle h_{(3,2)}, p_{(3,2)} \right\rangle = 1$
(c) $\left\langle h_{(3,2)}, p_{(2,2,1)} \right\rangle = 2$
(d) $\left\langle h_{(3,2)}, h_{(4,1)} \right\rangle = 3$
(e) $\left\langle h_{(3,2)}, h_{(3,1,1)} \right\rangle = 4$
(f) $\left\langle h_{(3,2)}, h_{(2,2,1)} \right\rangle = 5$

(3) (a) $\left\langle \sum_{\mu\vdash n} p_\mu/z_\mu, p_\lambda \right\rangle = 1$
(b) $\left\langle \sum_{\mu\vdash n} (-1)^{|\mu|-\ell(\mu)} p_\mu/z_\mu, p_\lambda \right\rangle = (-1)^{|\lambda|-\ell(\lambda)}$
(c) $\langle p_n, h_\lambda \rangle = \delta_{\lambda,(n)}$
(d) $\langle p_{1^n}, h_\lambda \rangle = z_{1^n}/\prod_{i=1}^{\ell(\lambda)} z_{1^{\lambda_i}} = \binom{n}{\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)}}$
(e) $\langle p_\lambda, h_\lambda \rangle = \prod_i m_i(\lambda)!$
(f) $\langle h_n, h_n \rangle = \left\langle \sum_{\mu\vdash n} p_\mu/z_\mu, \sum_{\lambda\vdash n} p_\lambda/z_\lambda \right\rangle = \sum_{\lambda\vdash n} 1/z_\lambda$ which is equal to 1 since $n!/z_\lambda =$ the number of permuations with cycle type λ and $\sum_{\lambda\vdash n} n!/z_\lambda = n!$.
(g) $\langle e_n, h_n \rangle = \sum_{\lambda\vdash n} (-1)^{|\lambda|-\ell(\lambda)}/z_\lambda = 0$ if $n > 1$ and is equal to 1 if $n = 1$. This follows since $\sum_{\lambda\vdash n} (-1)^{|\lambda|-\ell(\lambda)} n!/z_\lambda =$ the number of permutations of even length − the number of permutations of odd length. The numbers of these subsets of these permutations must be equal because composition with a permutation of length 1 is an involution which interchanges these sets.
(h) $\langle h_n, h_\lambda \rangle$

(4) For any partition $k > 0$, $\Delta\circ\omega(p_k) = (-1)^{k-1}\Delta(p_k) = (\omega\otimes\omega)\circ\Delta(p_k)$. Since ω and Δ are both ring homomorphisms, this formula holds on any $f \in \Lambda$. Using this identity, $\Delta(e_n) = \Delta\circ\omega(h_n) = (\omega\otimes\omega)\circ\Delta(h_n) = (\omega\otimes\omega)(\sum_{k=0}^n h_k\otimes h_{n-k}) = \sum_{k=0}^n e_k\otimes e_{n-k}$.

(5) Again we need only show that this property holds on a basis to conclude that it holds for all symmetric functions $\Delta'\circ\omega(p_\lambda) = \Delta'((-1)^{|\lambda|-\ell(\lambda)}p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)}(p_\lambda\otimes p_\lambda) = (\omega\otimes 1)(p_\lambda\otimes p_\lambda) = (\omega\otimes 1)\circ\Delta'(p_\lambda)$ Similarly we have $\Delta'\circ\omega(p_\lambda) = (1\otimes\omega)\circ\Delta'(p_\lambda)$.

(6) Let $M_n = [a_{i,j}]$ be the $n \times n$ matrix with $a_{i,j} = p_{n-i-j+2}$ if $n - i - j + 2 > 0$ and $a_{i,j} = i - 1$ if $n - i - j + 2 = 0$ and $a_{i,j} = 0$ otherwise. Show $\frac{(-1)^n}{n!} det M_n$ satisfies the relation of equation (3.13) by expanding the determinant about the first row. Notice that the $(1, k)$ minor $M_n^{(1,k)}$ (the minor formed by deleting the 1^{st} row and k^{th} column of M_n) has determinant equal to $(-1)^{n-1}(n-1)!$ if $k = 1$ and $(-1)^{n-k}(n-1)_k det M_{k-1}$ for $2 \leq k \leq n$. Therefore, $n\frac{(-1)^n}{n!} M_n = p_n p_n$

(7)

$$\begin{aligned} H(t) \otimes H(t) &= \left(\sum_{r \geq 0} h_r t^r \right) \otimes \left(\sum_{m \geq 0} h_m t^m \right) \\ &= \sum_{n \geq 0} t^n \sum_{k=0}^{n} h_{n-k} \otimes h_k \\ &= \sum_{n \geq 0} t^n \Delta(h_n) = \Delta(H(t)) \end{aligned}$$

Apply the result of problem 4 to show as well that $\Delta(E(t)) = E(t) \otimes E(t)$.

$$\begin{aligned} \Delta(P(t)) = \sum_{r \geq 1} \Delta(\frac{p_r}{r}) t^r &= \sum_{r \geq 1} \left(\frac{p_r}{r} \otimes 1 + 1 \otimes \frac{p_r}{r} \right) t^r \\ &= P(t) \otimes 1 + 1 \otimes P(t) \end{aligned}$$

(8) $p_k^\perp(p_\lambda) = m_k(\lambda) k p_{\lambda \ominus (k)}$ from equation (3.44). Notice that $\frac{\partial}{\partial p_k}(p_\lambda) = m_k(\lambda) p_{\lambda \ominus (k)}$, so $p_k^\perp(p_\lambda) = k \frac{\partial}{\partial p_k}(p_\lambda)$ and $p_k^\perp(f) = k \frac{\partial}{\partial p_k}(f)$.

(9)

$$H(t) = 1 + \sum_{\ell \geq 1} (-1)^\ell \left(\sum_{k \geq 1} e_k (-t)^k \right)^\ell$$

Now the coefficient of t^n on both sides of this equation will be h_n on the left, and on the right a term appears for every partition with length ℓ and a coefficient equal to -1 raised to the size of the partition times a multinomial coefficient.

$$\begin{aligned} h_n &= \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda)\ m_2(\lambda)\ \cdots} e_\lambda \\ &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_{i \geq 1} m_i(\lambda)!} e_\lambda \end{aligned}$$

(10)

$$P(t) = \sum_{\ell \geq 1} \frac{(-1)^{\ell - 1}}{\ell} \left(\sum_{m \geq 1} h_m t^m \right)^\ell$$

The coefficient of t^n on the left hand side of this equation is $\frac{p_n}{n}$ and on the right hand side for each partition there is a term with a multinomial coefficient which depends on the length and a factor of $(-1)^{\ell-1}$ over the length of the partition.

$$p_n = n \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)-1} \frac{(\ell(\lambda)-1)!}{\prod_{i \geq 1} m_i(\lambda)!} h_\lambda$$

(11) With the formula from the previous problem it is a matter of finding a good way of expressing the product p_λ. The coefficient of h_μ in p_λ will be positive if $\ell(\mu) - \ell(\lambda)$ even and negative otherwise. The coefficient of h_λ in p_μ is

$$(-1)^{\ell(\lambda)-\ell(\mu)} \sum_{\nu^{(1)} \uplus \nu^{(2)} \uplus \cdots \uplus \nu^{(\ell(\mu))} = \lambda} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i(\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}$$

where the sum is over all sequences of partitions with $\nu^{(i)} \vdash \lambda_i$.

(12) In order for the bialgebra structure to have a Hopf algebra structure it must hold that $m \circ (id \otimes S') \circ \Delta' = u \circ \varepsilon'$ where $m(f \otimes g) = f * g$. Act by this expression on p_1 and we must have that

$$p_1 S'(p_1) = 1$$

This cannot happen unless $S'(p_1) = 1/p_1$ which is not in our algebra.

(13) For $\lambda \vdash n$ we clearly have that $h_n * p_\lambda = \left(\sum_{\mu \vdash n} \frac{p_\mu}{z_\mu}\right) * p_\lambda = p_\lambda$ and $e_n * p_\lambda = \left(\sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \frac{p_\mu}{z_\mu}\right) * p_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda = \omega(p_\lambda)$. Therefore by linearity it holds that $h_n * f = f$ and $e_n * f = \omega(f)$.

(14) Note that $g = \sum_\mu c_\mu p_\mu$. Therefore, $g * p_\lambda = z_\lambda c_\lambda p_\lambda$ and $g * (g * p_\lambda) = z_\lambda^2 c_\lambda^2 p_\lambda = p_\lambda$, and so we know that $z_\lambda^2 c_\lambda^2 = 1$ or $z_\lambda c_\lambda = \pm 1$. We also have $\langle g, p_\lambda \rangle = c_\lambda z_\lambda = \pm 1$.

(15) Since each $p_k \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$ from equation (3.16) or problem number 10, we know that $p_\lambda \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$. It follows that each $f = \sum_\lambda c_\lambda p_\lambda$ with each $c_\lambda \in \mathbb{Z}$, then $f \in \mathbb{Z}[e_1, e_2, e_3, \ldots]$. Many counterexamples to the converse exists (e.g. $e_2 = p_2/2 + p_1^2/2$).

(16) From equation (3.12) or problem number 6b or 9 we know that $h_k \in \mathbb{Z}[e_1, e_2, \ldots, e_k]$ and with an application of ω on these equations we know equally that $e_k \in \mathbb{Z}[h_1, h_2, \ldots, h_k]$.

(17) (a) $\{f + \omega(f) : f \in \Lambda\}$ are the set of functions which are invariant under the involution ω. That property is clearly invariant under products since ω is a ring homomorphism.
(b) This is not invariant under the coproduct Δ, since for instance $\Delta(e_{(2,2)} + h_{(2,2)})$ in the degree $(2,2)$ tensor is not invariant under ω.
(c) Note that $p_\lambda + \omega(p_\lambda) = 2p_\lambda$ if $|\lambda| + \ell(\lambda)$ is even and it is equal to 0 if $|\lambda| + \ell(\lambda)$ is odd. Since p_λ is a linear basis of Λ, $\{p_\lambda : |\lambda| - \ell(\lambda) \; mod \; 2\}$ is a linear basis for $\{f + \omega(f) : f \in \Lambda\}$.
(d) If λ has $|\lambda| - \ell(\lambda)$ even, then $p_\lambda * p_\mu = 0$ or is a multiple of p_λ and hence the basis elements are closed. Δ also sends the basis elements p_λ to $p_\lambda \otimes p_\lambda$.

(18) The fact that this ring is a Hopf subalgebra and closed under the Kronecker product and coproduct is true for any subalgebra generated by a subcollection of the the p_k and this need only be checked by verifying these operations are closed on the p_λ basis where the parts of λ are taken from the p_k which generate the subalgebra.

CHAPTER 4

Symmetric polynomials

Our presentation of the ring of symmetric functions has so far been non-standard and revisionist in the sense that the motivation for defining the ring Λ was historically to study the ring of polynomials which are invariant under the permutation of the variables. In this chapter we consider the relationship between Λ and this ring.

In this section we wish to consider polynomials $f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ such that $f(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) = f(x_1, x_2, \ldots, x_n)$ for all $\sigma \in Sym_n$. These polynomials form a ring since clearly they are closed under multiplication and contain the element 1 as a unit.

We will denote this ring

$$\Lambda^{X_n} = \{f \in \mathbb{Q}[x_1, x_2, \ldots, x_n] : f(x_1, x_2, \ldots, x_n) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) \text{ for all } \sigma \in Sym_n\} \tag{4.1}$$

Now there is a relationship between Λ and Λ^{X_n} by setting $p_k[X_n] := \sum_{k=1}^{n} x_i^k$ and define a map $\Lambda \longrightarrow \Lambda^{X_n}$ by the linear homomorphism

$$p_\lambda \mapsto p_{\lambda_1}[X_n] p_{\lambda_1}[X_n] \cdots p_{\lambda_{\ell(\lambda)}}[X_n] \tag{4.2}$$

with the natural extension to linear combinations of the p_λ.

In a more general setting we will take the elements Λ to be a set of functors on polynomials $p_k[x_i] = x_i^k$ and $p_k[cE + dF] = cp_k[E] + dp_k[F]$ for $E, F \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ and coefficients $c, d \in \mathbb{Q}$ then $p_\lambda[E] := p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]$. This means that $f \in \Lambda$ will also be a function from $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ to itself with the additional property that if $E \in \Lambda^{X_n} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$ then $f[E] \in \Lambda^{X_n}$ since if $\sigma E = E$ for a $\sigma \in Sym_n$ then we will also have $\sigma p_k[E] = p_k[\sigma E] = p_k[E]$ (similarly, $p_\lambda[E]$ and $f[E]$ will be invariant under σ).

Example 18. As a sample computation we determine $p_2[X_3], e_2[X_3]$ and $h_2[X_3]$.

$$p_2[x_1 + x_2 + x_3] = x_1^2 + x_2^2 + x_3^2$$

$$\begin{aligned} e_2[x_1 + x_2 + x_3] &= p_{(1,1)}[x_1 + x_2 + x_3]/2 - p_{(2)}[x_1 + x_2 + x_3]/2 \\ &= (x_1 + x_2 + x_3)^2/2 - (x_1^2 + x_2^2 + x_3^2)/2 = x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

$$\begin{aligned} h_2[x_1 + x_2 + x_3] &= p_{(1,1)}[x_1 + x_2 + x_3]/2 + p_{(2)}[x_1 + x_2 + x_3]/2 \\ &= (x_1 + x_2 + x_3)^2/2 + (x_1^2 + x_2^2 + x_3^2)/2 \\ &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

Example 19. Calculate $e_4[X_3]$.

$$\begin{aligned}
e_4[X_3] &= \frac{p_{(1^4)}[X_3]}{24} - \frac{p_{(211)}[X_3]}{4} + \frac{p_{(22)}[X_3]}{8} + \frac{p_{(31)}[X_3]}{3} - \frac{p_{(4)}[X_3]}{4} \\
&= \frac{(x_1+x_2+x_3)^4}{24} - \frac{(x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)^2}{4} + \frac{(x_1^2+x_2^2+x_3^2)^2}{8} \\
&\quad + \frac{(x_1^3+x_2^3+x_3^3)(x_1+x_2+x_3)}{3} - \frac{x_1^4+x_2^4+x_3^4}{4} \\
&= 0
\end{aligned}$$

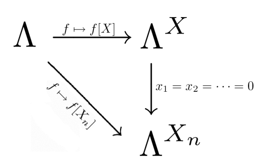
Remark 2. p_k is linear homomorphism because as we stated above $p_k[cE+dF] = cp_k[E] + dp_k[F]$ for $E, F \in \mathbb{Q}[x_1, x_2, \ldots x_n]$ and coefficients $c, d \in \mathbb{Q}$, but this is not true for p_λ in general (e.g. $p_{(2,1)}[x_1+x_2] = (x_1^2+x_2^2)(x_1+x_2) \neq p_{(2,1)}[x_1] + p_{(2,1)}[x_2] = x_1^3 + x_2^3$).

Remark 3. This notation is an extension of the linear homomorphism defined in equation (4.2) where we set $X_n := x_1 + x_2 + \cdots + x_n$.

In addition we will also consider Λ as functors on formal power series. Let $R = \mathbb{Q}[x_1, x_2, x_3, \ldots]$ and $R^{(k)}$ as the subspace of elements in R of degree k. $\widehat{R^{(k)}}$ will denote the completion of this subspace consisting of polynomials and formal series of monomials of degree k. Next define the ring

$$\begin{aligned}
(4.3) \qquad \Lambda^X = \{f(x_1, x_2, \ldots) \in \widehat{R^{(k)}} : f(x_1, x_2, \ldots) = \\
f(x_{\sigma_1}, x_{\sigma_2}, \ldots) \text{ for any permutation } \sigma, k \geq 0\}.
\end{aligned}$$

Just as we had for $\Lambda^{X_n} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$, $f \in \Lambda$ acts on $E \in \Lambda^X \subseteq \bigoplus_{k\geq 0} \widehat{R^{(k)}}$. Denote $X = x_1 + x_2 + x_3 + \cdots \in \Lambda^X$ so that $p_k[X] = \sum_{i\geq 1} x_i^k$. The operation of setting $x_{n+1} = x_{n+2} = \cdots = 0$ maps X to X_n and Λ^X to Λ^{X_n} such that the following diagram commutes.



Remark 4. This notation we have just introduced is quite useful, though there is one pitfall with which the reader should be aware.

Constants and variables are very different.

What we mean by this comment is that in polynomial notation where if $f(x) = x^k$ then $f(2) = 2^k$ and $f(-1) = (-1)^k$ (here constants have the same properties that variables do). In our notation, $p_k[2] = 2$ and $p_k[-1] = -1$ because $p_k[x_i] = x_i^k$, while p_k does nothing when it acts on constants. The reader should spend a few minutes to try to figure out a 'meaning' of $p_k[c]$ or $p_k[cX_n]$ because these **do not** represent $p_k(c)$ and $p_k(cx_1, cx_2, \ldots, cx_n)$ and it is important in doing calculations to be aware of this difference.

One interpretation of the expression $f[n]$ for a positive integer n should be thought of as $f[1+1+\cdots+1]$ and represents the symmetric function $f \in \Lambda$ with each p_k replaced by $x_1^k + x_2^k + \cdots + x_n^k$ and with each of these variables set to 1. We will derive more formulas for our symmetric functions below and using (4.5) we see that $e_k[n] = 0$ for $n = 0, 1, 2, \ldots, k-1$ and $e_k[k] = 1$. $f[c]$ when c is not a non negative integer does not have such a concrete realization and is instead a polynomial interpolation of $f[n]$.

Similarly, we have that $kX_n = X_n + X_n + \cdots + X_n$ (k-times) and hence $f[kX_n]$ represents the symmetric function f evaluated at a set of n variables repeated k times.

In some cases we will use a parameter q in some of our formulas. This parameter will act as a variable and has the property that $p_k[qX] = q^k p_k[X]$. The contrast between variables (q has the same properties as a variable) and constants can be seen here since $p_\lambda[qX] = q^{|\lambda|} p_\lambda[X]$ while for a constant c, $p_\lambda[cX] = c^{\ell(\lambda)} p_\lambda[X]$.

From now on we will denote $X_n := x_1 + x_2 + \cdots + x_n$ so that we have $p_k[X_n] = x_1^k + x_2^k + \cdots + x_n^k$. We are now ready to state and prove the fundamental theorem of symmetric functions which relates the algebra of symmetric functions and the algebra of symmetric polynomials.

We have defined $p_k[X]$ to be $\sum_i x_i^k$ and $p_k[X_n] = \sum_{i=1}^n x_i^k$. Therefore we realize $p_\lambda[X]$ and $p_\lambda[X_n]$ as just products of these elements. This does give us an explicit formula for $h_n[X]$ and $e_n[X]$ because they are just defined to be $h_n[X] = \sum_{\lambda \vdash n} p_\lambda[X]/z_\lambda$ and $e_n[X] = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} p_\lambda[X]/z_\lambda$. This formula is not at all adequate because we need only compute a few of these elements by hand or by computer to realize that the coefficient of any monomial of degree n in $h_n[X]$ is always 1. It is not immediately clear from the definition that the coefficients should even be integers.

Proposition 4.1. *For $n \geq 1$,*

$$h_n[X] = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{4.4}$$

$$e_n[X] = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{4.5}$$

These results are implied by the following expressions for the generating functions of $h_n[X]$ and $e_n[X]$

$$H(t)[X] = \sum_{n \geq 0} h_n[X] t^n = \prod_i \frac{1}{1 - t x_i} \tag{4.6}$$

$$E(t)[X] = \sum_{n \geq 0} (-1)^n e_n[X] t^n = \prod_i (1 - t x_i) \tag{4.7}$$

Proof. Consider the generating function $P(t)[X] = \sum_{r\geq 1} p_r[X]/rt^r$. This may be rewritten as

$$\begin{aligned} P(t)[X] &= \sum_{r\geq 1} \frac{p_r[X]}{r} t^r = \sum_{r\geq 1}\sum_i \frac{x_i^r}{r} t^r \\ &= \sum_i \sum_{r\geq 1} \frac{x_i^r}{r} t^r = -\sum_i log(1-tx_i) \\ &= log\left(\prod_i \frac{1}{1-tx_i}\right) \end{aligned} \tag{4.8}$$

However we have already seen in equation (1.9) that $H(t)[X] = exp(P(t))[X] = exp(P(t)[X])$ and a similar calculation yields $E(t)[X] = exp(-P(t)[X])$. This demonstrates equations (4.6) and (4.7).

The equation for $h_n[X]$ follows from taking the coefficient of t^n in (4.6). In each monomial there are n variables and each x_i can appear with repetition because $\frac{1}{1-tx_i} = 1+tx_i+(tx_i)^2+(tx_i)^3+\cdots$.

The equation for $e_n[X]$ can be arrived at by taking the coefficient of t^n in (4.7). In each monomial each x_i can appear at most once and each variable that appears contributes a factor of -1 and exactly n variables will appear in each monomial. □

We are now prepared to explicitly state the relationship between Λ and Λ^{X_n}. These spaces are not isomorphic, however the degree k components of each of these spaces is isomorphic as long as $k \leq n$.

Proposition 4.2. *Λ^{X_n} is algebraically generated by the elements $e_1[X_n]$, $e_2[X_n]$, ..., $e_n[X_n]$ and every $f(X_n) \in \Lambda^{X_n}$ is uniquely expressible as a linear combination of the elements $e_\lambda[X_n]$ for λ partitions with parts smaller or equal to n. In particular, the subspace of Λ^{X_n} of degree k is isomorphic to the subspace of degree k elements of Λ under the map which sends $f \mapsto f[X_n]$.*

Proof. For any $f(x_1, x_2, \cdots, x_n) \in \Lambda^{X_n}$ we note that the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$ must be the same if $\alpha = \sigma\lambda$ for some $\sigma \in Sym_n$. Therefore a linear basis of this space is given by the functions

$$\hat{m}_\lambda^{X_n} = \sum_{\alpha\sim\lambda} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \tag{4.9}$$

where λ is a partition with no more than n parts and the sum is over compositions α such that when the entries sorted in decreasing order the resulting partition is equal to λ. The dimension of Λ^{X_n} at degree m is the number of partitions of m with no more than n parts.

Now consider the elements $e_\lambda[X_n] \in \Lambda^{X_n}$ which are symmetric and hence can be expressed in the $\hat{m}_\mu^{X_n}$ basis. Notice that if $\lambda_1 \leq n$ then

$$e_\lambda[X_n] = \hat{m}_{\lambda'}^{X_n} + \text{ terms containing } \hat{m}_\mu^{X_n} \text{ with } \mu \text{ finer than } \lambda',$$

otherwise $e_\lambda[X_n] = 0$. Therefore $\{e_\lambda[X_n]\}_{\substack{\lambda\vdash n\\ \lambda_1\leq n}}$ is a basis for Λ^{X_n} and hence Λ^{X_n} is algebraically generated by the elements $e_1[X_n]$, $e_2[X_n]$, ..., $e_n[X_n]$. □

Since e_n is a linear combination of the h_λ with $\lambda \vdash n$ then we also have the following corollary.

COROLLARY 4.3. *Λ^{X_n} is algebraically generated by the elements $h_1[X_n]$, $h_2[X_n]$, ..., $h_n[X_n]$ and every $f(X_n) \in \Lambda^{X_n}$ is uniquely expressible as a linear combination of the elements $h_\lambda[X_n]$ for λ partitions with parts smaller or equal to n.*

Similarly, p_n can is a linear combination of the e_λ with $\lambda \vdash n$ and we can also state the previous corollary with $p_i[X_n]$ in place of $h_i[X_n]$. There is however a difference between the $p_i[X_n]$ and the $e_i[X_n]$ or $h_i[X_n]$ since if $f(X_n)$ is a symmetric polynomial with integer coefficients then when it is an expressed as a polynomial in either the $\{e_\lambda[X_n]\}_{\lambda_i\leq n}$ basis or the $\{h_\lambda[X_n]\}_{\lambda_i\leq n}$ basis it will have integer coefficients. This is not true in general of the $\{p_\lambda[X_n]\}_{\lambda_i\leq n}$ basis (see exercise 2.8).

This leads us to what we will refer to as the fundamental theorem of symmetric functions. It says essentially that Λ and Λ^X are isomorphic and as long as the degree of the symmetric functions you are working with is smaller than n then Λ, Λ^{X_n} and Λ^X are all the same.

THEOREM 4.4. *For $f, g \in \Lambda$ with deg$(f) \leq n$ and deg$(g) \leq n$, the following are equivalent:*

(1) $f = g$
(2) $f[E] = g[E]$ *for every expression* $E \in \bigoplus_{k\geq 0} R^{(k)}$
(3) $f[X] = g[X]$ *where* $X = x_1 + x_2 + x_3 + \cdots$
(4) $f[X_n] = g[X_n]$ *where* $X_n = x_1 + x_2 + \cdots + x_n$

PROOF. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (4). If $f[X] = g[X]$, then this expression holds independent of the values of x_i. In particular, if we set $x_{n+1} = x_{n+2} = \cdots = 0$, then we see that it must also hold that $f[X_n] = g[X_n]$.

(4) ⇒ (1). Assume that $f \neq g$ then $f - g \in \Lambda$ can be expressed in the e_λ basis with at least one coefficient not equal to 0. As we showed in the last proposition, $e_\lambda[X_n]$ is a basis of Λ^{X_n} and hence $f[X_n] - g[X_n]$ is not equal to 0. □

The formulas for $h_k[X]$ and $e_k[X]$ are interesting because it also gives us recurrences in terms of variables. Because any single variable appears in any monomial in $h_k[X]$ with exponent $0, 1, 2, \ldots, k$ and in $e_k[X]$ with exponent either 0 or 1, then we can grade $h_k[X+z]$ or $e_k[X+z]$ depending on the coefficient of z.

$$h_k[X + z] = \sum_{i=0}^{k} z^i h_{k-i}[X] \tag{4.10}$$

$$e_k[X+z] = e_k[X] + z e_{k-1}[X] \tag{4.11}$$

This is useful because it can be used to derive a formula for the homogeneous and elementary symmetric functions at $\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$

EXAMPLE 20.

$$p_2\left[\frac{1-q^3}{1-q}\right] = p_2[1+q+q^2] = 1 + q^2 + q^4 = \frac{1-q^6}{1-q^2}$$

$$\begin{aligned} e_2\left[\frac{1-q^3}{1-q}\right] &= p_{(1,1)}\left[\frac{1-q^3}{1-q}\right]/2 - p_{(2)}\left[\frac{1-q^3}{1-q}\right]/2 \\ &= \frac{(1-q^3)^2}{2(1-q)^2} - \frac{1-q^6}{2(1-q^2)} \\ &= \frac{(1-2q^3+q^6)(1+q) - (1-q^6)(1-q)}{2(1-q)(1-q^2)} = q\frac{1-q^3}{1-q} \end{aligned}$$

$$\begin{aligned} h_2\left[\frac{1-q^3}{1-q}\right] &= p_{(1,1)}\left[\frac{1-q^3}{1-q}\right]/2 - p_{(2)}\left[\frac{1-q^3}{1-q}\right]/2 \\ &= \frac{(1-q^3)^2}{2(1-q)^2} + \frac{1-q^6}{2(1-q^2)} \\ &= \frac{(1-2q^3+q^6)(1+q) + (1-q^6)(1-q)}{2(1-q)(1-q^2)} = \frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)} \end{aligned}$$

The use of $H(t)[X]$ as a generating function in which we can take coefficients a useful technique for deriving results in the theory of symmetric functions. To this end we define the special element $\Omega = \sum_{n\geq 0} h_n$ which lies in the completion of Λ. In some sense, Ω is still a generating function for the homogeneous generators like $H(t)$ from the previous section and we have $\Omega = H(1)$, $H(t)[X] = \Omega[tX]$, $E(t)[X] = \Omega[-X]$. This special element has some remarkable properties and we call it the Cauchy element.

PROPOSITION 4.5.

$$\Omega[X+Y] = \Omega[X]\Omega[Y] \tag{4.12}$$

and consequently

$$\Omega[-X] = \Omega[X]^{-1} \tag{4.13}$$

PROOF. Note that since $\Omega = H(1)$ so we have from equation (4.6) that

$$\Omega[X] = \prod_i \frac{1}{1-x_i}. \tag{4.14}$$

Therefore we also have

$$\Omega[X+Y] = \prod_i \frac{1}{1-x_i} \prod_i \frac{1}{1-y_i} = \Omega[X]\Omega[Y] \tag{4.15}$$

Notice that $\Omega[X-X] = 1$ since for $k > 0$, $p_k[X-X] = p_k[X] - p_k[X] = 0$. This implies that the operation of sending f to $f[X-X]$ gives the constant term of f and for Ω this is just 1. Therefore

$$\Omega[X-X] = \Omega[X]\Omega[-X] = 1 \tag{4.16}$$

and so $\Omega[-X] = \Omega[X]^{-1} = \prod_i 1 - x_i$. □

Algebra with infinite series sometimes has unusual consequences and there is one relation involving the element Ω which we shall exploit as often as possible.

PROPOSITION 4.6. *(The phantom relation) Let $\phi(z,u) = \sum_{k\in\mathbb{Z}} z^k u^{-k}$, then for any alphabet X,*

$$\phi(z,u)\Omega[zX]\Omega[-uX] = \phi(z,u) \tag{4.17}$$

What this means is that $\phi(z,u)\left(1 - \Omega[zX]\Omega[-uX]\right) = 0$ which is slightly unexpected since elements of our polynomial algebra are not zero divisors, however playing with these infinite series we can arrive at these unusual relations.

PROOF. Take the coefficient of $z^m u^n$ in $\phi(z,u)\Omega[zX]\Omega[-uX]$. This will be equal to

$$\cdots + (-1)^{n-1} h_{m+1}[X] e_{n-1}[X] + (-1)^n h_m[X] e_n[X] \\ + (-1)^{n+1} h_{m-1}[X] e_{n+1}[X] + (-1)^{n+2} h_{m-2}[X] e_{n+2}[X] + \cdots$$

Because $h_n[X] = e_n[X] = 0$ for $n < 0$, this sum is not infinite for each m and n, instead we have that for $m+n > 0$ the sum is equal to

$$\sum_{i=0}^{m+n} (-1)^i h_{m+n-i}[X] e_i[X] = 0. \tag{4.18}$$

If $m+n < 0$, then the sum is 0 simply because all terms are equal to 0, and if $m+n = 0$ then exactly one term is non-zero and it is equal to 1. This means that the coefficient of $z^m u^n$ in $\phi(z,u)\Omega[zX]\Omega[-uX]$ is equal to 1 if $m = -n$ and 0 otherwise and hence the series is equal to $\phi(z,u)$. □

Another remarkable property of the element Ω is that it plays the role of the identity element with respect to the Kronecker product. This means that for any symmetric function $\Omega * f = f * \Omega = f$. The bialgebra structure with product $*$ and coproduct Δ' has an identity element but that element does not lie in the algebra Λ, instead it is in the completion of Λ.

4.1. The monomial symmetric functions

For any given basis $\{a_\lambda\}_\lambda$ of Λ (so far we are essentially working with just the power, homogeneous and elementary) we can ask "what is the set of elements of Λ, $\{b_\lambda\}_\lambda$, such that $\langle a_\lambda, b_\mu\rangle = \delta_{\lambda\mu}$?"

It is a basic fact of linear algebra that the $\{b_\lambda\}_\lambda$ must also be a basis since if there is some linear dependence $\sum_\mu c_\mu b_\mu = 0$ with at least one $c_\lambda \neq 0$, then $\sum_\mu c_\mu b_\mu$ cannot be 0, because then $c_\lambda = \left\langle \sum_\mu c_\mu b_\mu, a_\lambda \right\rangle = \langle 0, a_\lambda \rangle = 0$. Therefore because the set $\{b_\lambda\}_\lambda$ has exactly the number of partitions of n elements at each degree and this set is linearly independent and therefore it spans and is a basis. We will call $\{b_\lambda\}_\lambda$ the basis dual to $\{a_\lambda\}_\lambda$. Notice also that this property is reflexive and $\{a_\lambda\}_\lambda$ is the dual basis to $\{b_\lambda\}_\lambda$ as well.

The bases $\{p_\lambda\}_\lambda$ and $\{p_\lambda/z_\lambda\}_\lambda$ are a pair of dual bases. As we have only just developed two other bases $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$, we should ask what their dual bases are. For this reason we develop the following amazing property of the element Ω. In the expression below, XY is the product of $X = \sum_i x_i$ and $Y = \sum_j y_j$ and hence $XY = \sum_{i,j} x_i y_j$. Therefore by definition, $\Omega[XY] = \prod_{i,j} \frac{1}{1-x_i y_j}$.

PROPOSITION 4.7. *Let $\{a_\lambda\}_\lambda$ be a basis for the symmetric functions then $\{b_\lambda\}_\lambda$ is the dual basis if and only if*

$$\Omega[XY] = \sum_\lambda a_\lambda[X] b_\lambda[Y] \tag{4.19}$$

It follows then that

$$\langle f[X], \Omega[XY] \rangle_X = f[Y] \tag{4.20}$$

PROOF. Since $\Omega = \sum_{n\geq 0} \sum_{\lambda \vdash n} p_\lambda / z_\lambda$, then we see that

$$\begin{aligned} \Omega[XY] &= \sum_{n\geq 0} \sum_{\lambda\vdash n} p_\lambda[X] p_\lambda[Y]/z_\lambda \\ &= \sum_{n\geq 0} \sum_{\lambda\vdash n} \sum_{\mu\vdash n} a_\mu[X] \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda \\ &= \sum_{n\geq 0} \sum_{\mu\vdash n} a_\mu[X] \sum_{\lambda\vdash n} \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda \qquad (4.21) \\ &= \sum_{n\geq 0} \sum_{\mu\vdash n} a_\mu[X] b_\mu[Y] \end{aligned}$$

The reverse implication can be seen from the same calculation since

$$\sum_{n\geq 0} \sum_{\lambda\vdash n} p_\lambda[X] p_\lambda[Y]/z_\lambda = \sum_{n\geq 0} \sum_{\lambda\vdash n} \sum_{\mu\vdash n} a_\mu[X] \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda$$

so we can conclude by taking the coefficient of $p_\lambda[Y]/z_\lambda$ that $p_\lambda = \sum_{\mu\vdash n} a_\mu \langle p_\lambda, b_\mu \rangle$. This means that if $a_\gamma = \sum_\lambda c_{\gamma\lambda} p_\lambda$,

$$a_\gamma = \sum_\lambda c_{\gamma\lambda} p_\lambda = \sum_\lambda c_{\gamma\lambda} \sum_{\mu\vdash n} a_\mu \langle p_\lambda, b_\mu \rangle = \sum_{\mu\vdash n} \langle a_\gamma, b_\mu \rangle a_\mu$$

Since $\{a_\lambda\}_\lambda$ is a basis, we can take the coefficient of a_λ on both sides of this equation and conclude that $\langle a_\gamma, b_\lambda \rangle = \delta_{\lambda\gamma}$ and hence $\{b_\mu\}_\mu$ is the dual basis to $\{a_\lambda\}_\lambda$.

To show the last result, we take for $f \in \Lambda$ and expand it in the power symmetric function basis using some coefficients c_λ, $f = \sum_\lambda c_\lambda p_\lambda$.

$$\begin{aligned} \langle f[X], \Omega[XY] \rangle &= \sum_\lambda c_\lambda \sum_\mu \langle p_\lambda[X], p_\mu[X]/z_\mu \rangle_X \, p_\mu[Y] \\ &= \sum_\lambda c_\lambda p_\lambda[Y] = f[Y]. \end{aligned} \tag{4.22}$$

□

We will define the basis dual to $\{h_\lambda\}_\lambda$ to be the monomial basis $\{m_\lambda\}_\lambda$ and the basis dual to the elementary symmetric functions $\{e_\lambda\}_\lambda$ are usually referred to as the forgotten symmetric functions. The last proposition can be used to find a direct formula for the monomial symmetric functions.

Proposition 4.8. *Let $\lambda \vdash n$,*

$$m_\lambda[X] = \sum_{\alpha \sim \lambda} \prod_i x_i^{\alpha_i} \tag{4.23}$$

where the sum is over all sequences $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and we have taken $\alpha \sim \lambda$ to mean the number of non-zero entries in α is $\ell(\lambda)$ and if they are sorted in decreasing order the sequence is equal to λ.

Proof. From the last propostion we know that $\Omega[XY] = \prod_i \frac{1}{1-x_i y_j} = \sum_\lambda h_\lambda[X] m_\lambda[Y]$. Now consider the coefficient of $y_{i_1}^{\alpha_1} y_{i_2}^{\alpha_2} \cdots y_{i_k}^{\alpha_k}$ in $\Omega[XY]$ is

$$\prod_j \frac{1}{1 - x_j y_{i_1}} \Big|_{y_{i_1}^{\alpha_1}} \prod_j \frac{1}{1 - x_j y_{i_2}} \Big|_{y_{i_2}^{\alpha_2}} \cdots \prod_j \frac{1}{1 - x_j y_{i_k}} \Big|_{y_{i_k}^{\alpha_k}} \tag{4.24}$$

which is equal to $h_{\alpha_1}[X] h_{\alpha_2}[X] \cdots h_{\alpha_k}[X]$. Therefore we may realize $\Omega[XY]$ as a sum over all sequences $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ with $\alpha_i \geq 0$ and a finite number of non-zero entries we find that

$$\Omega[XY] = \sum_\alpha h_\alpha[X] y^\alpha \tag{4.25}$$

where $h_\alpha = h_\lambda$ if $\alpha \sim \lambda$. This means that $m_\lambda[Y]$ is equal to the coefficient of $h_\lambda[X]$ in the expression above and hence it is equal to $\sum_{\alpha \sim \lambda} y^\alpha$. □

We have defined the monomial symmetric functions $\{m_\lambda\}_\lambda$ as the basis which is dual to the homogeneous basis $\{h_\lambda\}_\lambda$, but now knowing an explicit formula for $m_\lambda[X]$ allows us to easily deduce relations between these bases that are difficult to show otherwise. For instance, we immediately see that $m_{(k)}[X] = p_k[X]$ and $m_{(1^k)}[X] = e_k[X]$ and therefore $m_{(k)} = p_k$ and $m_{(1^k)} = e_k$ and so unlike our other bases m_λ is not generated as a product of elements. We can also see that $h_k = \sum_{\lambda \vdash k} m_\lambda$ either from Proposition 4.1 or by recalling that we have calculated $\langle h_k, h_\lambda \rangle = 1$ as an exercise. The formula for $m_\lambda[X]$ can also be used to derive a combinatorial rule for multiplying two monomial symmetric functions together.

PROPOSITION 4.9. *Let $\lambda \vdash n$ and $\mu \vdash k$*

$$m_\lambda \cdot m_\mu = \sum_{\nu \vdash n+k} r^\nu_{\lambda\mu} m_\nu \tag{4.26}$$

where $r^\nu_{\lambda\mu}$ is the number of pairs of sequences (α, β) with $\alpha_i, \beta_i \geq 0$ where $\alpha \sim \lambda$ and $\beta \sim \mu$ such that $\alpha + \beta = \nu$.

PROOF. This is easily seen in the expansion of $m_\lambda[X]m_\mu[X]$, we need only take the coefficient of x^ν in this expression. There is a contribution of weight 1 to each monomial of type x^ν in the product for each $\alpha \sim \lambda$ and $\beta \sim \mu$ such that $x^\alpha x^\beta = x^\nu$. This is equivalent to the condition that $\alpha + \beta = \nu$. □

EXAMPLE 21. We ask what the coefficient of $m_{(4,3,3)}$ is in $m^2_{(2,2,1)}$. This must be 2 because the only pairs $(\alpha, \beta) \sim ((2,2,1),(2,2,1))$ such that $\alpha + \beta = (4,3,3)$ are $((2,1,2),(2,2,1))$ and $((2,2,1),(2,1,2))$. As a more pictorial way of expressing this result, we may ask how many ways are there of coloring the Young diagram of the partition $(4,3,3)$ with two colors (the first color always lies to the left of the second) such that the horizontal pieces of the first color are of size $(2,2,1)$ and of the second color are of size $(2,2,1)$. The two diagrams are expressed as

Perhaps this combinatorial rule looks familiar since the coefficient of m_ν in $m_\lambda m_\mu$ will be the same as the coefficient of $h_\lambda \otimes h_\mu$ in the expression $\Delta(h_\nu)$ (a fact which we leave to the reader as an exercise). We used the same picture as appeared in chapter 1 to demonstrate exactly that connection.

From this we can arrive at a combinatorial method for computing the scalar product of $\langle h_\lambda, h_\mu \rangle$, $\langle e_\lambda, h_\mu \rangle$ or $\langle p_\lambda, h_\mu \rangle$. The scalar product of $\langle h_n, h_\lambda \rangle$ appears in the exercises of the last section, but the solution relied on the use of the $h_k^\perp$ operators on the h_μ basis. This time we give a proof that relies on a simple observation about symmetric functions given in terms of their variables. Since the homogeneous basis is dual to the monomial basis, we know that $\langle f, h_\mu \rangle$ is the coefficient of $m_\mu[X]$ in $f[X]$.

PROPOSITION 4.10. *For μ a partition of n, $\langle h_\lambda, h_\mu \rangle = A_{\lambda\mu}$ or*

$$h_\mu = \sum_\lambda A_{\lambda\mu} m_\lambda \tag{4.27}$$

where $A_{\lambda\mu}$ is the number of matricies with entries in $\mathbb{N}$ whose column sum is μ and row sum is equal to λ. $\langle h_\lambda, e_\mu \rangle = B_{\lambda\mu}$ or

$$e_\mu = \sum_\lambda B_{\lambda\mu} m_\lambda \tag{4.28}$$

where $B_{\lambda\mu}$ is the number of matricies with entries in $\{0,1\}$ whose column sum is μ and row sum is equal to λ. $\langle h_\lambda, p_\mu \rangle = C_{\lambda\mu}$ or

$$p_\mu = \sum_\lambda C_{\lambda\mu} m_\lambda \tag{4.29}$$

where $C_{\lambda\mu}$ is the number of matricies with entries in $\mathbb{N}$ whose column sum is μ and row sum is equal to λ and there is at most one non-zero entry for each column.

Proof. The coefficient of m_λ in h_μ will be equal to the coefficient of x^λ in $h_\mu[X]$ so we need only count the number of ways the coefficient x^λ may arise in $h_\mu[X]$. Since $h_n[X] = \sum_{|\alpha|=n} x^\alpha$ where the sum is over all compositions α whose entries sum to n, we see that the coefficient of x^λ will be the number of sequences $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\mu))})$ such that $x^{\alpha^{(1)}} x^{\alpha^{(2)}} \cdots x^{\alpha^{(\ell(\mu))}} = x^\lambda$ where $\alpha^{(i)}$ is a composition such that $|\alpha^{(i)}| = \mu_i$. We may think of $\alpha^{(i)}$ as a column vector of length $\ell(\mu)$ since the last non-zero entry occurs before $\ell(\mu)$ and the sum of the entries in that column are of course μ_i. The sum of the rows of the matrix $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\mu))})$ are exactly λ. Since there is a contribution of 1 to the coefficient of x^λ in $h_\mu[X]$ for every such matrix $A_{\lambda\mu}$ is exactly the number of such matricies.

The interpretation for $B_{\lambda\mu}$ and $C_{\lambda\mu}$ are very similar. Since the coefficient of m_λ in e_μ is equal to the coefficient of x^λ in $e_\mu[X]$, we are counting the number of ways that x^λ arises in $e_\mu[X]$. Since $e_n[X] = \sum_{|\alpha|=n} x^\alpha$ with the sum running over all compositions α with entries in $\{0,1\}$. This means that the coefficient of x^λ is again counting the number of matricies whose column sums are μ_i and whose row sums are λ_j, but with the additional restriction that the entries in these matricies are either 0 or 1.

Similarly, the interpretation for $C_{\lambda\mu}$ arises because $p_n[X] = \sum_i x_i^n = \sum_{|\alpha|=n} x^\alpha$, where the sum is over all compositions α with exactly one entry equal to n and the other enetries 0. This implies that the coefficient of x^λ in $p_\mu[X]$ is the number of matricies whose row sum λ_j and whose column sum is μ but at most one entry in the each column is allowed to be non-zero. □

Example 22. It is useful to see this proposition work in an example. We have established the coefficient of $m_{(2,2,2)}$ in $e_{(3,2,1)}$ is $B_{(2,2,2),(3,2,1)}$ and recall that this will also be the scalar product $\left\langle h_{(2,2,2)}, e_{(3,2,1)} \right\rangle$. One method for computing this scalar product could be to compute this directly by expanding both expressions in the power basis and using the definition of the scalar product, but we can also construct each of the $\{0,1\}$ matricies with row sums equal to $(2,2,2)$ and column sums equal to $(3,2,1)$. By exhaustively writing them out, we see there are exactly 3. They are

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.30}$$

This is not however the only combinatorial interpretation possible for these coefficients. We can provide another set of objects with the same number of elements as an interpretation that is perhaps easier to visualize. These interpretations are not significantly different however since there is a direct bijection between the elements in one set and the other.

Corollary 4.11. *Alternatively, $A_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are weakly increasing in the rows and there is no restriction on the relationship between the values in the columns.*

$B_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are strictly increasing in the rows and there is no restriction on the relationship between the values in the columns.

$C_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are weakly increasing in the rows and we require that all cells with label i must lie in the same row.

Example 23. We again compute the same coefficient $\langle h_{(2,2,2)}, e_{(3,2,1)} \rangle$ by giving the possible fillings of the Young diagram of shape $(2, 2, 2)$ with 3 1s, 2 2s and 1 3.

$$\begin{array}{|c|c|}\hline 1&3\\\hline 1&2\\\hline 1&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 1&3\\\hline 1&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline 1&3\\\hline\end{array} \tag{4.31}$$

Notice the relationship between these tabloid and to the matricies listed in the previous example. A bijection between the two sets of objects should be clear.

Example 24. To expand $h_{(3,2)}$ in terms of the monomial symmetric functions we examine all possible ways of filling the Young diagrams for the partitions of size 5 with 3 1s and 2 2s such that the entries are weakly increasing in the rows. We draw all of the possible tabloid as follows:

$$\begin{array}{lllll}1&1&1&2&2\end{array}\quad \begin{array}{llll}2\\1&1&1&2\end{array}\quad \begin{array}{llll}1\\1&1&2&2\end{array}\quad \begin{array}{lll}2&2\\1&1&1\end{array}\quad \begin{array}{lll}1&2\\1&1&2\end{array}\quad \begin{array}{lll}1&1\\1&2&2\end{array}$$

$$\begin{array}{lll}2\\2\\1&1&1\end{array}\quad \begin{array}{lll}2\\1\\1&1&2\end{array}\quad \begin{array}{lll}1\\2\\1&1&2\end{array}\quad \begin{array}{lll}1\\1\\1&2&2\end{array}\quad \begin{array}{ll}2\\2\\1\\1&1\end{array}\quad \begin{array}{ll}2\\1\\2\\1&1\end{array}$$

$$\begin{array}{ll}2\\1&2\\1&1\end{array}\quad \begin{array}{ll}2\\1&1\\1&2\end{array}\quad \begin{array}{ll}1\\1&1\\2&2\end{array}\quad \begin{array}{ll}1\\2&2\\1&1\end{array}\quad \begin{array}{ll}1\\1&2\\1&2\end{array}$$

$$\begin{array}{ll}1\\2\\2\\1&1\end{array}\quad \begin{array}{ll}2\\1\\1\\1&2\end{array}\quad \begin{array}{ll}1\\2\\1\\1&2\end{array}\quad \begin{array}{ll}1\\1\\2\\1&2\end{array}\quad \begin{array}{ll}1\\1\\1\\2&2\end{array}$$

There are also $\binom{5}{3} = 10$ ways of filling the Young diagram of shape (11111) in this manner. This implies that

$$h_{(32)} = m_{(5)} + 2m_{(41)} + 3m_{(32)} + 4m_{(311)} + 5m_{(221)} + 7m_{(2111)} + 10m_{(11111)}.$$

Example 25. To express $p_{(321)}$ in the monomial basis we need only examine partitions of size 6 such that the partition are sums of the parts of $(3, 2, 1)$. We list all of the possible tabloid for the partitions $(3, 2, 1), (3, 3), (4, 2), (5, 1), (6)$.

$$\begin{array}{lll}3\\2&2\\1&1&1\end{array}\quad \begin{array}{lll}2&2&3\\1&1&1\end{array}\quad \begin{array}{lll}1&1&1\\2&2&3\end{array}\quad \begin{array}{llll}2&2\\1&1&1&3\end{array}\quad \begin{array}{lllll}3\\1&1&1&2&2\end{array}\quad \begin{array}{llllll}1&1&1&2&2&3\end{array}$$

This implies that $p_{(3,2,1)}$ has the expansion

$$p_{(321)} = m_{(321)} + 2m_{(33)} + m_{(42)} + m_{(51)} + m_{(6)}.$$

The forgotten symmetric functions are the basis which is dual to the elementary symmetric functions. Because we have that $\langle \omega f, \omega g \rangle = \langle f, g \rangle$ where f and g are elements of Λ, we have that $\langle e_\mu, \omega(m_\lambda) \rangle = \langle h_\mu, m_\lambda \rangle = \delta_{\lambda\mu}$ and so $\{\omega(m_\lambda)\}_\lambda$ is the basis which is dual to the $\{e_\lambda\}_\lambda$. We name this basis $f_\lambda := \omega(m_\lambda)$, the forgotten symmetric functions.

Just by the definition, we have the following formulas:

$$\Omega[XY] = \sum_\lambda e_\lambda[X] f_\lambda[Y] \tag{4.32}$$

For a partition μ of n,

$$\begin{aligned} e_\mu &= \sum_\lambda A_{\lambda\mu} f_\lambda \\ h_\mu &= \sum_\lambda B_{\lambda\mu} f_\lambda \\ p_\mu &= (-1)^{|\mu| - \ell(\mu)} \sum_\lambda C_{\lambda\mu} f_\lambda \end{aligned} \tag{4.33}$$

where the coefficients $A_{\lambda\mu}, B_{\lambda\mu}$ and $C_{\lambda,\mu}$ are given in Proposition 4.10.

$$f_\lambda \cdot f_\mu = \sum_{\nu \vdash |\lambda| + |\mu|} r^\nu_{\lambda\mu} f_\nu \tag{4.34}$$

where the coefficients $r^\nu_{\lambda\mu}$ are given in Proposition 4.9.

To expand the monomial symmetric functions in terms of the forgotten basis we have the usual expansion

$$m_\lambda = \sum_{\mu \vdash |\lambda|} \langle m_\lambda, e_\mu \rangle f_\mu \tag{4.35}$$

Notice also that if we expand the elementary basis in terms of the homogeneous basis we see the same coefficients

$$e_\mu = \sum_{\lambda \vdash |\mu|} \langle m_\lambda, e_\mu \rangle h_\lambda \tag{4.36}$$

That is if we define the coefficient $D_{\lambda\mu} := \langle m_\lambda, e_\mu \rangle$, then we have the symmetry $m_\lambda = \sum_\mu D_{\lambda\mu} f_\mu$ and $e_\lambda = \sum_\mu D_{\mu\lambda} h_\mu$.

We have not exploited completely the formulas that we have derived for the coefficient of e_n in the h_λ basis. We can use this formula to give a rough combinatorial formula for the coefficient of h_λ in the expansion of e_μ. By the solution to exercise 1.9, we know that

$$e_n = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!} h_\lambda. \tag{4.37}$$

This implies that

$$e_\mu = \prod_{i=1}^{\ell(\mu)} \sum_{\nu \vdash \mu_i} (-1)^{\mu_i - \ell(\nu)} \frac{\ell(\nu)!}{\prod_i m_i(\nu)!} h_\nu. \tag{4.38}$$

Now in order to take the coefficient of h_λ in this equation we say that there will be a contribution to the coefficient of h_λ for every sequence of partitions $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})$ such that $\nu^{(1)} \uplus \nu^{(2)} \uplus \ldots \uplus \nu^{(\ell(\mu))} = \lambda$ and $\nu^{(i)} \vdash \mu_i$. We can see immediately that the sign of $e_\mu\Big|_{h_\lambda}$ is simply $(-1)^{n-\ell(\lambda)}$ because the sign of each contribution to the coefficient of h_λ in the product is always $\prod_{i=1}^{\ell(\mu)} (-1)^{\mu_i - \ell(\nu^{(i)})} = (-1)^{|\mu| - \ell(\lambda)}$. The contribution for each sequence $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})$ which satisfies these conditions is

$$\prod_{i=1}^{\ell(\mu)} \frac{\ell(\nu^{(i)})!}{\prod_{j=1}^{\nu_1^{(i)}} m_j(\nu^{(i)})!} \tag{4.39}$$

This implies the following proposition.

PROPOSITION 4.12.

$$\begin{array}{ll} e_\mu = \sum_{\lambda \vdash |\mu|} D_{\lambda\mu} h_\lambda & h_\mu = \sum_{\lambda \vdash |\mu|} D_{\lambda\mu} e_\lambda \\ m_\mu = \sum_{\lambda \vdash |\mu|} D_{\mu\lambda} f_\lambda & f_\mu = \sum_{\lambda \vdash |\mu|} D_{\mu\lambda} m_\lambda \end{array} \tag{4.40}$$

where

$$D_{\lambda\mu} = (-1)^{|\mu| - \ell(\lambda)} \sum_{(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})} \prod_{i=1}^{\ell(\mu)} \frac{\ell(\nu^{(i)})!}{\prod_{j=1}^{\nu_1^{(i)}} m_j(\nu^{(i)})!} \tag{4.41}$$

is the sum over all sequences of partitions such that $\nu^{(1)} \uplus \nu^{(2)} \uplus \ldots \uplus \nu^{(\ell(\mu))} = \lambda$ *and* $\nu^{(i)} \vdash \mu_i$.

The formula for $D_{\lambda\mu}$ is very similar to that for the coefficient of h_λ in p_μ, an explicit formula for these coefficients was calculated in exercise (1.11). These coefficients also appear in the expansion of the the monomial and forgotten bases in terms of the power basis.

PROPOSITION 4.13.

$$p_\mu = \sum_{\lambda \vdash |\mu|} E_{\lambda\mu} h_\lambda \qquad p_\mu = (-1)^{|\mu| - \ell(\mu)} \sum_{\lambda \vdash |\mu|} E_{\lambda\mu} e_\lambda$$

$$m_\mu = \sum_{\lambda \vdash |\mu|} E_{\mu\lambda} p_\lambda / z_\lambda \qquad f_\mu = \sum_{\lambda \vdash |\mu|} (-1)^{|\lambda| - \ell(\lambda)} E_{\mu\lambda} p_\lambda / z_\lambda$$

with

$$E_{\lambda\mu} = (-1)^{\ell(\lambda) - \ell(\mu)} \sum_{(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i (\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}$$

where the sum is over all sequences of partitions with $\nu^{(i)} \vdash \mu_i$ *and* $\nu^{(1)} \uplus \nu^{(2)} \uplus \cdots \uplus \nu^{(\ell(\mu))} = \lambda$.

PROOF. The justification of the expansion of p_μ in the homogeneous basis is exercise (1.10) and (1.11) from the previous chapter. An application of ω to this formula justifies the expansion of p_μ in the elementary basis.

Now to show the expansion of the monomial basis in the power basis, we recall that for $\mu \vdash n$

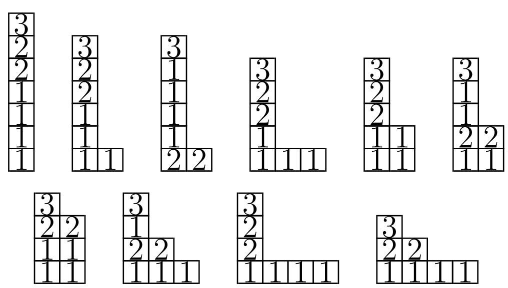
$$\begin{aligned} m_\mu &= \sum_{\nu \vdash n} \langle m_\mu, p_\lambda \rangle \, p_\lambda / z_\lambda \\ &= \sum_{\nu \vdash n} \left\langle m_\mu, \sum_{\nu \vdash n} E_{\nu\lambda} h_\nu \right\rangle p_\lambda / z_\lambda \\ &= \sum_{\nu \vdash n} E_{\mu\lambda} p_\lambda / z_\lambda \end{aligned}$$

The expansion of the forgotten basis in terms of the power basis also follows by an application of the involution ω on the previous formula. □

We should note that the only one of these formulas where the coefficients all have the same sign is the expansion of f_μ in the power sum basis. The coefficients of p_λ will be positive (or 0) if $|\mu| + \ell(\mu)$ is even and negative otherwise.

EXAMPLE 26. We will give an example of a computation of the expansion of $e_{(421)}$ and $p_{(421)}$ in the homogeneous basis. The sum is over the same set of objects so it is easy to both of the computations at the same time.

Each of the following pictures represents how to divide the partition λ in to sub-partitions $(\nu^{(1)}, \nu^{(2)}, \nu^{(3)})$ such that $\nu^{(1)} \vdash 4$, $\nu^{(2)} \vdash 2$ and $\nu^{(3)} \vdash 1$. For each of these tableaux we will count each with a weight.



Now in order to expand $e_{(421)}$ in terms of h_λ we count each of these tableaux with the weight

$$(-1)^{7-\ell(\lambda)} \prod_{i=1}^{3} \frac{\ell(\nu^{(i)})!}{\prod_{j\geq 1} m_j(\nu^{(i)})!},$$

where $\nu^{(i)}$ is the partition whose rows are labeled with i. This implies that

$$\begin{aligned} e_{(421)} = h_{(421)} &- h_{(41^3)} - 2\ h_{(3211)} + 2\ h_{(31^4)} - h_{(2^31)} + 4\ h_{(221^3)} \\ &- 4\ h_{(21^5)} + h_{(1^7)} \end{aligned}$$

In order to expand $p_{(421)}$ in terms of h_λ, we weight each of the tableaux listed above with the coefficient

$$(-1)^{3-\ell(\lambda)} 4 \cdot 2 \cdot 1 \prod_{i=1}^{3} \frac{(\ell(\nu^{(i)}) - 1)!}{\prod_{j\geq 1} m_j(\nu^{(i)})!}.$$

$$\begin{aligned} p_{(421)} = 8\ h_{(421)} &- 4\ h_{(41^3)} - 8\ h_{(3211)} + 4\ h_{(31^4)} - 4\ h_{(2^31)} + 10\ h_{(221^3)} \\ &- 6\ h_{(21^5)} + h_{(1^7)} \end{aligned}$$

From the previous discussion we have determined a formula or combinatorial interpretation for the coefficient of every one of the 5 bases in every other one of the 5 bases. The coefficients $A_{\lambda\mu}, B_{\lambda\mu}$ and $C_{\lambda\mu}$ can be found in Proposition 4.10 and Corollary 4.11. A formula/combinatorial interpretation for the coefficients $D_{\lambda\mu}$ is in Proposition 4.12 and the preceding discussion and $E_{\lambda\mu}$ is in Proposition 4.13. From these definitions we have the following table for the coefficient of b_λ in a_μ where a_μ represents the entry down the left side of the table and b_λ represents the label across the top of the table.

	p_λ	h_λ	e_λ	m_λ	f_λ
p_μ	$\delta_{\lambda\mu}$	$E_{\lambda\mu}$	$(-1)^{\lvert\mu\rvert-\ell(\mu)}E_{\lambda\mu}$	$C_{\lambda\mu}$	$(-1)^{\lvert\mu\rvert-\ell(\mu)}C_{\lambda\mu}$
h_μ	$C_{\mu\lambda}/z_\lambda$	$\delta_{\lambda\mu}$	$D_{\lambda\mu}$	$A_{\lambda\mu}$	$B_{\lambda\mu}$
e_μ	$(-1)^{\lvert\lambda\rvert-\ell(\lambda)}C_{\mu\lambda}/z_\lambda$	$D_{\lambda\mu}$	$\delta_{\lambda\mu}$	$B_{\lambda\mu}$	$A_{\lambda\mu}$
m_μ	$E_{\mu\lambda}/z_\lambda$	$F_{\lambda\mu}$	$G_{\lambda\mu}$	$\delta_{\lambda\mu}$	$D_{\mu\lambda}$
f_μ	$(-1)^{\lvert\lambda\rvert-\ell(\lambda)}E_{\mu\lambda}/z_\lambda$	$G_{\lambda\mu}$	$F_{\lambda\mu}$	$D_{\mu\lambda}$	$\delta_{\lambda\mu}$

This leaves two coefficients that we have not yet determined, $F_{\lambda\mu}$ and $G_{\lambda\mu}$, we leave it as an exercise to determine some sort of formula for these coefficients. For a more detailed account of the combinatorial interpretation of change of basis coefficients see [**?**].

4.2. Algebra operations and sets of variables

The notation that we have introduced allows us to express the operations of our Hopf algebra and bialgebra that we have already discussed as addition, subtraction and multiplication of alphabets.

Notice that $p_k[X+Y] = p_k[X] + p_k[Y]$ while $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$. Because we have defined $p_\lambda[X+Y] = \prod_i p_{\lambda_i}[X+Y] = \prod_i (p_{\lambda_i}[X] + p_{\lambda_i}[Y])$. It then follows that the coefficient of $p_\mu[X]p_\nu[Y]$ in $p_\lambda[X+Y]$ is equal to the coefficient of $p_\mu \otimes p_\nu$ in $\Delta(p_\lambda)$. More generally it follows that that if $\Delta(f) = \sum_i f_i \otimes g_i$, then $f[X+Y] = \sum_i f_i[X]g_i[Y]$..

This means that there is a clear isomorphism between $\Lambda \otimes \Lambda$ and Λ^{X+Y}, that is, a basis element $p_\lambda[X]p_\mu[Y]$ of Λ^{X+Y} is isomorphic to the basis element $p_\lambda \otimes p_\mu \in \Lambda \otimes \Lambda$. More generally, an element of Λ^{X+Y} $\sum_i f_i[X]g_i[Y]$ is isomorphic to $\sum_i f_i \otimes g_i$. That means that addition of two sets of variables encodes the coproduct Δ which we express in the following proposition.

PROPOSITION 4.14. *Given $f \in \Lambda$ such that $\Delta(f)$ is given by $\Delta(f) = \sum_i f_i \otimes g_i$, then*

$$f[X+Y] = \sum_i f_i[X]g_i[Y]. \tag{4.42}$$

Moreover, if $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are dual bases for Λ, then for all $f \in \Lambda$

$$f[X+Y] = \sum_{k\geq 0}\sum_{\lambda\vdash k}(a_\lambda^\perp f)[X]b_\lambda[Y] \tag{4.43}$$

PROOF. We know that for $f = p_\lambda$ we see that

$$
\begin{aligned}
p_\lambda[X+Y] &= \prod_{i=1}^{\ell(\lambda)} (p_{\lambda_i}[X] + p_{\lambda_i}[Y]) \\
&= \sum_{S \subseteq \{1,\ldots,\ell(\lambda)\}} \prod_{i \in S} p_{\lambda_i}[X] \prod_{i \notin S} p_{\lambda_i}[Y] \\
&= \sum_{k \geq 0} \sum_{\mu \vdash k} \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda \right) [X] p_\mu[Y]
\end{aligned}
\tag{4.44}
$$

Now for $f = \sum_\lambda c_\lambda p_\lambda$ we have that

$$
\begin{aligned}
f[X+Y] &= \sum_\lambda c_\lambda p_\lambda[X+Y] \\
&= \sum_\lambda c_\lambda \sum_{k \geq 0} \sum_{\mu \vdash k} \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda \right) [X] p_\mu[Y] \\
&= \sum_{k \geq 0} \sum_{\mu \vdash k} \sum_\lambda c_\lambda \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda \right) [X] p_\mu[Y] \\
&= \sum_{k \geq 0} \sum_{\mu \vdash k} \left(\frac{p_\mu^\perp}{z_\mu} f \right) [X] p_\mu[Y]
\end{aligned}
\tag{4.45}
$$

Now that we know that (4.43) holds for $a_\lambda = p_\lambda / z_\lambda$ and $b_\lambda = p_\lambda$, we show more generally that

$$
\begin{aligned}
f[X+Y] &= \sum_{k \geq 0} \sum_{\mu \vdash k} \sum_{\nu \vdash k} \left(\frac{p_\mu^\perp}{z_\mu} f \right) [X] < p_\mu, a_\nu > b_\nu[Y] \\
&= \sum_{k \geq 0} \sum_{\nu \vdash k} \sum_{\mu \vdash k} \left(< p_\mu, a_\nu > \frac{p_\mu^\perp}{z_\mu} f \right) [X] b_\nu[Y] \\
&= \sum_{k \geq 0} \sum_{\nu \vdash k} \left(a_\nu^\perp f \right) [X] b_\nu[Y].
\end{aligned}
\tag{4.46}
$$

□

Subtraction of variables is equivalent to addition of a negative set of variables and a symmetric function evaluated at a negative set of variables is equal to an application of the antipode map.

PROPOSITION 4.15. *For $f \in \Lambda$ such that f is homogeneous of degree k*

$$
f[-X] = S(f)[X] = (-1)^k \omega(f)[X]
\tag{4.47}
$$

PROOF. Recall that $S(p_k) = -p_k = (-1)^k \omega(p_k)$ and $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$. We also have $p_\lambda[-X] = (-1)^{\ell(\lambda)} p_\lambda[X] = S(p_\lambda)[X]$. This means that for $f = \sum_{\lambda \vdash k} c_\lambda p_\lambda$ for some coefficients c_λ,

$$f[-X] = \sum_{\lambda \vdash k} c_\lambda p_\lambda[-X] = \sum_{\lambda \vdash k} c_\lambda S(p_\lambda)[X] = S(f)[X].$$

□

REMARK 5. We can encode the antipode map S which is much like the involution ω, but not quite the same since it is off by -1 raised to the degree of the symmetric function it is acting on. It is possible to introduce notation which eliminates the sign. To this end one may introduce a variable q and at the end of the calculation set $q = -1$. This will be denoted ϵ. That is, for $f \in \Lambda$, f homogeneous of degree k

$$f[-\epsilon X] = (-1)^k \omega(f)[qX]\Big|_{q=-1} = (-1)^k q^k \omega(f)[X]\Big|_{q=-1} = \omega(f)[X].$$

Note that ϵ is very different from -1 because it is a variable. This notation is more useful when working with non-homogeneous symmetric functions since it allows us to encode the involution ω without referring to the degree of the symmetric function. We will not use this notation here.

Using this notation we can see multiplication as an operation that maps Λ^{X+Y} to Λ^X by setting the Y variables equal to the X variables. This can be seen since $\mu(f \otimes g) = f \cdot g$ while at the same time $f[X]g[Y]\Big|_{Y=X} = f[X]g[X] = (f \cdot g)[X]$. This means that we will be representing multiplication by the symbol $\Big|_{Y=X}$ which acts on the expression that lies to the left of this symbol by changing the Y variables to the X variables.

This notation implies that we have already computed such expressions as $h_m[X+Y]$ since we have already computed that $\Delta(h_m) = \sum_i h_i \otimes h_{m-i}$ in Proposition 1.15. Using the previous remark, this means $h_m[X+Y] = \sum_{i=0}^m h_i[X]h_{m-i}[Y]$, and similarly that $e_m[X+Y] = \sum_{i=0}^m e_i[X]e_{m-i}[Y]$.

Notice that for any $f, g \in \Lambda$,

$$\langle g[X+Y], f[Y] \rangle = (f^\perp g)[X]. \tag{4.48}$$

This is perhaps an unusual means for computing $f^\perp g$, but it is important interpretation of the operation $f[X+Y]$. We may also use this operation to compute specific operators $f^\perp$.

PROPOSITION 4.16. *For $k \in \mathbb{Z}$ and $f \in \Lambda$,*

$$h_k^\perp f[X] = f[X+z]\Big|_{z^k} \tag{4.49}$$

$$e_k^\perp f[X] = f[X-z]\Big|_{z^k}(-1)^k \tag{4.50}$$

PROOF. The first identity follows from equation (4.43),

$$f[X+z] = \sum_\lambda \left(h_\lambda^\perp f\right)[X] m_\lambda[z] \tag{4.51}$$

all terms of this sum are 0 unless λ has exactly one part. The coefficient of z^k in this equation will be $(h_k^\perp f)[X]$.

This same argument using the dual bases $\{e_\lambda\}_\lambda$ and $\{f_\lambda\}_\lambda$ and the relation $f_\lambda[-z] = (-1)^{|\lambda|} m_\lambda[z]$ shows that

$$f[X-z] = \sum_\lambda \left(e_\lambda^\perp f\right)[X] f_\lambda[-z] = \sum_\lambda (-1)^{|\lambda|}\left(e_\lambda^\perp f\right)[X] m_\lambda[z]. \tag{4.52}$$

The coefficient of z^k in this equation will be $(-1)^k (e_k^\perp f)[X]$. □

This last proposition implies that

$$f[X+z] = \sum_{k\geq 0} z^k (h_k^\perp f)[X] = \Omega[zX]^\perp f[X] \tag{4.53}$$

where by $\Omega[zX]^\perp$ is the operator which is dual to multiplication by the series $\Omega[zX]$ with respect to the scalar product in the X variables. As a manipulation, we know then for any two symmetric functions $f, g \in \Lambda$,

$$\langle f[X], g[X+z]\rangle = \langle \Omega[zX] f[X], g[X]\rangle . \tag{4.54}$$

The defining relation of the antipode, $\mu \circ (id \otimes S) \circ \Delta = u \circ \varepsilon$ can easily be seen in this notation.

$$\begin{aligned} \mu \circ (id \otimes S) \circ \Delta(f)[X] &= \mu \circ (id^X S^Y) f[X+Y] \\ &= \mu f[X-Y] = f[X-X] = f[0] \end{aligned} \tag{4.55}$$

This means that $u \circ \varepsilon(f)[X] = f[0]$, but this was something that we already knew since $u \circ \varepsilon(f) = f\Big|_{p_k=0}$.

Similarly, because we have that $p_k[XY] = p_k[X]p_k[Y]$ then just by the definition $p_\lambda[XY] = p_\lambda[X]p_\lambda[Y]$. Comparing this to $\Delta'(p_\lambda) = p_\lambda \otimes p_\lambda$, it follows that again the coefficient of $p_\mu \otimes p_\nu$ in $\Delta'(p_\lambda)$ is equal to the coefficient of $p_\mu[X]p_\nu[Y]$ in $p_\lambda[XY]$ (that is, they are both equal to $\delta_{\mu\lambda}\delta_{\nu\lambda}$). More generally this shows that if $\Delta'(f) = \sum_i f_i \otimes g_i$, then $f[XY] = \sum_i f_i[X] g_i[Y]$. This means that the coproduct Δ' is encoded in the multiplication of two sets of variables in the sense of the following propostion.

PROPOSITION 4.17. *For any symmetric function f inΛ, if $\Delta'(f) = \sum_i f_i \otimes g_i$, then*

$$f[XY] = \sum_i f_i[X] g_i[Y]. \tag{4.56}$$

Moreover for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$, we have that

$$f[XY] = \sum_{k\geq 0} \sum_{\lambda \vdash k} (a_\lambda * f)[X] b_\lambda[Y]. \tag{4.57}$$

PROOF. We have yet to show (4.57) which we know will hold for $f = p_\lambda$ $a_\mu = p_\mu/z_\mu$ and $b_\mu = p_\mu$ since

$$p_\lambda[XY] = p_\lambda[X]p_\lambda[Y] = \sum_\mu \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X]p_\mu[Y] \tag{4.58}$$

since all but one term of this sum is equal to 0. More generally, if $f = \sum_\lambda c_\lambda p_\lambda$ and $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are any pair of dual bases, then

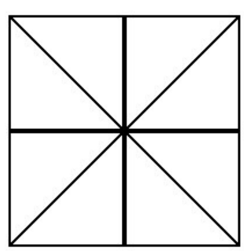
$$\begin{aligned} f[XY] &= \sum_\lambda \sum_\mu c_\lambda \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X]p_\mu[Y] \\ &= \sum_\lambda \sum_\mu \sum_\nu c_\lambda \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X] \langle p_\mu, a_\nu \rangle b_\nu[Y] \\ &= \sum_\lambda \sum_\nu \sum_\mu c_\lambda \left(\langle p_\mu, a_\nu \rangle \frac{p_\mu}{z_\mu} * p_\lambda\right)[X]b_\nu[Y] \\ &= \sum_\nu (a_\nu * f)[X]b_\nu[Y] \end{aligned} \tag{4.59}$$

□

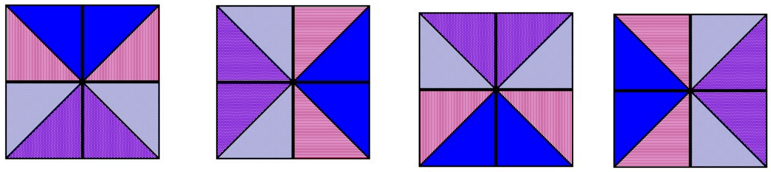
4.3. Application: Coloring enumeration

Starting with a figure like the one below, we can ask how many distinct ways there are of coloring the regions of the figure with k colors. If the figure is fixed in place and not allowed to move the answer is simply k^8 since there are 8 regions and each region can be colored independently with one of k different colors.

FIGURE 27. A square figure with 8 regions.

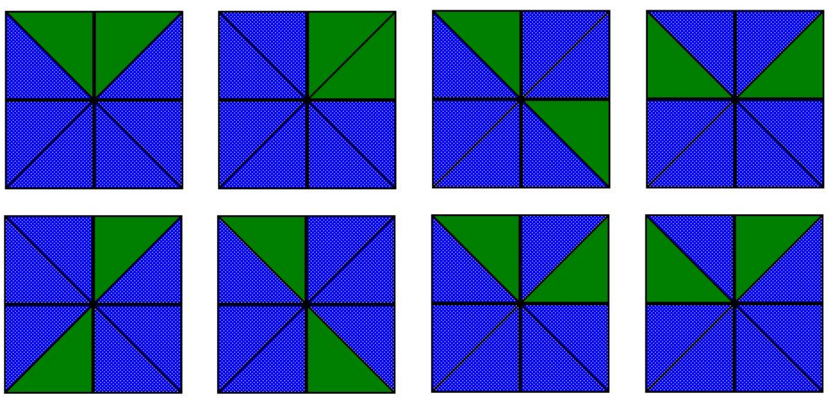


Now allow a group of isometries to act on the figure and say that two colorings are equal if there is some group element that transforms one to the other. With this condition there must be fewer than k^8 colorings because some of the k^8 colorings are now the equal. For instance, if we allow the group of four rotations to act on this figure then the following 4 colorings will be equivalent.



In general, to count the colorings we cannot simply divide by four since not all colorings will have the same order in their symmetry. In fact we will show that the number of colorings of figure 27 with k colors and this group of isometries acting on it is $\frac{1}{4}(k^8 + 2k^2 + k^4)$, something that is difficult to do with a simple counting argument.

A more general problem is to count the number of ways of coloring a figure like the one in figure 27 with a_1 regions blue, a_2 regions red, a_3 regions green, etc. such that $a_1+a_2+a_3+\cdots$ is equal to the number of regions. For example to color figure 27 with 6 blue regions and 2 green regions it is not difficult to determine that there are 8 distinct colorings like those given below. We wish to approach this problem in a general setting and give a formula for these enumerations.



It turns out that both of these types of problems can be solved using symmetric functions and the link between enumerating colorings and symmetric polynomials is a generating function for the number of colorings. Notice that if we assign a monomial weight $w(c)$ to each possible coloring c of a figure where $w(c)$ is x_1 raised to the number times the first color appears, x_2 to the number of times the second color appears, etc., then the sum over all possible colorings of $w(c)$ will be symmetric in the variables x_i and hence this expression will be a symmetric polynomial.

To begin we introduce some notation. A *group action* of a group G on a set R of regions to be colored satisfying the following properties:

(1) For the identity element $e \in G$, $e(r) = r$ for $r \in R$.
(2) For $g_1, g_2 \in G$, $g_1(g_2(r)) = (g_1 \cdot g_2)(r)$ for all $r \in R$.

Now the elements $g \in G$ act on the set R and permute the elements. The cycle type of G when it acts on R will be denoted by $\lambda_R(g)$ and is equal to the cycle type of the following permutation:

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ g(r_1) & g(r_2) & \cdots & g(r_n) \end{pmatrix}.$$

Define the cycle index symmetric function of a group G acting on a set R will be denoted

$$\mathcal{C}_R^G = \frac{1}{|G|} \sum_{g \in G} p_{\lambda_R(g)} \tag{4.60}$$

and it is a symmetric function of degree equal to the number of elements of R.

Example 28. Let $G = C_4$ be the cyclic group of permutations which rotate figure 27. That is, there are four elements of this group $C_4 = \{e, r, r^2, r^3\}$ where $r^4 = e$ and r acting on this figure is a rotation by 90 degrees. There are 8 regions of this figure to be colored and $\lambda_R(e) = (1^8)$, $\lambda_R(r) = \lambda_R(r^3) = (4,4)$, and $\lambda_R(r^2) = (2,2,2,2)$. Therefore

$$\mathcal{C}_R^{C_4} = \frac{1}{4}\left(p_1^8 + 2p_4^2 + p_2^4\right).$$

Example 29. Let $G = D_4$ be the group of rotations and reflections acting on figure 27. This time there are 8 elements in the group, $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ where $r^4 = e$ and $s^2 = e$ and $sr = r^3 s$. r acting on this figure is again a rotation by 90, s will be a flip across the horizontal passing through the center of the figure. We have already calculated the cycle type of the elements e, r, r^2, r^3 and we also have $\lambda_R(s) = \lambda_R(sr) = \lambda_R(sr^2) = \lambda_R(sr^3) = (2,2,2,2)$. Therefore

$$\mathcal{C}_R^{D_4} = \frac{1}{8}\left(p_1^8 + 2p_4^2 + 5p_2^4\right).$$

If we have a group action of G on a set X then the *orbit* of an element $x \in X$ is the set

$$Orbit(G;x) = \{g(x) : g \in G\}.$$

We also define the *stablilizer* of an element x to be the set

$$Stab(G;x) = \{g \in G : g(x) = x\}.$$

If $g \in Stab(G;x)$ then $g^{-1}(x) = g^{-1}(g(x)) = (g^{-1}g)(x) = x$ and hence $g^{-1} \in Stab(G;x)$. If $g, h \in Stab(G;x)$ then $(gh)(x) = g(h(x)) = x$ and so $gh \in Stab(G;x)$ and hence $Stab(G;x)$ is a subgroup of G. We also have that for $g \in G$, if $g(x) = y$ then $g^{-1}(y) = g^{-1}(g(x)) = (g^{-1}g)(x) = x$. The orbit and the stabilizer of an element x are related by the set $Orbit(G;x)$ is isomorphic to the set of left cosets of $Stab(G;x)$. We show this in the following lemma.

Proposition 4.18. *For $x \in X$, we have $Orbit(G;x)$ is isomorphic to the set of left cosets of $Stab(G;x)$ in G. This implies*

$$|Orbit(G;x)|\,|Stab(G;x)| = |G|.$$

PROOF. The correspondence between the orbit and the cosets of the stabilizer simply sends the element gx to the coset $g \cdot Stab(G;x)$. We show

$$\begin{aligned} g \cdot Stab(G;x) = h \cdot Stab(G;x) &\Leftrightarrow (h^{-1}g) \cdot Stab(G;x) = Stab(G;x) \\ &\Leftrightarrow h^{-1}g \in Stab(G;x) \\ &\Leftrightarrow (h^{-1}g)(x) = x \\ &\Leftrightarrow g(x) = h(x). \end{aligned}$$

This shows that the map is onto since for $g \in G$, $g(x) = g_i(x)$ for some representative element $g_i(x) \in Orbit(G;x)$ and $g \cdot Stab(G;x) = g_i \cdot Stab(G;x)$. This also implies that the map is one-to-one since $g(x) = h(x)$ implies $g \cdot Stab(G;x) = h \cdot Stab(G;x)$. Since the cosets of $Stab(G;x)$ partition the group G into equal parts and every element is in exactly one coset, we know that

$$|Orbit(G;x)| = \#\text{ cosets of } Stab(G;x) \text{ in } G = \frac{|G|}{|Stab(G;x)|}.$$

□

If we have a finite set X and G acts on X by permuting the elements, then there are a finite number of sets $Orbit(G;c)$ and every element of X will be in exactly one of the orbits (the set of orbits forms a partition of the set X). Let m be the number of elements in $\{Orbit(G;c) : c \in X\}$ and let $c_1, c_2, \ldots, c_m$ be representative elements of this set of orbits so that every $c \in X$ is in exactly one set $Orbit(G;c_i)$.

PROPOSITION 4.19. *If $c \in Orbit(G;d)$, then $Orbit(G;c) = Orbit(G;d)$ and consequently $|Stab(G;c)| = |Stab(G;d)|$.*

PROOF. If $c \in Orbit(G;d)$ then $c = g(d)$ for some $g \in G$. This mean that

$$Orbit(G;c) = \{h(c) : h \in G\} = \{h(g(d)) : h \in G\} = Orbit(G;d).$$

□

For a finite set R of regions, a *coloring* of R with k colors is a map $c : R \to \{1, 2, \ldots k\}$. If g acts on R then the definition of g on the coloring c is $g(c)(r) = c(g(r))$ for $r \in R$ (in other words, a group acts on a coloring of the regions by permuting the regions).

A coloring c will be invariant under the action of a group element g if every r in the same orbit of g is colored with the same value, that is, if $g(r) = r'$ then $c(r) = c(r')$. This implies that each cycle in the permutation

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ g(r_1) & g(r_2) & \cdots & g(r_n) \end{pmatrix}$$

may be assigned a value independently. The number of colorings invariant under the action of the element $g \in G$ is equal to k raised to the number of cycles in this permutation or $k^{\ell(\lambda_R(g))} = p_{\lambda_R(g)}[k]$. That is, we have

$$p_{\lambda_R(g)}[k] = \#\{c : R \to \{1, 2, \ldots k\} | g(c) = c\}.$$

This leads us to our first enumeration formula:

THEOREM 4.20. *(Burnside's formula) Let R be a set of regions to color and X be the set of colorings with k colors. Say that the set $\{Orbit(G;c) : c \in X\}$ has order m, then*

$$\mathcal{C}_R^G[k] = m.$$

PROOF. Let m be the number of elements in $\{Orbit(G;c) : c \in X\}$ and $c_1, c_2, \ldots, c_m$ be a set of representative elements so that every $c \in X$ is in exactly one of the sets $Orbit(G;c_i)$.

$$\begin{aligned}
\mathcal{C}_R^G[k] &= \frac{1}{|G|}\sum_{g\in G} p_{\lambda_R(g)}[k] \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{c\in X\\ g(c)=c}} 1 \\
&= \frac{1}{|G|}\sum_{c\in X}\sum_{\substack{g\in G\\ g(c)=c}} 1 \\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{c\in Orbit(G;c_i)} |Stab(G;c)| \qquad (4.61)\\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{c\in Orbit(G;c_i)} |Stab(G;c_i)| \\
&= \frac{1}{|G|}\sum_{i=1}^{m} |Orbit(G;c_i)||Stab(G;c_i)| \\
&= \frac{1}{|G|}\sum_{i=1}^{m} |G| = m.
\end{aligned}$$

□

EXAMPLE 30. We computed $\mathcal{C}_R^{C_4}$ and $\mathcal{C}_R^{D_4}$ for R equal to the regions in figure 27. Theorem 4.20 says that the number of distinct ways of coloring this figure k colors when the group C_4 acts on the figure is $\frac{1}{4}\left(k^8 + 2k^2 + k^4\right)$ and when D_4 acts on the figure is $\frac{1}{8}\left(k^8 + 2k^2 + 5k^4\right)$.

Let us consider a figure which we can verify by hand very easily that this formula does in fact work as advertised.

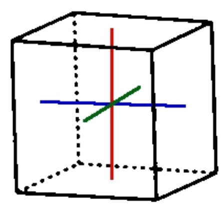
EXAMPLE 31. How many ways are there of coloring a cube with 2 colors such that two coloring are considered to be equal if they look the same by a rotation? The natural group of isometries which acts on this object is the group of rotations which permute the faces, edges and vertices of the cube.

It helps to have a cube to look at, if you can find a die or a Rubik's cube the calculations that we are about to do will seem easier.

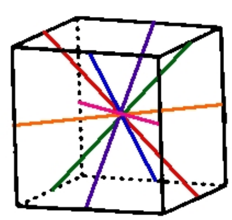
We note that the symmetry group of rotations the cube has 24 elements in it. This is easy to see because any rotation of the cube moves one of 6 faces to the face which is on top and then there are 4 choices for the face which is in front. The question is, what are these 24 group elements which act on the cube?

The first element to consider is the identity element. There is only 1 of these.

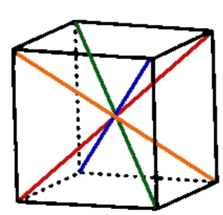
Next, there are 9 rotations of the cube with two faces left fixed. In the figure below this corresponds to the rotations about one of the three lines which pass through the center of the cube perpendicular to exactly two faces.



There are also 6 rotations of the cube by 180 degrees that leave two edges fixed. This corresponds to a rotation around one of the the six lines in the figure below that pass through the center of the cube and are perpendicular to two edges.



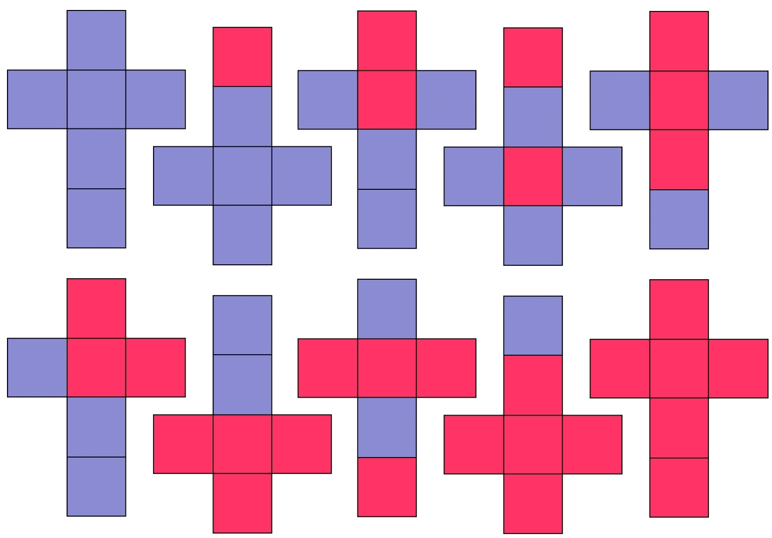
Finally there are 8 rotations by 120 degrees or 240 degrees around two opposite corners of the cube. This will be a rotation around one of the four lines in the figure below that pass through the center of the cube and connect two of the corners.



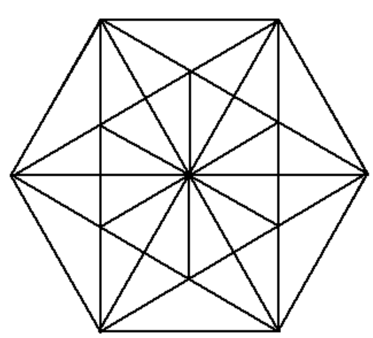
Let G be this group of isometries and R be the 6 faces of the cube. Calculating $\mathcal{C}_R^G$ is as easy as looking at each of the four types of permutations listed above. By looking at a cube and noticing what the cycle type of each of these permutations of the faces is we calculate

$$\frac{1}{24}\left(p_1^6 + 6p_4p_1^2 + 3p_2^2p_1^2 + 6p_2^3 + 8p_3^2\right).$$

Therefore if we wish to know the number of distinct ways of coloring the cube with 2 colors, it will be $\frac{1}{24}(2^6 + 6\cdot 2^3 + 3\cdot 2^4 + 6\cdot 2^3 + 8\cdot 2^2) = 10$. This number is something we can easily determine by exhaustively listing the 10 possible colorings. In the figure below we have folded out flat the 6 faces of the cube and colored them either red or blue.



Example 32. How many distinct ways are there of coloring the following figure with 3 colors where the group that acts on it is generated by a reflection across the vertical line and a rotation by 60 degrees?



The group acting on this figure is the dihedral group of order 12 since it is generated by two elements $x, y \in D_6$ satisfying $x^6 = y^2 = e$ and $xy = yx^5$. This means that $D_6 = \{e, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$. There are 30 regions in the figure and are permuted by the elements of this group. We compute the following cycle structures of these group elements on the regions of the figure.

$$\lambda_R(e) = (1^{30})$$
$$\lambda_R(x) = \lambda_R(x^5) = (6^5)$$
$$\lambda_R(x^2) = \lambda_R(x^4) = (3^{10})$$
$$\lambda_R(x^3) = \lambda_R(yx) = \lambda_R(yx^3) = \lambda(yx^5) = (2^{15})$$
$$\lambda_R(y) = \lambda_R(yx^2) = \lambda(yx^4) = (2^{14}, 1^2)$$

These calculations determine

$$\mathcal{C}_R^{D_6} = \frac{1}{12}\left(p_1^{30} + 2p_6^5 + 2p_3^{10} + 4p_2^{15} + 3p_2^{14}p_1^2\right)$$

It follows then that the number of ways of coloring this figure with 3 colors is equal to

$$\mathcal{C}_R^{D_6}[3] = \frac{1}{12}\left(3^{30} + 2\cdot 3^5 + 2\cdot 3^{10} + 4\cdot 3^{15} + 3\cdot 3^{16}\right) = 17157609895752.$$

Example 33. A simple graph is a set of vertices V (which we shall take as the set $=\{1, 2, \ldots n\}$) together with a set of edges $E \subseteq \{\{u, v\} : u, v \in V \text{ and } u \neq v\}$. The symmetric group acts on V by permuting the vertex set and on edges by $\sigma\{u, v\} = \{\sigma(u), \sigma(v)\}$. In our context, two graphs (V, E) and (V, E') will be isomorphic if there is a permutation $\sigma \in Sym_n$ such that $\sigma E = E'$.

A simple graph is represented by a set of labeled points and a line between two points u and v if $\{u, v\}$ is an edge in E.

We wish to count non-isomorphic simple graphs and this can be done by thinking of a graph (V, E) as a coloring of the two element subsets $R_n = \{\{i, j\} : 1 \leq i < j \leq n\}$ with two colors say white and black. The black edges will represent those which are in E and the white ones will represent those that are not in E.

This means that $\mathcal{C}_{R_n}^{Sym_n}$ will be a symmetric function of degree $\binom{n}{2}$. Lets compute the number of non-isomorphic graphs on with 2, 3, 4 and 5 vertices. To determine these symmetric functions we need to determine the action for each σ on the two element subsets but it is only necessary to look at one of each cycle type (we will list calculate λ_R of each of the group elements listed in cycle notation).

For $n = 2$, $\lambda_{R_2}((1)(2)) = (1)$ and $\lambda_{R_2}((12)) = (1)$. Therefore $\mathcal{C}_{R_2}^{Sym_2} = \frac{1}{2}(p_1 + p_1) = p_1$.

For $n = 3$, there are 3 two element subsets and $\lambda_{R_3}((1)(2)(3)) = (1, 1, 1)$, $\lambda_{R_3}((12)(3)) = (2, 1)$ and $\lambda_{R_3}((123)) = (3)$. We have, $\mathcal{C}_{R_3}^{Sym_3} = \frac{1}{6}\left(p_{(1^3)} + 3p_{(2,1)} + 2p_{(3)}\right) = h_3$.

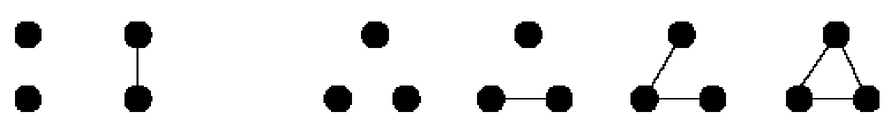
For $n = 4$, there are 6 two element subsets. $\lambda_{R_4}((1)(2)(3)(4)) = (1^6)$, $\lambda_{R_4}((12)(3)(4)) = (2, 2, 1, 1)$, $\lambda_{R_4}((12)(34)) = (2, 2, 1, 1)$, $\lambda_{R_4}((123)(4)) = (3, 3)$, $\lambda_{R_4}((1234)) = (4, 2)$. The cycle index symmetric function is

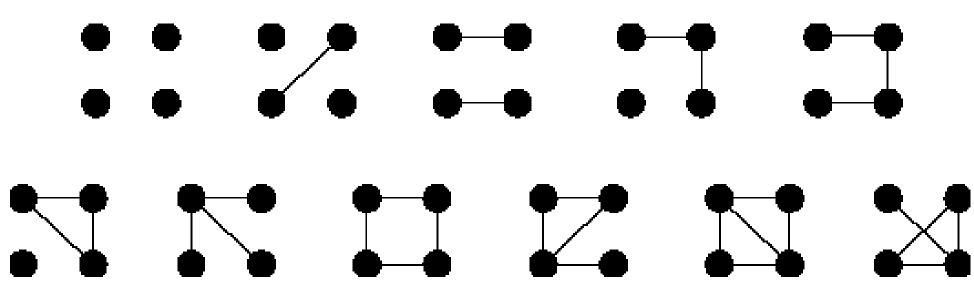
$$\mathcal{C}_{R_4}^{Sym_4} = \frac{1}{24}\left(p_{(1^6)} + 6p_{(2,2,1,1)} + 3p_{(2,2,1,1)} + 8p_{(3,3)} + 6p_{(4,2)}\right).$$

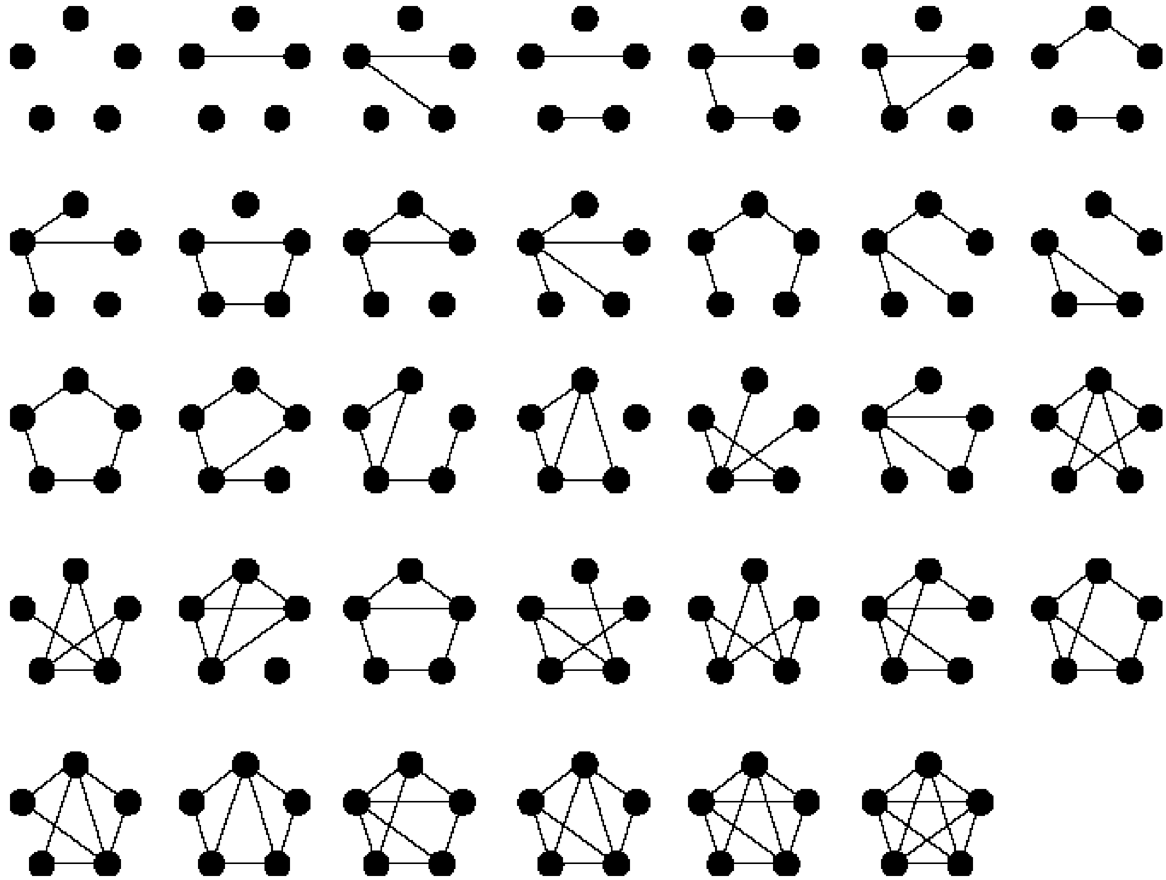
For $n = 5$ there are 10 two element subsets. For the most part, the order of the subset $\{u, v\}$ will be the l.c.m. of the length of the cycle that u is in and the length of the cycle that v is in (the exception being the sets $\{1, 3\}$ and $\{2, 4\}$ under the action of the element $(1234)(5)$). $\lambda_{R_5}((1)(2)(3)(4)(5)) = (1^{10})$, $\lambda_{R_5}((12)(3)(4)(5)) = (2^3, 1^4)$, $\lambda_{R_5}((12)(34)(5)) = (2^4, 1, 1)$, $\lambda_{R_5}((123)(4)(5)) = (3^3, 1)$, $\lambda_{R_5}((123)(45)) = (6, 3, 1)$, $\lambda_{R_5}((1234)(5)) = (4, 4, 2)$, $\lambda_{R_5}((12345)) = (5, 5)$.

$$\mathcal{C}_{R_5}^{Sym_5} = \frac{1}{120}\left(p_{(1^{10})} + 10p_{(2^3,1^4)} + 15p_{(2^4,1,1)} + 20p_{(3^3,1)} + 20p_{(6,3,1)} + 30p_{(4,4,2)} + 24p_{(5,5)}\right).$$

Now that we have the cycle index symmetric functions, it is quite easy to determine the number of non-isomorphic simple graphs there are, we just evaluate them at the value 2. This means that there are $\mathcal{C}_{R_2}^{Sym_2}[2] = 2$ graphs on two vertices, $\mathcal{C}_{R_3}^{Sym_3}[2] = 4$ graphs on three vertices, $\mathcal{C}_{R_4}^{Sym_4}[2] = 11$ graphs on four vertices, and $\mathcal{C}_{R_5}^{Sym_5}[2] = 34$ graphs on five vertices. Each of these values are not too difficult to verify by hand so we draw the sets of graphs below to see that they agree with the theory.







We turn our attention now to a more specific type of enumeration problem, that of counting the number of ways of coloring a figure using a prescribed number of colors. We gave an example of such a question when above when we posed the question how many ways the figure 27 could be colored with 6 blue regions and 2 green regions.

Let G be a group with an action on a set of regions R. For any coloring $c : R \longrightarrow \mathbb{N}$, we call the *weight function* the map w with $w(c) = \prod_{i=1}^{|R|} x_{c(r_i)}$ where the r_i are the elements of R listed in some order (since our variables commute this expression is independent of the order). The weight function sends a coloring to a monomial in $\mathbb{Q}[x_1, x_2, x_3, \ldots]$ The generating function $\sum_c w(c)$ where the sum is over all distinct colorings of R under the group G is called the *pattern inventory*.

The pattern inventory is a generating function which contains all the information necessary to count the number of patterns with given number of colors appearing, we need only take a coefficient in this generating function of $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ to find the number of colorings with color 1 appearing a_1 times, color 2 appearing a_2 times, color 3 appearing a_3 times, etc.

This generating function also encodes the total number of colorings using k colors. We can recover the information by setting $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = 0$. This give a clue to identifying the formula for the pattern inventory.

THEOREM 4.21. *(Pòlya's theorem) Let G be a group which acts on a set R and let C be a set of colorings mapping R to the set $\mathbb{N}$ which are all distinct under the action of G.*

$$\mathcal{C}_R^G[X] = \sum_{c \in C} w(c).$$

PROOF. This means □

4.4. Exercises:

(1) Show
$$e_k\left[[n]_q\right] = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$
where $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$.

(2) Show
$$h_k\left[[n]_q\right] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q.$$

(3) Show
$$p_k\left[[n]_q\right] = \frac{[nk]_q}{[k]_q}.$$

(4) Show directly (without appealing to a formula which we have derived for these values) that the coefficient of m_ν in $m_\lambda m_\mu$ will be the same as the coefficient of $h_\lambda \otimes h_\mu$ in the expression $\Delta(h_\nu)$.

(5) Show that $e_k[X - z] = \sum_{i=0}^{k} (-z)^i e_{k-i}[X]$ and $h_k[X - z] = h_k[X] - z h_k[X]$.

(6) Show $\Omega * f = f * \Omega = f$ for all $f \in \Lambda$.

(7) Show that $\Omega\left[\frac{x-y}{1-q}\right] = \prod_{i \geq 0} \frac{1-xq^i}{1-yq^i}$.

(8) Show that if $f(X_n)$ is in the linear span of the symmetric polynomials $m_\lambda[X_n]$ over $\mathbb{Z}$ then when it is expressed in the $\{e_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis it has coefficients in $\mathbb{Z}$ and when it is expressed in the $\{h_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis it has coefficients in $\mathbb{Z}$. Show that in general if it is expressed in the $\{p_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis the coefficients will be in $\mathbb{Q}$.

(9) Show that
$$\langle h_\mu, e_{\lambda'} \rangle \quad \begin{cases} > 0 & \text{if } \mu < \lambda \\ = 1 & \text{if } \lambda = \mu \\ = 0 & \text{otherwise.} \end{cases}$$

(10) Determine some sort of formula for the coefficient of h_λ in m_μ and the coefficient of e_λ in m_μ in terms of the other coefficients which we have already determined a formula.

CHAPTER 5

The Schur functions

The last of the six standard bases of the symmetric functions which have yet to give an account for are the Schur symmetric functions. We have saved the best for last. Since we have developed the symmetric functions as an algebra generated by elements $h_1, h_2, h_3, \ldots$ it seems natural to take a the following formula as a definition of the Schur function basis.

$$s_\lambda := det|h_{\lambda_i - i + j}|_{1 \le i,j \le \ell(\lambda)}. \tag{5.1}$$

This is known as the Jacobi-Trudi formula for the Schur functions.

Example 34.

$$s_{(1^n)} = det \begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ 1 & h_1 & h_2 & \cdots & h_{n-1} \\ 0 & 1 & h_1 & \cdots & h_{n-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}$$

This determinant we know to be equal to e_n by exercise 1.6(b).

$$s_{(n)} = det \begin{vmatrix} h_n \end{vmatrix} = h_n$$

$$s_{(n-k,1^k)} = det \begin{vmatrix} h_{n-k} & h_{n-k+1} & h_{n-k+2} & \cdots & h_n \\ 1 & h_1 & h_2 & \cdots & h_k \\ 0 & 1 & h_1 & \cdots & h_{k-1} \\ 0 & 0 & 1 & \cdots & h_{k-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}$$

By expanding this determinant about the first column and using the result above for $s_{(1^n)}$ and a short induction argument we can see that this is equal to $h_{n-k}e_k - h_{n-k+1}e_{k-1} + h_{n-k+2}e_{k-2} - \cdots + (-1)^k h_n$.

There are literally dozens of formulas for the Schur functions which we could have taken as a starting point to study these elements of the ring of symmetric functions. We chose the Jacobi-Trudi formula because it seems to be the simplest formulation for these functions as elements in the algebra $\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots]$. The original definition of the Schur function is due to Jacobi and was defined as the ratio of alternating polynomials as we present in (7) below.

The Schur functions $\{s_\lambda\}$ are *the* fundamental basis for the symmetric functions just as the irreducible characters of the symmetric group are the fundamental basis for the class functions of the symmetric group. Our presentation of them will begin as a deluge of formulas for the Schur functions by showing that 16 formulations of the definition are all an equivalent way of defining the basis s_λ. In this way we not only show that all of these formulas hold, but also that any one of them can be taken as the definition and the other formulas follow.

Before we begin we should note that some of the following definitions are obviously equivalent to each other and that it is not difficult to show one from the other so perhaps it is unfair to say that we are presenting 16 different definitions. At the same time the object of the following theorem is to try to take some of the mystery out of the Schur functions and indicate why they are an important basis.

THEOREM 5.1. *(A plethora of Schur function definitions) The following definitions are equivalent.*

(1)
$$s_\mu = det|h_{\mu_i - i + j}|_{1\le i,j\le \ell(\mu)}$$

(2) *Let* $n \ge \ell(\mu)$
$$s_\mu[X] = \Omega[XY_n] \prod_{1\le i<j\le n} (1 - y_j/y_i)\Big|_{y^\mu}$$

(3) *Let* $S_m = \sum_{i\ge 0} (-1)^i h_{m+i} e_i^\perp$*, then*
$$s_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{\ell(\mu)}} 1$$

(4) *Let* $\tilde{S}_m = \sum_{i\ge 0} (-1)^i e_{m+i} h_i^\perp$,
$$s_\mu = \tilde{S}_{\mu'_1} \tilde{S}_{\mu'_2} \cdots \tilde{S}_{\mu'_{\ell(\mu')}} 1$$

(5) *Let* $n \ge \ell(\mu')$
$$s_\mu[X] = \Omega[-XY_n] \prod_{1\le i<j\le n} (1 - y_j/y_i)\Big|_{y^{\mu'}}$$

(6)
$$s_\mu = det|e_{\mu'_i - i + j}|_{1\le i,j\le \ell(\mu')}$$

(7)
$$s_\lambda[X_n] = \Delta_{\lambda+\delta_n}(X_n)/\Delta_{\delta_n}(X_n)$$
where $\Delta_{(a_1,a_2,\ldots,a_n)}(X_n) = det|x_i^{a_j}|$ *and* $\delta_n = (n-1, n-2, \ldots, 1, 0)$.

(8) $s_\mu[X_n] = J_n(x^\mu)$*, where* $J_n(f(x_1,\ldots,x_n)) = W_n(f(x_1,\ldots,x_n))/W_n(1)$ *and*
$$W_n(f(x_1,\ldots,x_n)) = \sum_{\sigma\in Sym_n} \epsilon(\sigma)\sigma(x^\delta f(x_1,\ldots,x_n))$$

(9)
$$s_\mu[X_n] = \sum_{\sigma\in Sym_n} \sigma\left(x^\mu \prod_{i<j} \frac{x_i}{x_i - x_j}\right)$$

(10) $\{s_\mu\}_{\mu\vdash n}$ *is the set of elements of* Λ_n *which satisfy the equations*
$$h_\mu = \sum_{\lambda\vdash n} K_{\lambda\mu} s_\lambda$$
where $K_{\mu\lambda}$ *is the number of column strict tableaux of shape* μ *and content* λ.

(11)

$$s_\mu = \sum_\lambda K_{\mu\lambda} m_\lambda$$

where $K_{\mu\lambda}$ is the number of column strict tableaux of shape μ and content λ.

(12)

$$s_\mu[X_n] = \sum_T \prod_{i=1}^n x_i^{n_i(T)}$$

where the sum is over all column strict tableaux of shape μ and content with labels 1 through n and $n_i(T)$ is the number of cells of T labeled with an i.

(13) $\{s_\mu\}_\mu$ *is the unique basis which satisfies the following two properties*
   (a) $s_\mu = m_\mu + \sum_{\lambda <_{lex} \mu} c_{\lambda\mu} m_\lambda$ *for some coefficients* $c_{\lambda\mu}$
   (b) $\langle s_\mu, s_\lambda \rangle = \delta_{\lambda\mu}$

(14) $\{s_\mu\}_\mu$ *is the unique basis which satisfies the following two properties*
   (a) $s_\mu = m_\mu + \sum_{\lambda <_{lex} \mu} c_{\lambda\mu} m_\lambda$ *for some coefficients* $c_{\lambda\mu}$
   (b) $h_\mu = s_\mu + \sum_{\lambda >_{lex} \mu} d_{\lambda\mu} s_\lambda$ *for some coefficients* $d_{\lambda\mu}$

(15) $\{s_\mu\}_\mu$ *is the unique basis which satisfies the following two properties*
   (a) $s_\mu = m_\mu + \sum_{\lambda <_{lex} \mu} c_{\lambda\mu} m_\lambda$ *for some coefficients* $c_{\lambda\mu}$
   (b) $s_\mu = h_\mu + \sum_{\lambda >_{lex} \mu} g_{\lambda\mu} h_\lambda$ *for some coefficients* $d_{\lambda\mu}$

(16) *Fix $n > 0$. Let M^n_{mh} be the matrix $[A_{\lambda\mu}]_{\lambda,\mu \in \mathcal{P}ar(n)}$ where the coefficients $A_{\lambda\mu}$ are given in Proposition 2.10 so that*

$$[h_{(n)}, h_{(n-1,1)}, \ldots, h_{(1^n)}] = [m_{(n)}, m_{(n-1,1)}, \ldots, m_{(1^n)}] M^n_{mh}$$

Let L^n_{mh} and U^n_{mh} be the matricies in the LU-decomposition of the matrix M^n_{mh}, that is, L^n_{mh} is a lower triangular matrix and U^n_{mh} is an upper triangular matrix such that $M^n_{mh} = L^n_{mh} U^n_{mh}$.

$$\begin{aligned}[s_{(n)}, s_{(n-1,1)}, \ldots, s_{(1^n)}] &:= [h_{(n)}, h_{(n-1,1)}, \ldots, h_{(1^n)}] \left(U^n_{mh}\right)^{-1} \\ &= [m_{(n)}, m_{(n-1,1)}, \ldots, m_{(1^n)}] L^n_{mh}.\end{aligned}$$

Remark 6. We said very little about the expansion of m_μ in the elementary basis in the last chapter, but exercise 2.(8) says that

$$e_{\lambda'} = m_\lambda + \text{ terms containing } m_\mu \text{ with } \mu < \lambda$$

and so we must also have

$$m_\lambda = e_{\lambda'} + \text{ terms containing } e_\mu \text{ with } \mu < \lambda'. \tag{5.2}$$

This means that definitions (13), (14), (15) and (16) could all have been stated with the elementary basis (indexed by the conjugate partition) instead of the monomial basis.

We will show this theorem by a series of implications showing certain definitions imply others. We will try to outline clearly our sequence of reasoning showing that these definitions are all equivalent. To show these equivalences, we require a few lemmas. These sub-results will be useful in any case to demonstrate other properties of the Schur symmetric functions.

For any $m \in \mathbb{Z}$, let $S_m = \sum_{i\geq 0}(-1)^i h_{m+i} e_i^\perp$ and $\tilde{S}_m = \sum_{i\geq 0}(-1)^i e_{m+i} h_i^\perp$ as given in the proposition. Although they these operators are defined as an infinite sum, when they act on any given symmetric function, they are a finite operation because if i is sufficiently large (larger than the degree of the symmetric function that it is acting on) then $e_i^\perp$ and $h_i^\perp$ will certainly act as 0.

The operators S_m are usually referred to as Bernstein operators because A. Zelevinsky attributes their discovery to Bernstein around 1980. The operators $\tilde{S}_m$ are related to S_m by $\omega S_m \omega = \tilde{S}_m$.

We define a generating function for these operators by the following formula. Let $P[X]$ be any symmetric function in the set of variables X, then set

$$
\begin{aligned}
S(z)P[X] &= P[X - 1/z]\Omega[zX] \\
&= \left(\sum_{i\geq 0}(-1/z)^i e_i^\perp P[X]\right)\left(\sum_{k\geq 0} z^k h_k[X]\right) \\
&= \sum_{m\in\mathbb{Z}} z^m \sum_{i\geq 0}(-1)^i h_{m+i} e_i^\perp P[X] \qquad (5.3)\\
&= \sum_{m\in\mathbb{Z}} z^m S_m P[X].
\end{aligned}
$$

Similarly we can compute

$$
\tilde{S}(z)P[X] = P[X + 1/z]\Omega[-zX] = \sum_{m\in\mathbb{Z}}(-1)^m z^m \tilde{S}_m P[X]. \qquad (5.4)
$$

This means that in order to compute $S_m P$, we can use this generating function as alternative notation and we may equally compute $S_m P[X] = P[X - 1/z]\Omega[zX]\Big|_{z^m}$. As well, we can compute $\tilde{S}_m P[X] = (-1)^m P[X + 1/z]\Omega[-zX]\Big|_{z^m}$.

Example 35. As an example we compute $S_2 S_2 1$ and $\tilde{S}_1 \tilde{S}_2 1$ to see how the generating function definition works. To begin, $S_2 1 = \Omega[zX]\Big|_{z^2} = h_2[X]$.

Now $S_2 S_2 1 = S_2(h_2[X]) = h_2[X - 1/z]\Omega[zX]\Big|_{z^2}$. Recall from Proposition 2.15 that we have $P[X - 1/z] = \sum_{k\geq 0}(-\frac{1}{z})^k e_k^\perp P[X]$ and in particular $e_k^\perp h_j = (\delta_{k1} h_{j-1} + \delta_{k0} h_j)$. Therefore

$$
S_2 S_2 1 = h_2[X]\Omega[zX]\Big|_{z^2} - \frac{1}{z} h_1[X]\Omega[zX]\Big|_{z^2} = h_2[X]^2 - h_1[X]h_3[X].
$$

Similarly we can compute $\tilde{S}_2 1 = \Omega[-zX]\Big|_{z^2} = e_2[X]$.

$\tilde{S}_1\tilde{S}_2 1 = \tilde{S}_1(e_2[X]) = e_2[X+1/z]\Omega[-zX]\Big|_{z^1}$. $P[X+1/z] = \sum_{k\geq 0}(\frac{1}{z})^k h_k^{\perp} P[X]$ and $e_2[X+z] = e_2[X] + ze_1[X]$, so

$$\tilde{S}_1\tilde{S}_2 1 = e_2[X]\Omega[-zX]\Big|_{z^1} + \frac{1}{z}e_1[X]\Omega[-zX]\Big|_{z^1} = -e_2[X]e_1[X] + e_1[X]e_2[X] = 0.$$

We will use these generating functions to develop formulas for the Schur functions. To show that these operations work as advertised we demonstrate the following commutation relations in the next few lemmas.

Lemma 5.2. *For any* $m, n \in \mathbb{Z}$,

$$S_m S_n = -S_{n-1}S_{m+1} \tag{5.5}$$

$$\tilde{S}_m\tilde{S}_n = -\tilde{S}_{n-1}\tilde{S}_{m+1} \tag{5.6}$$

In particular, $S_mS_{m+1} = \tilde{S}_m\tilde{S}_{m+1} = 0$ *for all* m.

Proof. The second equation of this lemma follows directly from the first since we have the relationship that $\omega S_m \omega = \tilde{S}_m$ and so we need only conjugate equation (5.5) by ω to arrive at equation (5.6). In addition, the last relation follows by setting $n = m+1$, then we have that $S_mS_{m+1} = -S_mS_{m+1} = 0$.

The proof of the first equation follows by computing a relationship between the generating functions $S(z)S(u)$ and $S(u)S(z)$. To find this formula we choose an arbitrary symmetric function $P[X]$ and act on this by $S(z)S(u)$. It follows that

$$\begin{aligned} S(z)S(u)P[X] &= S(z)P[X-1/u]\Omega[uX] \\ &= P[X-1/z-1/u]\Omega[u(X-1/z)]\Omega[zX] \\ &= P[X-1/u-1/z]\Omega[z(X-1/u)]\Omega[uX]\frac{1-u/z}{1-z/u} \\ &= S(u)S(z)P[X]\frac{1-u/z}{1-z/u} \end{aligned} \tag{5.7}$$

Note that $\frac{1-u/z}{1-z/u} = \frac{u(z-u)}{z(u-z)} = -\frac{u}{z}$. We may therefore conclude that $S(z)S(u) = -\frac{u}{z}S(u)S(z)$. Equation (5.5) follows by taking the coefficient of $z^m u^n$ in both sides of this equation. □

Lemma 5.3. *For any* $m, n \in \mathbb{Z}$,

$$S_m h_n = h_n S_m - h_{n-1}S_{m+1} \tag{5.8}$$

$$h_n S_m = \sum_{i=0}^{n} S_{m+i}h_{n-i} \tag{5.9}$$

$$\tilde{S}_m e_n = e_n\tilde{S}_m - e_{n-1}\tilde{S}_{m+1} \tag{5.10}$$

$$e_n\tilde{S}_m = \sum_{i=0}^{n} \tilde{S}_{m+i}e_{n-i} \tag{5.11}$$

PROOF. We need only show the first two of these formulas because the last two follow by a conjugation by the operator ω on both sides of the equations just as in the previous lemma.

These formulas follow by showing a relationship between $S(z)\Omega[uX]$ and $\Omega[uX]S(z)$ by acting again these two generating functions on a arbitrary symmetric function $P[X]$. We calculate that

$$\begin{aligned} S(z)\Omega[uX]P[X] &= \Omega[u(X-1/z)]P[X-1/z]\Omega[zX] \\ &= (1-u/z)\Omega[uX]S(z)P[X] \end{aligned} \tag{5.12}$$

Now if we take the coefficient of $z^m u^n$ in both sides of these equations we arrive at equation (5.8). But now by dividing both sides of equation (5.12) it follows that $\Omega[uX]S(z) = \frac{1}{1-u/z}S(z)\Omega[uX] = (1+u/z+(u/z)^2+(u/z)^3+\cdots)S(z)\Omega[uX]$. By taking the coefficient of $z^m u^n$ in both sides of this equation we arrive at (5.9). □

LEMMA 5.4. *For any* $m, n \in \mathbb{Z}$,

$$S_m e_n = \sum_{i=0}^{n} (-1)^i e_{n-i} S_{m+i} \tag{5.13}$$

$$e_n S_m = S_m e_n + S_{m+1} e_{n-1} \tag{5.14}$$

$$\tilde{S}_m h_n = \sum_{i=0}^{n} (-1)^i h_{n-i} \tilde{S}_{m+i} \tag{5.15}$$

$$h_n \tilde{S}_m = \tilde{S}_m h_n + \tilde{S}_{m+1} h_{n-1} \tag{5.16}$$

PROOF. Again we have that the last two equations follow from the first two since they are related by a conjugation by the operation ω. We will demonstrate the first two equations by arriving at a relation between $S(z)\Omega[-uX]$ and $\Omega[-uX]S(z)$.

$$\begin{aligned} S(z)\Omega[-uX]P[X] &= \Omega[-u(X-1/z)]P[X-1/z]\Omega[zX] \\ &= \Omega[-uX]S(z)P[X]\frac{1}{1-u/z} \\ &= \Omega[-uX]S(z)P[X](1+u/z+(u/z)^2+\cdots) \end{aligned} \tag{5.17}$$

Now if we take the coefficient of $z^m u^n$ in both sides of this equation (remembering that the coefficient of u^n in $\Omega[-uX]$ is $(-1)^n e_n$, then we arrive at equation (5.13). It also follows that $\Omega[-uX]S(z) = (1-u/z)S(z)\Omega[-uX]$ and again taking the coefficient of $z^m u^n$ in both sides of this equation we arrive at equation (5.14). □

LEMMA 5.5. *For* $m, n \in \mathbb{Z}$,

$$S_m \tilde{S}_n = \tilde{S}_{n+1} S_{m-1} + (-1)^n \delta_{m,-n}, \tag{5.18}$$

$$\tilde{S}_n S_m = S_{m+1} \tilde{S}_{n-1} + (-1)^n \delta_{m,-n}. \tag{5.19}$$

PROOF. This relation follows in the same way that we showed the previous commutation relations, by acting the generating functions for these operators on an arbitrary symmetric function, $P[X]$. We will skip a few steps and state that

$$S(z)\tilde{S}(u)P[X] = P[X + 1/u - 1/z]\Omega[zX]\Omega[-uX]\frac{1}{1 - u/z} \tag{5.20}$$

and

$$\tilde{S}(u)S(z)P[X] = P[X + 1/u - 1/z]\Omega[zX]\Omega[-uX]\frac{1}{1 - z/u}. \tag{5.21}$$

Recall from the phantom relation (Proposition 2.6), that we defined $\phi(z, u) = \sum_{k\in\mathbb{Z}} z^k u^{-k}$ and we see this generating function if we add $S(z)\tilde{S}(u)$ and $(z/u)\tilde{S}(u)S(z)$. We develop this expression and apply the phantom relation and the dual to the phantom relation $\phi(z, u)\Omega[zX]^{\perp}\Omega[-uX]^{\perp} = \phi(z, u)$.

$$\begin{aligned} S(z)\tilde{S}(u)P[X]+&(z/u)\tilde{S}(u)S(z)P[X] \\ &= P[X + 1/u - 1/z]\Omega[zX]\Omega[-uX]\phi(z, u) \\ &= P[X]\phi(z, u). \end{aligned} \tag{5.22}$$

Now take the coefficient of $z^m u^n$ on each side of this equation. On the right hand side the answer is simply 0 unless $m = -n$, and then the answer is $P[X]$. On the left hand side of this generating function, the coefficient of $z^m u^n$ is $(-1)^n S_m\tilde{S}_n P[X]+(-1)^{n+1}\tilde{S}_{n+1}S_{m-1}P[X]$. If we multiply each side of the equation by the appropriate power of (-1) we arrive at the relation stated in the proposition. ☐

Next we consider polynomials $f(x_1, x_2, \dots, x_n)$ with the property that $\sigma f(x_1, x_2, \dots, x_n) = \epsilon(\sigma)f(x_1, x_2, \dots, x_n)$. These are called alternating polynomials and notice that it is sufficient to show that for the basic transpositions $s_i f(x_1, x_2, \dots, x_n) = -f(x_1, x_2, \dots, x_n)$. We will show that alternating polynomials are always a symmetric polynomial times the special element $\Delta_{\delta_n}(x_1, x_2, \dots, x_n) = det|x_j^{n-i}|_{1\le i,j\le n}$.

LEMMA 5.6. *If $f(x_1, x_2, \dots, x_n)$ is an alternating polynomial then it is divisible by $\prod_{1\le i<j\le n}(x_i - x_j)$.*

PROOF. For any polynomial $p(x)$ if $p(a) = 0$, then $p(x)$ is divisible by $(x - a)$. This is a well known fact about polynomials, but it follows for the reason that the ring of polynomials forms a Euclidean domain. For any polynomials $p(x)$ and $f(x)$ we know that there exists polynomials $q(x)$ and $r(x)$ such that $p(x) = f(x)q(x) + r(x)$ and $deg(r(x)) < deg(f(x))$. Therefore $p(x) = (x - a)q(x) + r(x)$ where $deg(r(x)) < deg(x - a) = 1$. Therefore $r(x)$ is a constant and by evaluating this polynomial at $x = a$, we see that $r(x) = p(a)$. Since $p(a) = 0$ then $p(x) = (x - a)q(x)$ and hence $p(x)$ is divisible by $x - a$.

Let $f(x_1, x_2, \dots, x_n)$ be an alternating polynomial. Since

$$f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = -f(x_1, \dots, x_j, \dots, x_i, \dots, x_n),$$

if $x_j = x_i$ then

$$f(x_1, \dots, x_i, \dots, x_i, \dots, x_n) = -f(x_1, \dots, x_i, \dots, x_i, \dots, x_n) = 0.$$

From this we conclude that $f(x_1, x_2, \ldots, x_n)$ is divisible by $x_j - x_i$ for any i and j. If $i < j$ and $k < \ell$ then $x_i - x_j$ does not divide $x_k - x_\ell$ but it does divide $f(x_1, x_2, \ldots, x_n)$, therefore $\prod_{i<j}(x_i - x_j)$ divides $f(x_1, x_2, \ldots, x_n)$. ☐

Recall that interchanging two rows of a matrix has the effect of changing the sign of the determinant of the matrix. This means that a matrix such as $\Gamma_{(a_1,a_2,\ldots,a_n)} = [x_j^{a_i}]_{1\le i,j\le n}$ which has the property that the permutation of the variables in the entries also permutes the columns of the matrix, then the determinant of the matrix will be an alternating polynomial (and hence by the last lemma will be divisible by $\prod_{1\le i<j\le n}(x_i - x_j)$). For any sequence $(a_1, a_2, \ldots, a_n)$ with $a_i \ge 0$, we set $\Delta_{(a_1,a_2,\ldots,a_n)}(x_1, x_2, \ldots, x_n) = det\left(\Gamma_{(a_1,a_2,\ldots,a_n)}\right)$.

Corollary 5.7. *(the Vandermonde determinant) For $n \ge 0$, let $\delta_n = (n-1, n-2, \ldots, 1, 0)$, then*

$$\Delta_{\delta_n}(x_1, x_2, \ldots, x_n) := det \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots & \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{1\le i<j\le n}(x_i - x_j)$$

Proof. Note that $\Delta_{\delta_n}(x_1, x_2, \ldots, x_n) = det|x_j^{n-i}|_{1\le i,j\le n}$ is a polynomial of degree $\binom{n}{2}$ and by the previous remark it is alternating and therefore divisible by $\prod_{1\le i<j\le n}(x_i - x_j)$ by the previous Lemma. Since $\prod_{1\le i<j\le n}(x_i - x_j)$ is also of degree $\binom{n}{2}$, this means that $\Delta_{\delta_n}(x_1, x_2, \ldots, x_n) = c\prod_{1\le i<j\le n}(x_i - x_j)$ for some constant c. It remains to show that $c = 1$, but this is easy to do since we clearly have that the coefficient of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ is 1 in both $\Delta_{\delta_n}(x_1, x_2, \ldots, x_n)$ and $\prod_{1\le i<j\le n}(x_i - x_j)$. ☐

Corollary 5.8. *Again we let $\delta_n = (n-1, n-2, \ldots, 1, 0)$, then*

$$\Delta_{\delta_n}(X_n)\Delta_{\delta_n}(Y_n) = \Omega[-X_nY_n]det\left|\frac{1}{1 - x_iy_j}\right|_{1\le i,j\le n}. \tag{5.23}$$

Proof. $\Delta_{\delta_n}(X_n)\Delta_{\delta_n}(Y_n)$ is a polynomial of degree $2\binom{n}{2}$ and since $\Omega[-X_nY_n] = \prod_{i,j=1}^n(1 - x_iy_j)$, the right hand side is also a polynomial of degree $n^2 - n$. $\Omega[-X_nY_n]det\left|\frac{1}{1-x_iy_j}\right|_{1\le i,j\le n}$ is alternating in the x variables and hence is divisible by $\Delta_{\delta_n}(X_n)$ and in the y variables and hence is divisible by $\Delta_{\delta_n}(Y_n)$. Since the degree is the same as $\Delta_{\delta_n}(X_n)\Delta_{\delta_n}(Y_n)$ these polynomials must be equal up to a constant factor. It is slightly more difficult than in the previous proposition to justify that the constant factor equals 1, but we need only examine the coefficient of one monomial, namely that of $x^{\delta_n}y^{\delta_n}$, and show that it is the same on both sides of the equation.

On the left hand side of (5.23) this coefficient is easily seen to be 1. On the right hand side of (5.23), we expand this expression

$$\Omega[-X_nY_n]det\left|\frac{1}{1-x_iy_j}\right|_{1\le i,j\le n} = \sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\ne\sigma(i)}}^{n}(1-x_iy_j).$$

In order to have a monomial in this sum with x_1^{n-1} so that y_n does not occur, it must be that it comes from the product, $(x_1y_1)(x_1y_2)\cdots(x_1y_{n-1})$ and $\sigma(1)=n$. Since x_2 occurs with power $n-2$, the only way that this can happen and have the correct powers of y_i is from the product $(x_2y_1)(x_2y_2)\cdots(x_2y_{n-2})$ and so $\sigma(2)=n-1$. This continues and it is easy to see that $x^{\delta_n}y^{\delta_n}$ only appears as a monomial in the term which has $\sigma(i)=n-i+1$. The sign of that terms will be $\epsilon(\sigma)(-1)^{\binom{n}{2}}$ and since σ itself has length $\binom{n}{2}$, this coefficient is equal to 1. Therefore (5.23) holds as stated. □

PROPOSITION 5.9. *For $m\in\mathbb{Z}$,*

$$\langle S_mf,g\rangle = \left\langle f,(-1)^m\tilde{S}_{-m}g\right\rangle \tag{5.24}$$

PROOF.

$$\begin{aligned}\langle S_mf,g\rangle &= \left\langle \sum_{i\ge0}(-1)^ih_{m+i}e_i^\perp f,g\right\rangle\\ &= \sum_{i\ge0}\left\langle f,(-1)^ie_ih_{m+i}^\perp g\right\rangle\\ &= \left\langle f,\sum_{i\ge m}(-1)^{i-m}e_{i-m}h_i^\perp g\right\rangle\\ &= \left\langle f,(-1)^m\sum_{i\ge0}(-1)^ie_{-m+i}h_i^\perp g\right\rangle\\ &= \left\langle f,(-1)^m\tilde{S}_{-m}g\right\rangle\end{aligned} \tag{5.25}$$

□

PROPOSITION 5.10.

$$h_n^\perp\tilde{S}_m = \tilde{S}_mh_n^\perp+\tilde{S}_{m-1}h_{n-1}^\perp \tag{5.26}$$

$$h_n^\perp S_m = \sum_{i=0}^{n}(-1)^iS_{m-i}h_{n-i}^\perp \tag{5.27}$$

$$e_n^\perp S_m = S_me_n^\perp+S_{m-1}e_{n-1}^\perp \tag{5.28}$$

$$e_n^\perp\tilde{S}_m = \sum_{i=0}^{n}(-1)^i\tilde{S}_{m-i}e_{n-i}^\perp \tag{5.29}$$

Proof. We can show these commutation relation in the same way that we showed Propositions 5.2, 5.3 and 5.4, by computing the commutation relation of the generating series $H^{\perp}(u)P[X] = P[X+u] = \sum_{i\geq 0} u^i h_i^{\perp} P[X]$ and $E^{\perp}(u)P[X] = P[X-u] = \sum_{i\geq 0}(-u)^i e_i^{\perp} P[X]$ with $S(z)$ and $\tilde{S}(z)$. Alternatively we will use the last proposition and calculate that

$$\begin{aligned} \left\langle h_n^{\perp}\tilde{S}_m f, g\right\rangle &= \left\langle \tilde{S}_m f, h_n g\right\rangle \\ &= \langle f, (-1)^m S_{-m} h_n g\rangle \\ &= \langle f, (-1)^m (h_n S_{-m} g - h_{n-1} S_{-m+1} g)\rangle \\ &= \left\langle \tilde{S}_m h_n^{\perp} f, g\right\rangle + \left\langle \tilde{S}_{m-1} h_{n-1}^{\perp} f, g\right\rangle \end{aligned} \tag{5.30}$$

Therefore since this relationship holds for all symmetric functions g, it is true that $h_n^{\perp}\tilde{S}_m f = \tilde{S}_m h_n^{\perp} f + \tilde{S}_{m-1} h_{n-1}^{\perp} f$ and this shows the first equation. All of the other commutation relations in this proposition can be shown by similar means and we leave them as an exercise to the reader. □

Proof. (of Theorem 5.1)

We are now ready to prove that the definitions listed in Theorem 5.1 are all equivalent. We will start by showing that the first six are equal in that order. After that we will take a break, prove some related results and continue to show that the other results are equivalent.

Proof of (1) ⇔ (2) :

We compute the following expression for the product $\prod_{1\leq i<j\leq n}(1-y_j/y_i)$ using Corollary 5.7.

$$\begin{aligned} \prod_{1\leq i<j\leq n}(1-y_j/y_i) &= \Delta_{\delta_n}(y_1,y_2,\ldots,y_n) y^{-\delta_n} \\ &= det\left|y_j^{n-i}\right|_{1\leq i,j\leq n} y^{-\delta_n} \\ &= \sum_{\sigma\in Sym_n} \epsilon(\sigma)\prod_{i=1}^{n} y_i^{-(n-i)+(n-\sigma(i))} \end{aligned} \tag{5.31}$$

Now the expression for (2) is the same as (1) by the following computation.

$$\begin{aligned} \Omega[XY_n]\prod_{1\leq i<j\leq n}(1-y_j/y_i)\Big|_{y^{\mu}} &= \Omega[XY_n]\sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n} y_i^{i-\sigma(i)}\Big|_{y^{\mu}} \\ &= \sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n} h_{\mu_i-i+\sigma(i)}[X] \\ &= det|h_{\mu_i-i+j}[X]|_{1\leq i,j\leq n} \end{aligned} \tag{5.32}$$

implying that the definitions are equivalent.

Proof of (2) $\Leftrightarrow$ (3) :

We know that $S_m = \sum_{i\geq 0}(-1)^i h_{m+i} e_i^\perp$, but we will be using equation (5.3). Assume by induction that all partitions with $\ell(\mu) \leq k \leq n$ satisfy

$$s_\mu[X] = \Omega[XY_n] \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{y^\mu} \tag{5.33}$$

For a partition with $\ell(\mu) = k$, we calculate

$$\begin{aligned} S_m(s_\mu[X]) &= S_m\left(\Omega[XY_n] \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{y^\mu}\right) \\ &= \Omega\left[\left(X - \frac{1}{z}\right) Y_n\right] \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{y^\mu} \Omega[zX]\Big|_{z^m} \\ &= \Omega[XY_n]\Omega[zX]\Omega\left[-\frac{1}{z}Y_n\right] \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{z^m y^\mu} \\ &= \Omega[X(z + Y_n)] \prod_{i=1}^{n}(1 - y_i/z) \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{z^m y^\mu} \end{aligned} \tag{5.34}$$

This last expression is exactly the formula for the symmetric function $s_{(m,\mu_1,\ldots,\mu_k)}[X]$ using the alphabet $z + Y_n$ in definition (2). Therefore we know by induction that

$$S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k} 1 = \Omega[XY_n] \prod_{1\leq i\leq j\leq n} (1 - y_j/y_i)\Big|_{y^\mu} \tag{5.35}$$

for all partitions λ.

Remark 7. We have not so far assumed anywhere that μ is a partition in these proofs and the first three definitions make sense for μ equal to any sequence of integers. It is the case the Schur function may be defined for any sequence of integers, but to obtain a basis we examine only the set of Schur functions which are indexed by partitions. Notice however that relation (5.5) can be used to 'straighten' any sequence which is not a partition to one which is in the following sense:

For any sequence $(a_1, a_2, \cdots, a_k)$, if $a_i \geq a_{i+1}$ for all $1 \leq i \leq k-1$, then $s_{(a_1,a_2,\cdots,a_k)}$ is a Schur function indexed by a partition. If not then find the smallest i for which $a_i < a_{i+1}$. If $a_i + 1 = a_{i+1}$, then $s_{(a_1,a_2,\cdots,a_k)} = S_{a_1} \cdots S_{a_i} S_{a_i+1} \cdots S_{a_k} 1 = -S_{a_1} \cdots S_{a_i} S_{a_i+1} \cdots S_{a_k} 1 = 0$. If $a_{i+1} > a_i + 1$, then $s_{(a_1,a_2,\cdots,a_k)} = -s_{(a_1,a_2,\cdots,a_{i+1}-1,a_i+1,\ldots,a_k)}$ and now at least the first i entries of this sequence are in order. In this way we have defined a bubble sort method for 'straightening' a sequence indexing a Schur function into a partition indexing a Schur function.

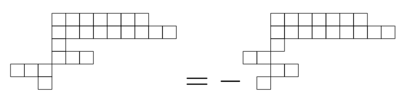
Example 36. The Schur function indexed by the sequence $(2, 1, 5, 3, 1, 1)$ is equal to 0 because $S_2S_1S_5S_3S_1S_11 = -S_2S_4S_2S_3S_1S_11 = 0$ since $S_2S_3 = 0$. The sequence need not consist of positive integers either, for instance the Schur function indexed by the sequence

$(-1,-3,3,1,9,7)$ is equal to

$$\begin{aligned} S_{-1}S_{-3}S_3S_1S_9S_71 &= -S_{-1}S_2S_{-2}S_1S_9S_71 = S_1S_0S_{-2}S_1S_9S_71 \\ &= -S_1S_0S_0S_{-1}S_9S_71 = S_1S_0S_0S_8S_0S_71 \\ &= \ldots = -S_5S_2S_1S_1S_0S_71 \\ &= S_5S_2S_1S_1S_6S_11 = S_5S_2S_1S_5S_2S_11 \\ &= \ldots = S_5S_3S_3S_2S_2S_11 \end{aligned} \tag{5.36}$$

Each of these sequences may be represented by an image, one row of length a_i for a sequence $(a_1, a_2, \ldots, a_{\ell(a)})$. This gives the algebraic formula a combinatorial interpretation by recognizing that the each exchange in the sequence has the effect of swapping two rows increasing the length of one of these rows by 1 and decreases the length of the other row by 1.

Each operator in the expression above can be represented by a row of cells so that when the operators are applied in decreasing order the image will be a Young diagram for a partition. Each exchange of operators represents a corresponding change in the diagram and Lemma 5.2 defines an equivalence class of signed diagrams. For example the first exchange of operators in equation (5.36) above is represented by the following diagram.



There is another way that one can compute the Schur functions indexed by a sequence of integers. Notice that for each of the generators of the symmetric group τ_i for $1 \leq i < n$ where $\tau_i = (i\; i+1)$ acts on the sequences indexing a Schur function. $(\tau_i(a_1+k-1, a_2+k-2, \ldots, a_k)) - (k-1, k-2, \ldots, a_k) = (a_1+k-1, \ldots, a_{i+1}+k-i-1, a_i+k-i, \ldots, a_k) - (k-1, \ldots, k-i, k-i-1, \ldots, a_k) = (a_1, \ldots, a_{i+1}-1, a_i+1, \ldots, a_k)$. This means that we can define an action of the symmetric group generators on the sequences indexing the symmetric functions. That is,

$$s_{(a_1,a_2,\ldots,a_k)} = -s_{(a_1,\ldots,a_{i+1}-1,a_i+1,\ldots,a_k)} = -s_{\tau_i(a+\delta_k)-\delta_k}. \tag{5.37}$$

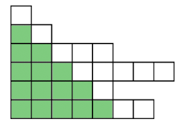
Since this formula holds for the generators of the symmetric group, then it must be that

$$\begin{aligned} s_{(a_1,a_2,\ldots,a_k)} &= \epsilon(\sigma) s_{\sigma(a+\delta_k)-\delta_k} \\ &= \epsilon(\sigma) s_{(a_{\sigma(1)}-\sigma(1)+1, a_{\sigma(2)}-\sigma(2)+2, \ldots, a_{\sigma(k)}-\sigma(k)+k)} \end{aligned} \tag{5.38}$$

for any permutation $\sigma \in Sym_k$.

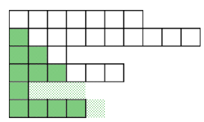
This is a very useful and combinatorial method for computing the Schur function indexed by a sequence. The formula says that if we add a staircase to the indexing sequence and two of the parts of the result have the same length, then the Schur function is 0. If the result of adding a staircase shape to the partition has distinct parts, then there is a unique permutation which sorts the sequence to a strict partition. The Schur function indexed by this sequence is equal (up to a sign) to the Schur function indexed by the partition of this sorted sequence minus the staircase shape. This result may again be 0 if this staircase shape has parts of negative length.

EXAMPLE 37. For example, the sequence $(2, 1, 5, 3, 1, 1)$ plus the staircase shape $(5, 4, 3, 2, 1, 0)$ can be can be seen visually by the diagram below.

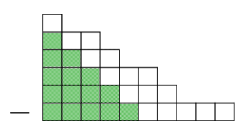


Since this diagram has two rows that are the same length, the Schur function $s_{(2,1,5,3,1,1)} = 0$.

EXAMPLE 38. The Schur function indexed by the sequence $(-1, -3, 3, 1, 9, 7)$ can also be calculated by adding the sequence $(5, 4, 3, 2, 1, 0)$ which we represent by the following diagram.



Since all of the rows have different lengths, there is a permutation that sorts the rows in decreasing order and this has been drawn below, this time with the staircase which now fits below the partition. The permutation that sorts the sequence $(4, 1, 6, 3, 10, 7)$ to $(10, 7, 6, 4, 3, 1)$ this has length 11 and the following diagram shows that $s_{(-1,-3,3,1,9,7)} = -s_{(5,3,3,2,2,1)}$.



Proof of $(3) \Leftrightarrow (4)$:

We must assume that since the operation of conjugation of a partition only makes sense for partitions (and not for a sequence of integers) that this equivalence will only hold for partitions. We can assume by induction that for a partition of length k that we know

$$S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k} 1 = \tilde{S}_{\mu'_1} \tilde{S}_{\mu'_2} \cdots \tilde{S}_{\mu'_{\mu_1}} 1 \tag{5.39}$$

Consider an $m \geq \mu_1$. We know that

$$S_m S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k} 1 = S_m \tilde{S}_{\mu'_1} \tilde{S}_{\mu'_2} \cdots \tilde{S}_{\mu'_{\mu_1}} \tilde{S}_0 \cdots \tilde{S}_0 1 \tag{5.40}$$

where there are $m - \mu_1$ copies of $\tilde{S}_0$ on this trailing composition of operators. We may place as many as we want since $\tilde{S}_0 1 = 1$. Now by applying Lemma 5.5 this is equal to

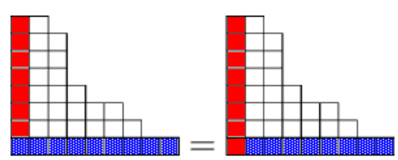
$$\begin{aligned}
&= \tilde{S}_{\mu'_1+1} S_{m-1} \tilde{S}_{\mu'_2} \cdots \tilde{S}_{\mu'_{\mu_1}} \tilde{S}_0 \cdots \tilde{S}_0 1 \\
&= \tilde{S}_{\mu'_1+1} \tilde{S}_{\mu'_2+1} S_{m-2} \cdots \tilde{S}_{\mu'_{\mu_1}} \tilde{S}_0 \cdots \tilde{S}_0 1 \\
&= \quad \vdots \\
&= \tilde{S}_{\mu'_1+1} \tilde{S}_{\mu'_2+1} \cdots \tilde{S}_{\mu'_{\mu_1}+1} \tilde{S}_1 \cdots \tilde{S}_1 S_0 1 \\
&= \tilde{S}_{\mu'_1+1} \tilde{S}_{\mu'_2+1} \cdots \tilde{S}_{\mu'_{\mu_1}+1} \tilde{S}_1 \cdots \tilde{S}_1 1
\end{aligned} \tag{5.41}$$

and $(\mu'_1+1, \mu'_2+1, \ldots, \mu'_{\mu_1}+1, 1, \ldots, 1)$ is equal to the partition μ' with a column of length m placed on the left of the partition.

This shows that for any partition μ,

$$\tilde{S}_{\mu'_1} \tilde{S}_{\mu'_2} \cdots \tilde{S}_{\mu'_{\mu_1}} 1 = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{\ell(\mu)}} 1. \tag{5.42}$$

Remark 8. The commutation relation in Lemma 5.5 has a clear combinatorial interpretation at this point. We have shown that adding a column of length n followed by a row of length m on a partition is the same as adding a row of length $m-1$ followed by a column of length $n+1$. Perhaps this is best seen in the following picture to see that if S_m and $\tilde{S}_n$ are operators which add rows and columns to the partitions, then of course they must satisfy this relation.



We should note that the algebraic formula which we have proven is even a stronger statement than the picture suggests because the formula says that the commutation relation holds between these operators independent of what they are acting on while our picture really only makes sense when the red and blue cells surround a partition μ with $\mu_1 \leq m$ and $\ell(\mu) \leq n$.

Proof of (4) $\Leftrightarrow$ (5) :

Recall that $\tilde{S}_m P[X] = (-1)^m P[X + 1/z]\Omega[zX]\Big|_{z^m}$. We will show by induction on the length of the sequence μ (which again we do not need to assume is a partition) that

$$\tilde{S}_{\mu_1} \tilde{S}_{\mu_2} \cdots \tilde{S}_{\mu_{\ell(\mu)}} 1 = (-1)^{|\mu|} \Omega[-XZ_n] \prod_{1 \leq i < j \leq n} (1 - z_i/z_j)\Big|_{z^\mu} \tag{5.43}$$

Assume that equation (5.43) holds for all sequences μ such that $\ell(\mu) \leq k \leq n$. Then

$$\begin{aligned}
&\tilde{S}_m \tilde{S}_{\mu_1} \tilde{S}_{\mu_2} \cdots \tilde{S}_{\mu_{\ell(\mu)}} 1 \\
&\quad = (-1)^{|\mu|} \tilde{S}_m \Omega[-Z_n X] \prod_{1 \leq i < j \leq n} (1 - z_i/z_j) \Big|_{z^\mu} \\
&\quad = (-1)^{|\mu|+m} \Omega[-(X + 1/u) Z_n] \Omega[-uX] \prod_{1 \leq i < j \leq n} (1 - z_i/z_j) \Big|_{z^\mu} \Big|_{u^m} \qquad (5.44) \\
&\quad = (-1)^{|\mu|+m} \Omega[-(u + Z_n) X] \prod_{i=1}^{n} (1 - z_i/u) \prod_{1 \leq i < j \leq n} (1 - z_i/z_j) \Big|_{u^m z^\mu}.
\end{aligned}$$

This is exactly formula (5.43) in the alphabet consisting of the variables $u, z_1, z_2, \ldots, z_n$ and the sequence $(m, \mu_1, \mu_2, \ldots, \mu_n)$. The base case for this induction argument is a sequence of length 0 and the formula holds trivially, hence it is true for all sequences.

Proof of (5) $\Leftrightarrow$ (6) :

The proof for this equivalence proceeds much as it did in the proof of (1) $\Leftrightarrow$ (2).

We have from equation (5.31),

$$\prod_{1 \leq i < j \leq n} (1 - y_j/y_i) = \sum_{\sigma \in Sym_n} \epsilon(\sigma) \prod_{i=1}^{n} y_i^{i - \sigma(i)}. \qquad (5.45)$$

Now showing that these two definitions are equal proceeds as we did in the nearly identical proof of (2) $\Leftrightarrow$ (3).

$$\begin{aligned}
\Omega[-XY_n] \prod_{1 \leq i < j \leq n} (1 - y_j/y_i) \Big|_{y^\mu} &= \sum_{\sigma \in Sym_n} \epsilon(\sigma) \Omega[-XY_n] \prod_{i=1}^{n} y_i^{i-\sigma(i)} \Big|_{y^\mu} \\
&= \sum_{\sigma \in Sym_n} \epsilon(\sigma) \prod_{i=1}^{n} (-1)^{\mu_i - i + \sigma(i)} e_{\mu_i - i + \sigma(i)}[X] \qquad (5.46) \\
&= (-1)^{|\mu|} det \, |e_{\mu_i - i + j}[X]|_{1 \leq i,j \leq n}
\end{aligned}$$

Therefore the definitions are equivalent.

Proof of (2) $\Leftrightarrow$ (7)

We will show that $\Delta_{\delta_n}(X_n) s_\mu[X] = \Delta_{\mu + \delta_n}(X_n)$ by direct calculation. Choose an n such that $n \geq \ell(\mu)$. Recall from Corollary 5.8 we have a formula for $\Delta_{\delta_n}(X_n) \Delta_{\delta_n}(Y_n) \Omega[X_n Y_n]$.

$$\begin{aligned}\Delta_{\delta_n}(X_n)s_\mu[X_n] &= \Delta_{\delta_n}(X_n)\Omega[X_nY_n]\prod_{1\le i\le j\le n}(1-y_j/y_i)\Big|_{y^\mu} \\ &= \Delta_{\delta_n}(X_n)\Omega[X_nY_n]y^{-\delta_n}\prod_{1\le i\le j\le n}(y_i-y_j)\Big|_{y^\mu} \\ &= \Delta_{\delta_n}(X_n)\Delta_{\delta_n}(Y_n)\Omega[X_nY_n]\Big|_{y^{\mu+\delta_n}} \\ &= det\left|\frac{1}{1-x_iy_j}\right|_{1\le i,j\le n}\Big|_{y^{\mu+\delta_n}}.\end{aligned} \tag{5.47}$$

Now expand the determinant in the last line of this calculation where we arrive at the following expression.

$$\begin{aligned} &= \left(\sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^n\frac{1}{1-x_{\sigma(i)}y_i}\right)\Big|_{y^{\mu+\delta_n}} \\ &= \sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^n\left(\sum_{k\ge 0}(x_{\sigma(i)}y_i)^k\Big|_{y_i^{\mu_i+n-i}}\right) \\ &= \sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^n x_{\sigma(i)}^{\mu_i+n-i} \\ &= \Delta_{\mu+\delta_n}(X_n)\end{aligned} \tag{5.48}$$

Therefore $s_\mu[X_n]\Delta(X_n) = \Delta_{\mu+\delta_n}(X_n)$.

Remark 9. Definitions (7), (8) and (9) are three ways of writing the same formula and there is little to prove that these are equivalent.

Proof of (7) $\Leftrightarrow$ (8)

Note that

$$W_n(x^\mu) = \sum_{\sigma\in Sym_n}\epsilon(\sigma)\sigma(x^{\mu+\delta_n}) = \Delta_{\mu+\delta_n}(x_1,x_2,\ldots,x_n) \tag{5.49}$$

and in particular $W_n(1) = \Delta_{\delta_n}(x_1,x_2,\ldots,x_n)$. Therefore $J(x^\mu) = \Delta_{\mu+\delta_n}(X_n)/\Delta_{\delta_n}(X_n) = s_\mu[X_n]$.

Proof of (7) $\Leftrightarrow$ (9)

$\Delta_{\delta_n}(x_1, x_2, \ldots, x_n) = \prod_{1\leq i<j\leq n}(x_i - x_j)$ and hence has the property that $\sigma(\prod_{1\leq i<j\leq n}(x_i - x_j)) = \epsilon(\sigma)\prod_{1\leq i<j\leq n}(x_i - x_j)$ for all permutations σ. This implies that

$$\begin{aligned}
\sum_{\sigma\in Sym_n} \sigma\left(x^\mu \prod_{1\leq i<j\leq n} \frac{x_i}{x_i - x_j}\right) &= \sum_{\sigma\in Sym_n} \sigma\left(x^\mu \frac{x_n^\delta}{\Delta_{\delta_n}(x_1, x_2, \ldots, x_n)}\right)\\
&= \frac{1}{\Delta_{\delta_n}(x_1, x_2, \ldots, x_n)} \sum_{\sigma\in Sym_n} \epsilon(\sigma)\sigma\left(x^{\delta_n+\mu}\right) \qquad (5.50)\\
&= \frac{\Delta_{\delta_n+\mu}(x_1, x_2, \ldots, x_n)}{\Delta_{\delta_n}(x_1, x_2, \ldots, x_n)}
\end{aligned}$$

Remark 10. So far we have shown that the numbered definitions (1) through (9) listed in the theorem are equivalent because the expressions for the basis that they produce are the same. Another way that we can show that two definitions are equivalent is to show that one of the definitions consists of conditions which uniquely define a basis of the symmetric functions and that the other definition defines a basis which satisfies these conditions. It follows that these definitions will be equivalent.

Now that we have shown that definitions (1) through (9) are equivalent, we will demonstrate the following product rule.

Proposition 5.11. *(The Pieri Rule)*

$$h_m s_\lambda = \sum_\mu s_\mu \qquad (5.51)$$

where the sum is over all partitions $\mu \vdash |\lambda| + m$ which contain the partition λ and there is at most one cell for each column in μ/λ (alternatively, $0 \leq \mu'_i - \lambda'_i \leq 1$ for all i). We say that the partitions in this sum are those such that μ/λ is a horizonal m-strip (which we will denote as $\mu/\lambda \in \mathcal{H}_m$).

Proof. What we will show is that defintion (4) and the commutation relations in Propostion 5.4 are enough to demonstrate this result. Recall that equation (5.16) says that

$$h_n\tilde{S}_m = \tilde{S}_m h_n + \tilde{S}_{m+1}h_{n-1} \qquad (5.52)$$

so that

$$\begin{aligned}
h_n s_\lambda &= h_n\tilde{S}_{\lambda'_1}\tilde{S}_{\lambda'_2}\cdots\tilde{S}_{\lambda'_{\lambda_1}}1\\
&= \sum_\delta \tilde{S}_{\lambda'_1+\delta_1}\tilde{S}_{\lambda'_2+\delta_2}\cdots\tilde{S}_{\lambda'_{\lambda_1}+\delta_{\lambda_1}}h_{n-|\delta|} \qquad (5.53)\\
&= \sum_\delta \tilde{S}_{\lambda'_1+\delta_1}\tilde{S}_{\lambda'_2+\delta_2}\cdots\tilde{S}_{\lambda'_{\lambda_1}+\delta_{\lambda_1}}(\tilde{S}_1)^{n-|\delta|}1
\end{aligned}$$

where the sum here is over sequences δ of length λ_1 of 0s and 1s. The terms that have $\lambda'_i = \lambda'_{i+1}$ and $\delta_i = 0$ and $\delta_{i+1} = 1$ vanish because in that case part of this composition contains $\tilde{S}_{\lambda'_i}\tilde{S}_{\lambda'_i+1} = 0$. All other terms correspond to partitions μ which contain λ and have at most one cell in each column of μ/λ. □

By an application of the involution ω we can show the following corollary, that the product of e_m and s_λ is the conjugate of the result for h_m times $s_{\lambda'}$. Alternatively we can show that defintion (3) and the commutation relation between e_m and S_n can be used to show the following conjugate Pieri rule.

Corollary 5.12. *(Conjugate Pieri rule)*

$$e_m s_\lambda = \sum_\mu s_\mu \tag{5.54}$$

where the sum is over all partitions $\mu \vdash |\lambda| + m$ *which contain the partition* λ *and there is at most one cell for each row in* μ/λ *(alternatively,* $0 \leq \mu_i - \lambda_i \leq 1$ *for all* i*). We say that the partitions in this sum are those such that* μ/λ *is a vertical* m*-strip and denote this by* $\mu/\lambda \in \mathcal{V}_m$*.*

We leave the following two additional results as exercises. They can be proven in the same manner that we used to show the Pieri rule using the commutation relations between $h_m^\perp$ and $\tilde{S}_n$ (alternatively $e_m^\perp$ and S_n).

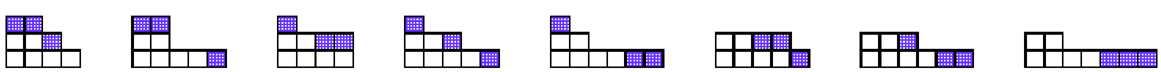
Proposition 5.13. *(Dual Pieri rule)*

$$h_m^\perp s_\lambda = \sum_{\mu:\lambda/\mu\in\mathcal{H}_m} s_\mu \tag{5.55}$$

Corollary 5.14. *(Dual conjugate Pieri rule)*

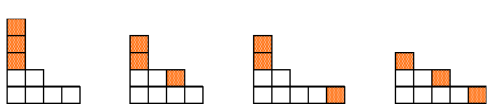
$$e_m^\perp s_\lambda = \sum_{\mu:\lambda/\mu\in\mathcal{V}_m} s_\mu \tag{5.56}$$

Example 39. The product of h_3 with the Schur function $s_{(4,2)}$ will be the sum of Schur functions indexed by set of partitions of size 9 that are in the following diagrams.



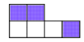
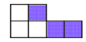
This means that $h_3 s_{(4,2)} = s_{(4,3,2)} + s_{(5,2,2)} + s_{(4,4,1)} + s_{(5,3,1)} + s_{(6,2,1)} + s_{(5,4)} + s_{(6,3)} + s_{(7,2)}$.

By contrast, if we multiply e_3 by $s_{(4,2)}$ there will be one Schur function in the result for each partition in the following diagrams.



Therefore we have that $e_3 s_{(4,2)} = s_{(4,2,1,1,1)} + s_{(4,3,1,1)} + s_{(5,2,1,1)} + s_{(5,3,1)}$.

In order to calculate $h_3^\perp s_{(4,2)}$, there will be one term in the expression for each of the following two partitions.

Therefore we have that $h_3^\perp s_{(4,2)} = s_{(2,1)} + s_{(3)}$.

Note that $e_3^\perp s_{(4,2)} = 0$ since there is no way of removing a vertical strip of size 3 from a partition which has only height 2. For this reason $e_m^\perp s_\lambda = 0$ if and only if $m > \ell(\lambda)$ and similarly $h_m^\perp s_\lambda = 0$ if and only if $m > \lambda_1$.

For the next three definitions we will have to refer to the section on tableaux, in particular we are interested in facts about the number of column strict tableaux of shape λ and content μ, $K_{\lambda\mu}$. These coefficients arise as the entries in the change of basis matrix between the homogeneous and Schur basis and the Schur and monomial basis. This fact will follow directly from the Pieri rule which we have established that the basis defined in (1) through (6) satisfies. We recall in particular that $K_{\lambda\mu} = 1$ if $\lambda = \mu$ and $K_{\lambda\mu} = 0$ if $\lambda < \mu$ in dominance order of partitions.

Proof of (4) $\Leftrightarrow$ (10) :

By the facts that we have established about column strict tableaux, we know that if s_μ is a basis of the symmetric functions and $h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu} s_\lambda$, then $h_\mu = s_\mu +$ terms containing s_λ with partitions $\lambda > \mu$.

For any family of symmetric functions indexed by partitions, we will denote $\overrightarrow{b^n}$ as the row vector of $\left[b_{(n)}, b_{(n-1,1)}, \ldots, b_{(1^n)}\right]$. Therefore the change of basis matrix M_{sh}^n defined as the matrix

$$\overrightarrow{h^n} = \overrightarrow{s^n} M_{sh}^n \tag{5.57}$$

is clearly upper triangular and hence is invertible. So by stating that $h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda$ for all partitions $\mu \vdash n$, we establish a formula for s_λ as an entry in $\overrightarrow{s^n} = \overrightarrow{h^n} {M_{sh}^n}^{-1}$.

We show that any basis that satisfies the Pieri rule (in particular, the Schur functions) also satisfies definition (10) by induction on the length of the partition μ. Let $\tilde{\mu} = (\mu_1, \ldots, \mu_{\ell(\mu)-1})$ and assume that

$$h_{\tilde{\mu}} = \sum_{\lambda \vdash |\tilde{\mu}|} K_{\lambda\tilde{\mu}} s_\lambda. \tag{5.58}$$

Now since $h_\mu = h_{\mu_{\ell(\mu)}} h_{\tilde{\mu}}$ so

$$\begin{aligned} h_\mu &= \sum_{\lambda \vdash |\tilde{\mu}|} K_{\lambda\tilde{\mu}} \sum_{\gamma : \gamma/\lambda \in \mathcal{H}_m} s_\gamma \\ &= \sum_{\gamma \vdash |\mu|} \sum_{\lambda : \gamma/\lambda \in \mathcal{H}_m} K_{\lambda\tilde{\mu}} s_\gamma \end{aligned} \tag{5.59}$$

Every column strict tableau T of shape λ and content $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ is isomorphic to a column strict tableau of shape γ and content $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)-1})$ with $\lambda/\gamma \in \mathcal{H}_{\mu_{\ell(\mu)}}$ since each of the cells labeled with a $\ell(\mu)$ must lie in a horizontal strip in T. This shows that

$$\sum_{\lambda:\gamma/\lambda \in \mathcal{H}_m} K_{\lambda\tilde{\alpha}} = K_{\gamma\alpha}, \tag{5.60}$$

and hence in particular when $\alpha = \mu$, we have

$$h_\mu = \sum_{\gamma \vdash |\mu|} K_{\gamma\mu} s_\gamma. \tag{5.61}$$

Example 40. This establishes that h_μ is equal to the sum over all column strict tableaux of content μ counted with weight $s_{\lambda(T)}$ for each column strict tableau T of shape $\lambda(T)$. So for instance, if we would like to compute $h_{(2,2,2)}$ we draw all column strict tableaux of content $(2,2,2)$, as in:

$$\begin{array}{ll}3&3\\2&2\\1&1\end{array}\quad \begin{array}{lll}3&&\\2&2&\\1&1&3\end{array}\quad \begin{array}{lll}3&&\\2&3&\\1&1&2\end{array}\quad \begin{array}{llll}3&&&\\2&&&\\1&1&2&3\end{array}\quad \begin{array}{llll}2&3&&\\1&1&2&3\end{array}\quad \begin{array}{llll}2&2&&\\1&1&3&3\end{array}$$

$$\begin{array}{lll}2&3&3\\1&1&2\end{array}\quad \begin{array}{lllll}2&&&&\\1&1&2&3&3\end{array}\quad \begin{array}{llll}3&3&&\\1&1&2&2\end{array}\quad \begin{array}{lllll}3&&&&\\1&1&2&2&3\end{array}\quad \begin{array}{llllll}1&1&2&2&3&3\end{array}.$$

This shows that $h_{(2,2,2)} = s_{(2,2,2)} + 2s_{(3,2,1)} + s_{(4,1,1)} + 3s_{(4,2)} + 2s_{(5,1)} + s_{(6)}$.

Example 41. $h_{(1^n)}$ when expanded in the Schur basis will have exactly one term for each standard tableau. For example, when we compute $h_{(1^4)}$, all of the standard tableaux of size 4 are

$$\begin{array}{l}4\\3\\2\\1\end{array}\quad \begin{array}{ll}4&\\3&\\1&2\end{array}\quad \begin{array}{ll}4&\\2&\\1&3\end{array}\quad \begin{array}{ll}3&\\2&\\1&4\end{array}\quad \begin{array}{ll}3&4\\1&2\end{array}\quad \begin{array}{ll}2&4\\1&3\end{array}$$

$$\begin{array}{lll}4&&\\1&2&3\end{array}\quad \begin{array}{lll}3&&\\1&2&4\end{array}\quad \begin{array}{lll}2&&\\1&3&4\end{array}\quad \begin{array}{llll}1&2&3&4\end{array}.$$

Therefore

$$h_{(1^4)} = s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} + s_{(1,1,1,1)}.$$

Proof of (4) $\Leftrightarrow$ (11) :

The coefficient of m_λ in s_μ is given by the scalar product $\langle s_\mu, h_\lambda \rangle$. Let α be a sequence of non-negative integers of length equal to k. Assume that for all sequences of partitions β of length less than k we have that $\langle s_\nu, h_\beta \rangle = K_{\nu\beta}$. We set $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{\ell(\alpha)-1})$ and notice that

$$\begin{aligned}\langle s_\mu, h_\alpha \rangle &= \left\langle h^\perp_{\alpha_{\ell(\alpha)}} s_\mu, h_{\tilde{\alpha}} \right\rangle \\ &= \sum_{\mu/\gamma \in \mathcal{H}_{\alpha_{\ell(\alpha)}}} \langle s_\gamma, h_{\tilde{\alpha}} \rangle \\ &= \sum_{\mu/\gamma \in \mathcal{H}_{\alpha_{\ell(\alpha)}}} K_{\gamma\tilde{\alpha}} = K_{\mu\alpha}\end{aligned} \tag{5.62}$$

which follows from equation (5.60). In particular, when $\alpha = \lambda$ (a partition), $\langle s_\mu, h_\lambda \rangle = K_{\mu\lambda}$ and therefore

$$s_\mu = \sum_{\lambda \vdash |\mu|} \langle s_\mu, h_\lambda \rangle \, m_\lambda = \sum_{\lambda \vdash |\mu|} K_{\mu\lambda} m_\lambda. \tag{5.63}$$

Example 42. We have just shown that the Schur function s_μ is given as the sum over all column strict tableaux T of shape μ and partition content with a weight of $m_{\mu(T)}$ where $\mu(T)$ is the content of the tableau T. This means that in order to compute $s_{(2,2,2)}$ we draw all column strict tableaux of shape $(2,2,2)$ and partition content:

$$\begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&4\\\hline 2&2\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&5\\\hline 2&3\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&5\\\hline 2&4\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 5&6\\\hline 3&4\\\hline 1&2\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 4&6\\\hline 3&5\\\hline 1&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 5&6\\\hline 2&4\\\hline 1&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&6\\\hline 2&5\\\hline 1&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&6\\\hline 2&5\\\hline 1&4\\\hline\end{array}$$

Therefore we know that $s_{(2,2,2)} = m_{(2,2,2)} + m_{(2,2,1,1)} + 2m_{(2,1,1,1,1)} + 5m_{(1,1,1,1,1,1)}$.

Proof of (11) $\Leftrightarrow$ (12):

We have already shown in equation (5.60) that if α is a sequence of integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_i \geq 0$ such that sorting α to a partition λ (with dropping the 0s at the end of the sequence) then the number of column strict tableaux of shape μ and content λ is equal to the number of column strict tableaux of shape μ and content α. Therefore if we let CST_μ^α represent the set of column strict tableaux of shape μ and content α then,

$$\sum_{T \in CST_\mu} \prod_{i=1}^{n} x_i^{n_i(T)} = \sum_{\alpha : |\alpha| = |\mu|} \sum_{T \in CST_\mu^\alpha} x^\alpha \tag{5.64}$$

where the sum on the right hand side of this equation is over all sequences α of length n of non-negative integers whose sum is $|\mu|$. This Say that $\alpha \sim \lambda$ if there exists a permutation $\sigma \in S_n$ such that $\sigma\alpha = \lambda$ (ignoring any 0s at the end of the partition). We have shown that if $\alpha \sim \lambda$, then $K_{\mu\alpha} = K_{\mu\lambda}$. This implies that the previous expression is equal to

$$\begin{aligned} &= \sum_{\lambda \vdash |\mu|} \sum_{T \in CST_\mu^\lambda} \sum_{\alpha \sim \lambda} x^\alpha \\ &= \sum_{\lambda \vdash |\mu|} \sum_{T \in CST_\mu^\lambda} m_\lambda[X_n] \\ &= \sum_{\lambda \vdash |\mu|} K_{\lambda\mu} m_\lambda[X_n]. \end{aligned} \tag{5.65}$$

This last expression is equivalent to definition (11) for any $n \geq |\mu|$.

Example 43. If we want to evaluate a Schur function at an alphabet with a finite number of variables, this formula is useful (by taking a limit we can also use it to evaluate Schur functions on an infinite number of variables as well). We take as an example, $s_{(2,2,2)}[\frac{1-q^4}{1-q}] = s_{(2,2,2)}[1+q+q^2+q^3]$. For each tableau T of shape $(2,2,2)$ we count the term with a monomial of $\prod_{i=1}^{4} q^{(i-1)n_i(T)}$ which is equal to q to the power of the sum of the entries in the tableau -6.

$$\begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 2&2\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 3&3\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 3&3\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&4\\\hline 2&2\\\hline 1&1\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 3&4\\\hline 2&3\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&4\\\hline 2&3\\\hline 1&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 2&3\\\hline 1&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 2&3\\\hline 1&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 4&4\\\hline 3&3\\\hline 1&2\\\hline\end{array}$$

This means that we have evaluated

$$\begin{aligned} s_{(2,2,2)}\left[\frac{1-q^4}{1-q}\right] &= q^6+q^8+q^{10}+q^{12}+q^7+q^8+q^9+q^9+q^{10}+q^{11} \\ &= q^6 \begin{bmatrix}5\\2\end{bmatrix} \end{aligned}$$

We leave it as an exercise to generalize this result.

Proof of (3) $\Leftrightarrow$ (13):

The conditions listed in (13) (a) and (b) are exactly what is necessary to calculate a self-dual basis using the Grahm-Schmit orthonormalization procedure. We start with $s_{(1^n)} = m_{(1^n)}$ taking each successive partition of size n in lex order we define

$$s_\mu = m_\mu - \sum_{\lambda <_{lex} \mu} \langle m_\mu, s_\lambda\rangle\, s_\lambda \tag{5.66}$$

This uniquely determines the basis which satisfies conditions (a) and (b) of (13).

Now in order to show that the definition (3) is equivalent to (13), we verify that if $s_\mu = S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{\ell(\mu)}}1$ for all partitions μ, then $\langle s_\mu, s_\lambda\rangle = \delta_{\lambda\mu}$. Since the scalar product is symmetric, without loss of generality we may assume that $\mu_1 \geq \lambda_1$. Let $\tilde\lambda = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ and $\tilde\mu = (\mu_2, \ldots, \mu_{\ell(\mu)})$. Recall that proposition 5.9 says

$$\langle s_\mu, s_\lambda\rangle = \langle S_{\mu_1} s_{\tilde\mu}, s_\lambda\rangle = \left\langle s_{\tilde\mu}, (-1)^{\mu_1}\tilde S_{-\mu_1} s_\lambda\right\rangle. \tag{5.67}$$

Now for a partition ν and $m > \nu_1$, applying Lemma 5.5 we have

$$\tilde S_{-m}(s_\nu) = S_{\nu_1+1}S_{\nu_2+1}\cdots S_{\nu_{\ell(\nu)}+1}\tilde S_{-m-\ell(\nu)}1 = 0. \tag{5.68}$$

This implies that if $\mu_1 > \lambda_1$, then $\langle s_\mu, s_\lambda\rangle = 0$. If $\mu_1 = \lambda_1$, then $\mu_1 + 1 > \lambda_2$ and

$$\langle s_\mu, s_\lambda\rangle = \left\langle s_{\tilde\mu}, s_{\tilde\lambda} + (-1)^{\mu_1} S_{\lambda_1+1}\tilde S_{-\mu_1-1} s_{\tilde\lambda}\right\rangle = \langle s_{\tilde\mu}, s_{\tilde\lambda}\rangle. \tag{5.69}$$

Therefore, by an inductive argument we have shown $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$.

Example 44. We shall use definition (13) to calculate explicitly the symmetric functions s_λ for $\lambda \vdash 3$ in terms of the monomial basis. By the triangularity relations, we know that $s_{(111)} = m_{(111)}$.

$s_{(21)} = m_{(21)} - \left\langle m_{(21)}, s_{(111)} \right\rangle s_{(111)}$. Since we have already calculated $s_{(111)} = m_{(111)}$, we need only calculate one scalar product by brute force: $\left\langle m_{(21)}, m_{(111)} \right\rangle = -2$ and $s_{(21)} = m_{(21)} + 2m_{(111)}$.

$s_{(3)} = m_{(3)} - \left\langle m_{(3)}, s_{(21)} \right\rangle s_{(21)} - \left\langle m_{(3)}, s_{(111)} \right\rangle s_{(111)}$. Now in order to expand $s_{(3)}$ explicitly we need to calculate two scalar products. We calculate then that $s_{(3)} = m_{(3)} + s_{(21)} - s_{(111)} = m_{(3)} + m_{(21)} + m_{(111)}$.

Proof of (13) $\Leftrightarrow$ (14):

We have already said that condition (13) uniquely defines a basis of the symmetric functions. It does not seem immediately clear that condition (14) even determines the basis uniquely, yet it does. We will count the following example as the outline of a proof of how definition (14) defines the Schur functions.

EXAMPLE 45. To compute the Schur functions of degree 4, we set

$$s_{(4)} := h_{(4)} = m_{(4)} + m_{(31)} + m_{(22)} + m_{(211)} + m_{(1111)} \tag{5.70}$$

then since $h_{(31)}$ has leading term equal to $m_{(4)}$, then we set

$$\begin{aligned} s_{(31)} &= h_{(31)} - s_{(4)} = h_{(31)} - h_{(4)} \\ &= m_{(31)} + m_{(22)} + 2m_{(211)} + 3m_{(1111)}. \end{aligned}$$

To compute $s_{(22)}$, since the leading term of $h_{(22)}$ is $m_{(4)}$ we eliminate it by subtracting $s_{(4)}$ and $h_{(22)} - s_{(4)}$ has a leading term of $m_{(31)}$ so we must have

$$\begin{aligned} s_{(22)} &= h_{(22)} - s_{(4)} - s_{(31)} = h_{(22)} - h_{(31)} \\ &= m_{(22)} + m_{(211)} + 2\,m_{(1111)}. \end{aligned}$$

Next we have that $h_{(211)}$ has a leading term of $m_{(4)}$, and the leading term of $h_{(211)} - s_{(4)}$ is $2m_{(31)}$ and the leading term of $h_{(211)} - s_{(4)} - 2s_{(31)}$ is equal to $m_{(22)}$ and hence

$$\begin{aligned} s_{(211)} &= h_{(211)} - s_{(4)} - 2s_{(31)} - s_{(22)} \\ &= h_{(211)} - h_{(22)} - h_{(31)} + h_{(1111)} \\ &= m_{(211)} + 3m_{(1111)}. \end{aligned}$$

Finally we know already that $s_{(1111)} = m_{(1111)} = h_{(1111)} - 3h_{(211)} + h_{(22)} + 2h_{(31)} - h_{(4)}$.

Assume the functions $\{s_\mu\}_\mu$ satisfy (13) (a) and (b), then clearly they also satisfy (14) (a) since these conditions are the same and the coefficient of s_λ in h_μ will be equal to

$$\langle h_\mu, s_\lambda \rangle = \left\langle h_\mu, m_\lambda + \sum_{\nu <_{lex} \lambda} c_{\lambda\nu} m_\nu \right\rangle \tag{5.71}$$

and this value must clearly be equal to 0 unless $\mu <_{lex} \lambda$ since the monomial and homogeneous bases are self dual. In fact we see that the coefficient of s_λ in h_μ is equal to $c_{\lambda\mu}$ and hence

$$h_\mu = s_\mu + \sum_{\lambda >_{lex} \mu} c_{\lambda\mu} s_\lambda. \tag{5.72}$$

Therefore the functions $\{s_\mu\}_\mu$ also satisfy (14) (b).

Proof of (14) $\Leftrightarrow$ (15):

These two conditions are exactly equivalent, (14) (a) and (15) (a) are the same and if $h_\lambda = s_\lambda +$ terms which are lower in lex order then $s_\lambda = h_\lambda +$ terms which are lower in lex order.

Proof of (14) $\Leftrightarrow$ (16)

The LU decomposition of a matrix is unique and so (16) defines the Schur functions as a basis in terms of the homogeneous and monomial bases.

The fact that these definitions are equivalent is a matter of expressing conditions (14) (a) and (b) in terms of matricies.

□

We have justified a large number of properties of the Schur functions by verifying that these conditions all defined the same set of symmetric functions. A few of these properties may not yet be clear so we identify the most important ones here.

Corollary 5.15.

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \tag{5.73}$$

$$\omega(s_\lambda) = s_{\lambda'} \tag{5.74}$$

$$f[X_n]\Big|_{s_\lambda[X_n]} = f[X_n] \prod_{1 \le i < j \le n} (1 - x_i/x_j)\Big|_{x^\lambda} \tag{5.75}$$

$$s_\mu = m_\mu + \textit{ terms } m_\lambda \textit{ with } \lambda < \mu \tag{5.76}$$

$$s_\mu = h_\mu + \textit{ terms } h_\lambda \textit{ with } \lambda > \mu \tag{5.77}$$

$$s_\mu[X_n] = 0 \textit{ if } n < \ell(\mu) \tag{5.78}$$

We have yet to compute a formula for the expansion of the Schur functions in the power symmetric functions. In order to do this we will examine the commutation relations between $p_k^\perp$ and the operators S_m.

Lemma 5.16. *For $k > 0$ and any $m \in \mathbb{Z}$, we have*

$$p_k^\perp S_m = S_{m-k} + S_m p_k^\perp. \tag{5.79}$$

equivalently

$$p_k S_m = S_m p_k + S_{m+k} \tag{5.80}$$

Proof. The commutation relation of $p_k^\perp$ and h_m is easy to compute given that $p_k^\perp h_m = p_k^\perp(h_m) + h_m p_k^\perp$. Recall that $p_k^\perp(h_m) = h_{m-k}$, hence

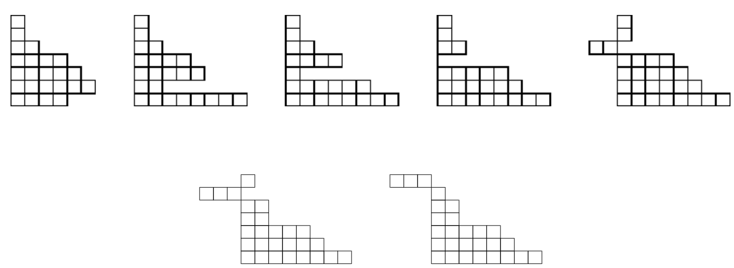
$$\begin{aligned} p_k^\perp S_m &= p_k^\perp \sum_{i\geq 0}(-1)^i h_{m+i} e_i^\perp \\ &= \sum_{i\geq 0}(-1)^i p_k^\perp(h_{m+i}) e_i^\perp + \sum_{i\geq 0}(-1)^i h_{m+i} e_i^\perp p_k^\perp \\ &= S_{m-k} + S_m p_k^\perp \end{aligned}$$

The second identity is proven in a similar manner. □

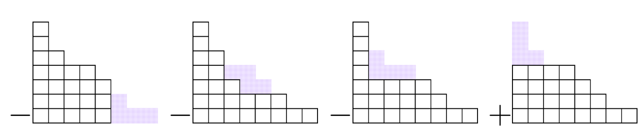
From this we can calculate the action of $p_k^\perp$ when it acts on the Schur function s_λ. For any partition λ, we will set $\lambda^{rc} = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$.

To indicate how this is done we do an example because it explains clearly how the proof proceeds and how to express the action of $p_k^\perp$ on the Schur function s_λ.

Example 46. To demonstrate this Lemma we compute $p_4^\perp$ on $s_{(8,6,5,4,2,1,1)}$. In order to do this we consider this Schur function as the composition of operators $S_8S_6S_5S_4S_2S_1S_11$. We will represent compositions of operators by a diagram. This means that $p_4^\perp s_{(8,6,5,4,2,1,1)}$ will be the sum of the compositions of operators represented by all of the following 7 diagrams.



Some of these compositions of operators are 0. All of them can be reduced by straightening relations in equation (5.5) or we can use the method described in Example 37. The following sequence of operators survive (corresponding to the 1^{st}, 3^{rd}, 4^{th} and 5^{th} digarams), the others are 0 either because they end in a negative part or in the case of the 2^{nd} diagram there is a relation of the form $S_mS_{m+1} = 0$. The sign next to the diagram comes from the number of applications of the relation (5.5) that were use to straighten the composition to a partition.



Notice that the resulting diagrams are partitions which are contained in the partition $(8, 6, 5, 4, 2, 1, 1)$ and differ from it by removing a 'slinky' from the border (also known as a rim hook). What we have shown is that

$$p_4^\perp s_{(8654211)} = -s_{(5554211)} - s_{(8632211)} - s_{(8651111)} + s_{(8654)}.$$

Proposition 5.17. *For $k > 0$,*

$$p_k^\perp s_\lambda = \sum_\mu (-1)^{\ell(\lambda/\mu)-1} s_\mu \tag{5.81}$$

where the sum is over all partitions μ such that λ/μ is connected and of size k and $\lambda^{rc} \subseteq \mu \subseteq \lambda$. Note that we are using the notation $\ell(\lambda/\mu)$ to represent the number of rows that the skew partition λ/μ occupies.

Proof. By the previous lemma, we have that

$$p_k^\perp s_\lambda = S_{\lambda_1-k} S_{\lambda_2} \cdots S_{\lambda_{\ell(\lambda)}} 1 + S_{\lambda_1} S_{\lambda_2-k} \cdots S_{\lambda_{\ell(\lambda)}} 1 + \cdots + S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_{\ell(\lambda)}-k} 1$$

Now each of these terms is (up to sign) equal to a Schur function or 0 and the commutation relation of Lemma 5.2 allows us to see that the terms which correspond to partitions μ will have λ/μ will be a slinky while all others will be 0. The sign in front of the Schur function indexed by a partition μ will be the number of exchanges required to straighten the composition of operators. This will be 1 less than the number of rows that λ/μ occupies. □

Using the self-duality property of the Schur functions we may easily derive from this equation the expansion of the product $p_k s_\lambda$ in the Schur basis.

Corollary 5.18.

$$p_k s_\lambda = \sum_\mu (-1)^{\ell(\mu/\lambda)-1} s_\mu \tag{5.82}$$

where the sum is over all parititions μ such that μ/λ is connected and of size k and $\mu^{rc} \subseteq \lambda \subseteq \mu$.

Proof.

$$\begin{aligned} p_k s_\lambda &= \sum_{\mu \vdash |\lambda|+k} \langle p_k s_\lambda, s_\mu \rangle s_\mu \\ &= \sum_{\mu \vdash |\lambda+k|} \left\langle s_\lambda, p_k^\perp s_\mu \right\rangle s_\mu \\ &= \sum_{\mu \vdash |\lambda|+k} \sum_{\nu : \mu^{rc} \subseteq \nu \subseteq \mu} \left\langle s_\lambda, (-1)^{\ell(\nu/\mu)-1} s_\nu \right\rangle s_\mu \\ &= \sum_{\mu : \mu^{rc} \subseteq \lambda \subseteq \mu} (-1)^{\ell(\lambda/\mu)-1} s_\mu \end{aligned} \tag{5.83}$$

the the sum over partitions ν in the third line is such that $\mu^{rc} \subseteq \nu \subseteq \mu$ and μ/λ is connected and of size k. Similar conditions hold on the final expression for $p_k s_\lambda$. □

Proposition 5.17 and Corollary 5.18 are often referred to as the Murnaghan-Nakayama rule but they are also attributed to Littlewood.

This leads us to a combinatorial interpretation for the coefficient of p_λ/z_λ in s_μ which will be equal to $\langle p_\lambda, s_\mu \rangle$. Define $\overline{\chi}^\mu(\lambda)$ to be this quantity, it then follows that

$$s_\mu = \sum_\lambda \overline{\chi}^\mu(\lambda) \frac{p_\lambda}{z_\lambda}$$

and

$$p_\lambda = \sum_\mu \overline{\chi}^\mu(\lambda) s_\mu.$$

For fixed partitions λ and μ we can calculate $\overline{\chi}^\mu(\lambda)$ by successively removing a row at a time from the partition λ.

EXAMPLE 47. To calculate the coefficient $\overline{\chi}^{(44)}(4211)$ we are determining the scalar product $\langle p_{(4211)}, s_{(44)} \rangle = \langle p_{(421)}, p_1^\perp s_{(44)} \rangle$. Since $p_1 = h_1$ it follows either from Proposition 5.17 or 5.13 that $p_1^\perp s_{(44)} = s_{(43)}$. Therefore $\overline{\chi}^{(44)}(4211) = \langle p_{(42)}, p_1^\perp s_{(43)} \rangle = \langle p_{(42)}, s_{(42)} + s_{(33)} \rangle$. Now we can apply the Proposition 5.17 to determine the following coefficients. We can determine from Corollary 5.18 that for $k > 0$, $\langle p_k, s_{(k-i,1^i)} \rangle = (-1)^i$ and $\langle p_k, s_\lambda \rangle = 0$ if λ is not a hook shaped partition.

$$\begin{aligned} \overline{\chi}^{(44)}(4211) &= \langle p_{(4)}, p_2^\perp s_{(42)} \rangle + \langle p_{(4)}, p_2^\perp s_{(33)} \rangle \\ &= \langle p_{(4)}, s_{(4)} + s_{(22)} \rangle + \langle p_{(4)}, s_{(31)} - s_{(22)} \rangle = 1 - 1 \end{aligned}$$

At each successive step of this procedure we can label each of the terms that are Schur functions by a Young diagram with cells labeled at each step they were removed. The terms which contribute to the sum in the end are the Young diagrams such that they have been filled with λ_1 1s, λ_2 2s, etc. such that the cells of the diagram labeled with an i are a slinky (or rim hook) of size λ_i. Call this combinatorial object a rim hook tableau (or slinky tableau) of content λ. The sign of a rim hook tableau will be the product of each of the signs associated to the cells labeled with an i and so the value of the sign of a rim hook tableau will be ± 1.

In the example above the two terms which contributed to the value of the coefficient are represented by the following diagrams.

$$\begin{array}{|c|c|c|c|} \hline 2&2&3&4 \\ \hline 1&1&1&1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 1&2&2&4 \\ \hline 1&1&1&3 \\ \hline \end{array}$$

The one on left has a weight of $+1$ while the weight of the diagram on the right will be -1. This is different from the combinatorial interpretations for other coefficients which we have presented so far in the fact that the sign of each of the terms is not always the same and so it is difficult to gain much information about the sign of the coefficients $\overline{\chi}^\mu(\lambda)$ without computing the coefficient explicitly.

Stated more precisely, the last example gives

PROPOSITION 5.19. $\overline{\chi}^\mu(\lambda) = \sum_T sgn(T)$ *where the sum is over all rim hook tableaux (slinky tableaux)* T *of shape* μ *and content* λ.

The following proposition shows that the functions $s_\lambda^\perp s_\mu$ have a simple determinantal formula which generalizes the Jacobi-Trudi identity. We will develop a formula for the expansion of these elements in the Schur basis in the next section.

PROPOSITION 5.20.

$$s_\mu^\perp s_\lambda = det\left|h_{\lambda_i-\mu_j-i+j}\right|_{1\leq i,j\leq n}$$

This is an easy calculation which shall be done in two steps because the intermediate step will be useful for another formula that we will derive for these elements.

LEMMA 5.21.

$$s_\mu^\perp s_\lambda[X] = \Omega[Z_nX]s_\mu[Z_n]\prod_{1\leq i<j\leq n}(1-z_j/z_i)\Big|_{z^\lambda}$$

PROOF. To expand a formula for $f^\perp g[X]$ we will use the coproduct structure on variables that was discussed at the end of the last chapter through the identity $f^\perp g[X] = \langle f[Y], g[X+Y]\rangle_Y$. Now,

$$\begin{aligned} s_\mu^\perp s_\lambda[X] &= \left\langle s_\mu[Y], \Omega[Z_n(X+Y)]\prod_{1\leq i<j\leq n}(1-z_j/z_i)\Big|_{z^\lambda}\right\rangle \\ &= \langle s_\mu[Y], \Omega[Z_nY]\rangle\,\Omega[Z_nX]\prod_{1\leq i<j\leq n}(1-z_j/z_i)\Big|_{z^\lambda} \qquad (5.84)\\ &= s_\mu[Z_n]\Omega[Z_nX]\prod_{1\leq i<j\leq n}(1-z_j/z_i)\Big|_{z^\lambda}. \end{aligned}$$

□

The second step of this proof will be to change this to a determinant much like in the proof of the equivalence between definition (1) and (2). The key step in the proof will be the use of Corollary 5.8.

PROOF. Recall from Proposition 5.7 that $\prod_{1\leq i<j\leq n}(1-x_j/x_i) = x^{-\delta_n}\Delta_{\delta_n}(X_n)$ where $\delta_n = (n-1, n-2, \ldots, 1, 0)$.

We begin with the identity from the previous lemma.

$$
\begin{aligned}
s_\mu^\perp s_\lambda[X] &= \Omega[Z_nX]s_\mu[Z_n] \prod_{1\le i<j\le n}(1-z_j/z_i)\Big|_{z^\lambda} \\
&= \Omega[Z_nX]\Omega[Z_nY_n] \prod_{1\le i<j\le n}(1-y_j/y_i) \prod_{1\le i<j\le n}(1-z_j/z_i)\Big|_{z^\lambda y^\mu} \\
&= \Omega[Z_nX]\Omega[Z_nY_n]y^{-\delta_n}z^{-\delta_n}\Delta_{\delta_n}(Y_n)\Delta_{\delta_n}(Z_n)\Big|_{z^\lambda y^\mu} \\
&= \Omega[Z_nX]det \left|\frac{1}{1-z_iy_j}\right|_{1\le i,j\le n}\Big|_{z^{\lambda+\delta_n}y^{\mu+\delta_n}} \qquad (5.85)\\
&= \Omega[Z_nX]\sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n}\frac{1}{1-z_iy_{\sigma(i)}}\Big|_{z^{\lambda+\delta_n}y^{\mu+\delta_n}} \\
&= \Omega[Z_nX]\sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n}z_i^{\mu_{\sigma(i)}+n-\sigma(i)}\Big|_{z^{\lambda+\delta_n}} \\
&= \sum_{\sigma\in Sym_n}\epsilon(\sigma)\prod_{i=1}^{n}h_{\lambda_i-\mu_{\sigma(i)}+n-i-(n-\sigma(i))}[X] \\
&= det\left|h_{\lambda_i-\mu_j-i+j}[X]\right|_{1\le i,j\le n}
\end{aligned}
$$

□

5.1. The Irreducible Characters of Sym_n

There is a connection between the class functions of Sym_n that we have alluded to in this presentation, that the Schur functions are the Frobenius image of the irreducible characters of Sym_n. Recall the following basic facts about the irreducible characters of the symmetric group that we derived for the characters of an arbitrary finite group.

(1) The number of irreducible characters of Sym_n is equal to the number of conjugacy classes in Sym_n (the number of partitions of n).
(2) If $\{\chi^\lambda\}_{\lambda\vdash n}$ is a complete set of irreducible characters, then $\chi^\lambda(e)$ is the dimension of the module corresponding to this irreducible character where e is the identity element in Sym_n.
(3) $\sum_{\lambda\vdash n}\chi^\lambda(e)^2 = n!$.
(4) $\langle\chi^\lambda,\chi^\mu\rangle = \delta_{\lambda\mu}$.
(5) $\sum_{\lambda\vdash n}\chi^\lambda(\pi)\chi^\lambda(\sigma^{-1}) = z_\gamma$ if both π and σ are both of cycle type γ and the result is 0 if π and σ are not of the same cycle type.
(6) Since π and π^{-1} are of the same cycle type, then $\chi^\lambda(\pi) = \chi^\lambda(\pi^{-1}) = \overline{\chi^\lambda(\pi)}$ and hence χ^λ are all real valued.

These properties correspond to the following facts about the Schur symmetric functions.

(1) $\{s_\lambda\}_{\lambda \vdash n}$ is a linear basis for the symmetric functions of degree n.
(2) $s_\lambda\Big|_{p_1^n} = \left\langle s_\lambda, p_{(1^n)}\right\rangle = \left\langle s_\lambda, h_{(1^n)}\right\rangle = K_{\lambda(1^n)}$ is equal to the number of standard tableaux of shape λ.
(3) In particular, $n! = \left\langle p_{(1^n)}, p_{(1^n)}\right\rangle = \sum_{\lambda,\mu}\left\langle K_{\lambda(1^n)}s_\lambda, K_{\mu(1^n)}s_\mu\right\rangle = \sum_{\lambda\vdash n} K^2_{\lambda(1^n)}$.
(4) $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$
(5) $p_\lambda = \sum_{\mu\vdash|\lambda|}\overline{\chi}^\mu(\lambda)s_\mu$ and in particular $z_\lambda\delta_{\lambda\mu} = \langle p_\lambda, p_\gamma\rangle = \sum_{\mu\vdash|\lambda|}\overline{\chi}^\mu(\lambda)\overline{\chi}^\mu(\gamma)$.
(6) $\overline{\chi}^\mu(\lambda) = \sum_T sgn(T)$ where the sum is over all rim-hook tableaux T of shape μ and content λ and the sign of the tableau is the product of the signs of each of the hooks (see Proposition 5.19) hence these coefficients are clearly integers.

The real block here is we have not established an association between the irreducible characters and partitions. To this end we establish the following Proposition.

Proposition 5.22. *Let $Sym_\lambda = Sym_{\lambda_1} \times Sym_{\lambda_2} \times \cdots \times Sym_{\lambda_{\ell(\lambda)}} \subseteq Sym_{|\lambda|}$. Represent the trivial character on Sym_λ by χ^{T_λ} and the sign character on Sym_λ by χ^{sgn_λ}, then for $\lambda, \mu \vdash n$*

$$\left\langle \chi^{T_\lambda}\uparrow^{Sym_n}_{Sym_\lambda}, \chi^{sgn_\mu}\uparrow^{Sym_n}_{Sym_\mu}\right\rangle > 0$$

if and only if $\lambda \geq \mu'$, $\left\langle \chi^{T_\lambda}\uparrow^{Sym_n}_{Sym_\lambda}, \chi^{sgn_\mu}\uparrow^{Sym_n}_{Sym_\mu}\right\rangle = 0$ otherwise, and $\left\langle \chi^{T_\lambda}\uparrow^{Sym_n}_{Sym_\lambda}, \chi^{sgn_\mu}\uparrow^{Sym_n}_{Sym_\mu}\right\rangle = 1$ if $\lambda = \mu'$.

Proof. We have already mostly established this proposition except for a few details because $\mathcal{F}(\chi^{T_\mu}\uparrow^{Sym_n}_{Sym_\mu}) = h_\mu$ and $\mathcal{F}(\chi^{sgn_\lambda}\uparrow^{Sym_n}_{Sym_\lambda}) = e_\lambda$ hence we are making a statement about the scalar product $\langle h_\mu, e_\lambda\rangle$. By Proposition 2.10, this is the number of matrices with entries in 0 and 1 with column sum μ and row sum λ and there is a second combinatorial interpretation which we have discussed. These combinatorial interpretations or a proof by induction using the dual multiplication operators can be used to establish the corresponding statements in the proposition for the scalar product of symmetric functions. This is left as an exercise in the previous chapter. □

The fact that $\left\langle \chi^{T_\lambda}\uparrow^{Sym_n}_{Sym_\lambda}, \chi^{sgn_{\lambda'}}\uparrow^{Sym_n}_{Sym_{\lambda'}}\right\rangle = 1$ implies that there is a single common irreducible constituent to both of these characters. Define χ^λ to be this character and set $\tilde{s}_\lambda = \mathcal{F}(\chi^\lambda)$.

$\{\chi^\mu\}_{\mu\vdash n}$ is then a complete family of irreducible characters since there are the correct number which must be linearly independent. By the definition of the Frobenius map,

$$\tilde{s}_\mu = \mathcal{F}(\chi^\mu) = \sum_{\lambda\vdash n}\chi^\mu(\lambda)p_\lambda/z_\lambda$$

where $\chi^\mu(\lambda)$ is the value of the class function evaluated at a permutation of cycle type λ. We would like to say that $\mathcal{F}(\chi^\mu)$ are in fact the Schur functions, since they are certainly an orthonormal family of symmetric functions given that

$$\langle \tilde{s}_\mu, \tilde{s}_\lambda\rangle = \left\langle \chi^\mu, \chi^\lambda\right\rangle = \delta_{\lambda\mu}. \tag{5.86}$$

If χ is a character, then it will have the property that the coefficient of $\tilde{s}_\lambda$ in $\mathcal{F}(\chi)$ will always be non-negative since it is multiplicity of the irreducible representation corresponding to the partition λ in the module/matrix representation corresponding to χ. For this reason families of symmetric functions which have non-negative coefficients when expanded in the Schur basis are interesting because it is possible that they correspond to family of representations.

Now we look closer to see what the proposition above says about the functions $\tilde{s}_\mu$. Since $\langle \tilde{s}_\mu, e_{\mu'} \rangle = 1$, we know that all characters will have non-negative scalar products. Therefore if $\lambda \not\leq \mu$ then by exercise 2.(8),

$$0 \leq \langle \tilde{s}_\mu, h_\lambda \rangle \leq \langle e_{\mu'}, h_\lambda \rangle = 0.$$

This implies that

$$\tilde{s}_\mu = m_\mu \ + \ \text{terms containing } m_\lambda \text{ with } \lambda < \mu.$$

Similarly, since $\langle \tilde{s}_\mu, h_\mu \rangle = 1$, then for $\mu \not\leq \lambda$,

$$0 \leq \langle \tilde{s}_\mu, e_{\lambda'} \rangle \leq \langle h_\mu, e_{\lambda'} \rangle = 0.$$

We also know from Remark 6 that $m_\lambda = e_{\lambda'}$ + terms containing $e_{\gamma'}$ with $\gamma < \lambda$. Therefore

$$\langle \tilde{s}_\mu, m_\lambda \rangle = 0.$$

This implies the result which we are looking to prove, that

$$\tilde{s}_\mu = h_\mu \ + \ \text{terms containing } h_\lambda \text{ with } \lambda > \mu.$$

5.2. Exercises

(1) Define the generating function of $h_n^\perp$ operators $H^\perp(u)$ by the formula $H^\perp(u)P[X] = P[X+u]$ and for the $(-1)^n e_n^\perp$ operators $E^\perp(u)P[X] = P[X-u]$. Compute the commutation relations between these generating functions and $S(z)$ defined in equation (5.3) and equation (5.4). Use these relations to derive the dual Pieri rules.

(2) Show that

$$s_\lambda \left[\frac{1-q^n}{1-q} \right] = q^{n(\lambda)} H^n(\lambda'; q)$$

where $H^n(\lambda; q) = \prod_{s \in \lambda} \frac{1-q^{n-c(s)}}{1-q^{h(s)}}$ and for $s = (i,j)$, $c(s) = j - i$ and $h(s) = \lambda_i + \lambda'_j - i - j + 1$.

(3) Show the identity $p_k S_m = S_m p_k + S_{m+k}$. Use scalar products to show how this implies $\tilde{S}_m p_k^\perp - (-1)^k \tilde{S}_{m-k} = p_k^\perp \tilde{S}_m$ and use this identity to derive the Murnaghan-Nakayama rule.

(4) If λ is a partition such that $\lambda = \lambda'$ then prove that for all $f \in \Lambda$ such that f is of degree $|\lambda|$, $s_\lambda * f = \omega(s_\lambda * f)$.

CHAPTER 6

The Littlewood-Richardson Rule

We started by developing the symmetric functions and developed three different 'multiplication' operations: product, the Kronecker or inner product and plethysm. These were defined on the power basis because there they are the easiest to state and understand.

$$p_\lambda \cdot p_\mu = p_{\lambda \uplus \mu}$$

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \frac{p_\nu}{z_\nu}$$

$$p_\lambda \circ p_\mu = \prod_{i=1}^{\ell(\lambda)} p_{(\lambda_i \mu_1, \lambda_i \mu_2, \ldots, \lambda_i \mu_{\ell(\lambda)})}$$

Each of these operations have an interpretation in terms of representation theory that makes them a curious object to study in the theory of symmetric functions. The definition of these operations leads to three natural questions that arise. What is the coefficient of s_ν in the expression $s_\lambda \odot s_\mu$ where $\odot$ is one of the operations $\cdot$, $*$ or $\circ$. This means that we are looking for some expression/formula/combinatorial interpretation/ algorithm or method of computation for the following three expressions:

$$\langle s_\lambda s_\mu, s_\nu \rangle, \langle s_\lambda * s_\mu, s_\nu \rangle, \langle s_\lambda \circ s_\mu, s_\nu \rangle. \tag{6.1}$$

Since we have developed means of expanding the Schur function in terms of the power basis in equations (**??**), a formula for each of these expressions can be found by expanding the Schur functions in the expressions above and giving an expression for these quantities in terms of sums of expressions involving the coefficients $\chi^\mu(\lambda)$. This is our starting point for each of these coefficients. This formula however is fairly unsatisfactory because it does little to explain why the coefficients are positive integers or what they might represent.

A goal of exposition will be to arrive at a 'satisfactory' explanation for the coefficients above.

One of the most important aspects of the Schur functions are the coefficients which appear when a product of two Schur functions are again expanded in the Schur basis.

Example 48. We may compute the product $s_{(21)}s_{(32)}$ by expanding these Schur functions in terms of the homogeneous basis and then converting this expression into the back into the Schur basis by computing the Schur functions of degree 8 which appear in this expression. We find that

$$\begin{aligned} s_{(21)}s_{(32)} = s_{(53)} &+ s_{(521)} + s_{(44)} + 2\, s_{(431)} + s_{(422)} \\ &+ s_{(4211)} + s_{(332)} + s_{(3311)} + s_{(3221)} \end{aligned}$$

It is not obvious from the method which we used to compute this expansion that the coefficients which appear should be non-negative integers. This is a property which always occurs in the expansion of a product of Schur functions and the purpose of this section will be to arrive at a combinatorial rule for computing each coefficient. This combinatorial rule is known as the Littlewood-Richardson rule.

The coefficient of s_μ in the product of Schur functions s_λ and s_ν is typically denoted as $c^\mu_{\lambda\nu}$, that is,

$$s_\lambda s_\nu = \sum_{\mu \vdash |\lambda|+|\nu|} c^\mu_{\lambda\nu} s_\mu. \tag{6.2}$$

This means that $c^\mu_{\lambda\nu} = \langle s_\lambda s_\nu, s_\mu \rangle$ and by the definition of $s^\perp_\lambda$ this means as well that $c^\mu_{\lambda\nu} = \left\langle s_\nu, s^\perp_\lambda s_\mu \right\rangle$ and hence

$$s^\perp_\lambda s_\mu = \sum_{\nu \vdash |\mu|-|\lambda|} c^\mu_{\lambda\nu} s_\nu \tag{6.3}$$

Since the product of two Schur functions is commutative, we note that it must be the case that $c^\mu_{\lambda\nu} = c^\mu_{\nu\lambda}$ and since $\omega(s_\lambda) = s_{\lambda'}$ then it follows that $c^{\mu'}_{\lambda'\nu'} = c^\mu_{\lambda\nu}$ since $\langle s_{\lambda'} s_{\nu'}, s_{\mu'} \rangle = \langle \omega(s_\lambda s_\nu), \omega(s_\mu) \rangle = \langle s_\lambda s_\nu, s_\mu \rangle$. We also note that $c^\mu_{\lambda\nu}$ is non-zero only if $|\mu| = |\lambda| + |\nu|$.

To begin with, we will derive a combinatorial rule for computing the coefficient $c^\mu_{\lambda\nu}$. This rule is not the Littlewood-Richardson rule, but is instead a precursor since will be a sum which contains negative components and we will arrive at the Littlewood-Richardson rule by refining this combinatorial procedure.

Proposition 6.1.

$$s^\perp_\lambda s_\mu = \sum_{T \in CST_\lambda} s_{\mu - n(T)} \tag{6.4}$$

where the sum is over all column strict tableau of shape λ and with cells labeled with entries in $\{1, 2, \ldots, \ell(\mu)\}$ and $n(T) = (n_1(T), n_2(T), \ldots, n_{\ell(\mu)}(T))$ where $n_i(T)$ is the number of cells in T labeled with an i.

PROOF. Recall that by definition (2) we know that $s_\mu[X] = \Omega[XZ_n]\prod_{1\le i<j\le n}(1 - z_i/z_j)\Big|_{z_\mu}$. Since we have in general that $(g^\perp f)[X] = \langle g[Y], f[X+Y]\rangle$ then

$$\begin{aligned}
(s_\lambda^\perp s_\mu)[X] &= \left\langle s_\lambda[Y], \Omega[(X+Y)Z_n] \prod_{1\le i<j\le n}(1-z_i/z_j)\Big|_{z_\mu}\right\rangle_Y \\
&= \langle s_\lambda[Y], \Omega[YZ_n]\rangle_Y\, \Omega[XZ_n]\prod_{1\le i<j\le n}(1-z_i/z_j)\Big|_{z_\mu} \\
&= s_\lambda[Z_n]\Omega[XZ_n]\prod_{1\le i<j\le n}(1-z_i/z_j)\Big|_{z_\mu} \qquad (6.5)\\
&= \sum_{T\in CST_\lambda} z_n^{n(T)}\Omega[XZ_n]\prod_{1\le i<j\le n}(1-z_i/z_j)\Big|_{z_\mu} \\
&= \sum_{T\in CST_\lambda} s_{\mu-n(T)}[X]
\end{aligned}$$

This last equality follows by using the extended definition of the Schur functions indexed by a composition since as we remarked in the proof of the equivalence of definitions (1), (2) and (3) that this relation holds for Schur functions indexed by an arbitrary sequence. □

Although it seems like this is a sum over a positive set of objects, $\mu - n_i(T)$ is not always a partitions and hence some of the terms in the sum could be negative when expanded in the Schur basis. We will give an example to demonstrate this combinatorial formula.

EXAMPLE 49. Let us show how this equation can be used to compute the expression $s_{(21)}^\perp s_{(442)}$. The formula says that there is one term in this sum for each column strict tableau of shape $(2,1)$ with labels in the set $\{1,2,3\}$. Below we list all 8 tableau as well as the composition of integers representing the content and the Schur function indexed by $(4,4,2)$ minus the content composition.

$\begin{smallmatrix}2\\1&1\end{smallmatrix}$	$\begin{smallmatrix}3\\1&1\end{smallmatrix}$	$\begin{smallmatrix}2\\1&2\end{smallmatrix}$	$\begin{smallmatrix}3\\1&3\end{smallmatrix}$
$(2,1,0)$	$(2,0,1)$	$(1,2,0)$	$(1,0,2)$
$s_{(232)} = 0$	$s_{(241)} = -s_{(331)}$	$s_{(322)}$	$s_{(340)} = 0$
$\begin{smallmatrix}3\\2&2\end{smallmatrix}$	$\begin{smallmatrix}3\\2&3\end{smallmatrix}$	$\begin{smallmatrix}3\\1&2\end{smallmatrix}$	$\begin{smallmatrix}2\\1&3\end{smallmatrix}$
$(0,2,1)$	$(0,1,2)$	$(1,1,1)$	$(1,1,1)$
$s_{(421)}$	$s_{(430)} = s_{(43)}$	$s_{(331)}$	$s_{(331)}$

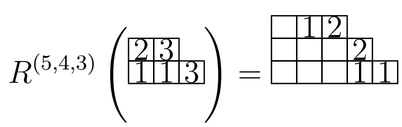
This implies that the Schur function expansion of $s_{(21)}^\perp s_{(442)}$ is given by $s_{(322)} + s_{(421)} + s_{(43)} + s_{(331)}$.

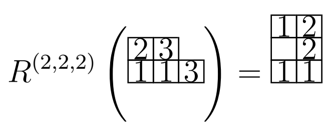
Notice that the one term that represents a negative Schur function when indexed by a partition in this example cancels with one of the two terms coming from the tableaux of content $(1,1,1)$. It is not clear from the way that we have presented this example which terms survive in this sum but add an additional combinatorial element and we will be able to identify exactly what the terms are which survive.

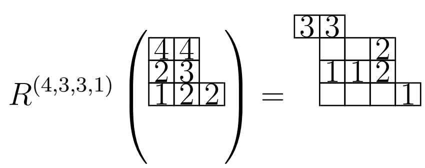
For a partition μ and a column strict tableau T we define the recording diagram for T with respect to the partition μ which we will denote by $R^\mu(T)$. $R^\mu(T)$ will have $n_i(T)$ cells in the i^{th} row and these cells will be right justified so that the rightmost cell lies in the μ_i^{th} column of the diagram (we will allow these cells to trickle into the $(-x, +y)$-quadrant if necessary). In the i^{th} row the cells will be labeled in increasing order and will contain a label k for each label i in the k^{th} row of the tableau T.

This is best demonstrated with a few examples.

EXAMPLE 50.





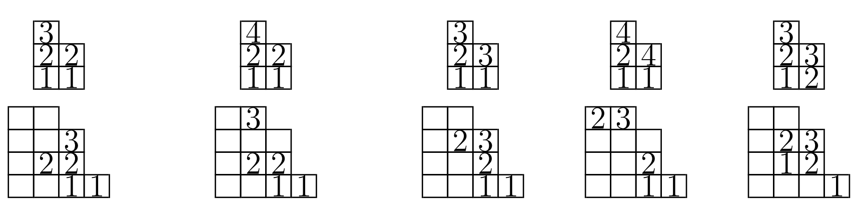


The Littlewood-Richardson rule can now be stated as follows.

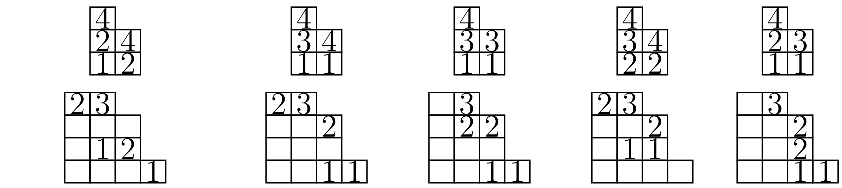
THEOREM 6.2. *$c^\mu_{\lambda\nu}$ is the number of column strict tableaux of shape λ such that $R^\mu(T)$ is a column strict tableau of shape μ/ν.*

Before we proceed with the proof we will give a second example of a computation with equation (6.4) and this time we will list all of the column strict tableaux as well as the recording tableau to demonstrate that the Littlewood-Richardson rule works as advertised.

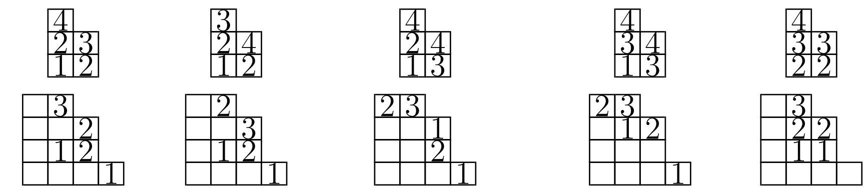
EXAMPLE 51. As an example we will compute $s^\perp_{(221)} s_{(4332)}$. There are 20 tableaux of shape $(2, 2, 1)$ with content in the labels $\{1, 2, 3, 4, 5\}$.



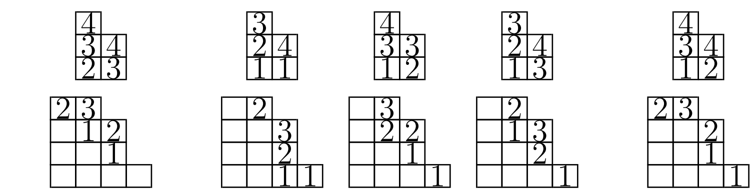
$s_{(2122)} = 0$ $\quad s_{(2131)} = -s_{(2221)}$ $\quad s_{(2212)} = 0$ $\quad s_{(2230)} = 0$ $\quad s_{(3112)} = 0$



$s_{(3130)} = -s_{(322)}$ $\quad s_{(2320)} = 0$ $\quad s_{(2311)} = 0$ $\quad s_{(4120)} = 0$ $\quad s_{(2221)}$



$s_{(3121)} = 0$ $\quad s_{(3121)} = 0$ $\quad s_{(3220)} = s_{(322)}$ $\quad s_{(3310)} = s_{(331)}$ $\quad s_{(4111)}$



$s_{(3310)} = s_{(331)}$ $\quad s_{(4111)}$ $\quad s_{(2221)}$ $\quad s_{(3211)}$ $\quad s_{(3220)} = s_{(322)}$

There are two terms in this collection with negative weight and two terms with positive weight such that $R^{(4332)}(T)$ is not a column strict tableau and these negative and positive terms cancel. 9 of the 20 terms in this sum have 0 weight and the only terms which survive are those such that $R^{(4332)}(T)$ is a skew column strict tableau. This calculation shows that

$$s^{\perp}_{(221)} s_{(4332)} = s_{(421)} + s_{(4111)} + s_{(331)} + s_{(322)} + 2s_{(3211)} + s_{(2221)}$$