List of Exercises

Exercise 1 *Check if the following sets are ideals:*

1. $S_1 = \{p(x^2) | p(x) \in \mathbb{C}[x]\}$
   Polynomials in $\mathbb{C}[x]$ *with only even exponents*
2. $S_2 = \{p(x, y) \in \mathbb{C}[x, y] | p(0, 0) = 0\}$
   Polynomials whose total degree is greater than zero.
3. $S_3 = \{p(x, y) \in \mathbb{C}[x, y] | p(0, y) = 0\}$
   Polynomials whose degree in x *is greater than zero.*
4. $S_4 = \{p(x_1, \ldots, x_{n-1}) \in \mathbb{C}[x_1, \ldots, x_n]\}$
   Polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ *which do not depend on* x_n.

Exercise 2 *Prove that the intersection of two ideals is always an ideal. When is the union of two ideals an ideal?*

Exercise 3 *Let* I_1, I_2 *be ideals of* $\mathbb{C}[x_1, \ldots, x_n]$ *and* $f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_n]$. *Prove that the set* $\{g_1 f_1 + g_2 f_2 | g_1 \in I_1, g_2 \in I_2\}$ *is an ideal.*

Exercise 4 *Let* $e_1, \ldots, e_n$ *be a basis of a vector space* V *and let* $\phi_1, \ldots, \phi_n$ *be the dual basis i.e.*

$$\phi_i(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases}$$

i) *Prove that* $\phi_i \wedge \phi_j$, $1 \leq i < j \leq n$ *is a basis of* $\bigwedge^2(V^*)$.

   Hint: Prove that for each exterior two form ω *one has*

$$\omega = \sum_{1 \leq i < j \leq n} \omega(e_i, e_j) \phi_i \wedge \phi_j$$

   and that $\phi_i \wedge \phi_j$, $1 \leq i < j \leq n$ *are linearly independent.*

ii) *Prove that* $\phi_{i_1} \wedge \ldots \wedge \phi_{i_k}$, $1 \leq i_1 < \ldots < i_k \leq n$ *is a basis of* $\bigwedge^k(V^*)$.

Exercise 5 *Let* $v_1, v_2 \in \mathbb{R}^4$ *be linearly independent. Prove that* $v_1 \wedge v_2 = \rho w_1 \wedge w_2$ *with* $\rho \in \mathbb{R}^*$ *if and only if* w_1 *and* w_2 *span the same subspace as* v_1 *and* v_2.