1 Ring of invariants

Exercise 1. *Is* $1 \in k[x^2]$*? Is* $x^3 \in k[x^2]$*? Is* $x \in k[x,y]$*? Is* $xy \in k[x,y]$*?*

Exercise 2. *Let* $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Find* Mf *for the following polynomials:*

1. $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 - 11x_1^2x_2^2$.

2. $f(x_1, x_2, x_3) = 2x_1^2 - 11x_2x_3$

3. $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 - 11x_1x_2x_3 + 2x_3^2$.

Exercise 3. *Which of the examples in Exercise 2 are invariant under* $G = S_3$*?*

Exercise 4. *Give another example of an* S_3*-invariant. Can you guess generators for the ring:* $k[x,y,z]^{S_3}$.

2 Monomial ordering

1. Order the following terms in lex order:

   (a) $(1, 2, 3, 0), (0, 2, 5, 0)$

   (b) $(1, 2, 3, 4), (1, 2, 5, 2)$

2. Consider "degree-lex" order defined by $\alpha \leq_{dlex} \beta$ if

   i) $\alpha = \beta$ or

   ii) $|\alpha| = \sum_i \alpha_i < \sum_i \beta_i = |\beta|$ or

   iii) $|\alpha| = |\beta|$ and $\alpha <_{lex} \beta$.

   Order the following terms in dlex order:

   (a) $(1, 2, 3, 0), (0, 2, 5, 0)$

   (b) $(1, 2, 3, 4), (1, 2, 5, 2)$

3. Write the following polynomials in $\mathbb{R}[x, y, z]$ in decreasing term order using the lexicographic order with $x > y > z$. Then determine the multideg, LM, and LT.

   (a) $3xy - 5yz + 7xz$

   (b) $5 + 3x^2z - 2xy^4z^3 + 3z - 5x + 2y$

3 Ideals and the division algorithm

1. What elements make up the ideal $\langle x^2 \rangle$ in $\mathbb{R}[x]$? What elements make up the ideal $\langle x^2 \rangle$ in $\mathbb{R}[x, y]$?

2. Is $1 \in \langle x^2 \rangle$. Is $x^3 \in \langle x^2 \rangle$? Explain.

3. Is $y - 1 \in \langle x + 1, xy + 1 \rangle$?

4. Show that the set of polynomials in $\mathbb{Z}[x]$ whose coefficients are even integers make up an ideal of $\mathbb{Z}[x]$.

5. Calculate the remainder on dividing f by the given sets of polynomials S:

(a) $f = x^2yz + xz^2 - yz$ and $S = \{x^2 - y, y - z\}$

(b) $f = x^2yz + xz^2 - yz$ and $S = \{y - z, x^2 - y\}$

(c) $f = x^3y^2 - xyz + yz^2$ and $S = \{y - z, x^2 - yz, x + z^2\}$

4 Gröbner Bases

4.1 Buchberger's algorithm

1. Calculate the S-polynomials $S(f, g)$:

   (a) $f = x - y + z, g = x + y - 2z$

   (b) $f = x^2y - xy^2, g = xy - x$

   (c) $f = e_1 - y_1, g = e_2 - y_2$.

2. Construct a Gröbner basis for the following ideals in $\mathbb{R}[x, y, z]$:

   (a) $I = \langle x - y + z, x + y - 2z, 3x - y + 3z \rangle$

   (b) $I = \langle x^2 y - xy^2, xy - x \rangle$

   (c) $I = \langle e_1 - y_1, e_2 - y_2, e_3 - y_3 \rangle$

5 The ring of invariants

1. Is $f = -2x^3 + x^2y + x^2z + y^2x - 2y^3 + y^2z + z^2x + z^2y - 2z^3$ in $k[x, y, z]^{S_3}$?

2. Express f in terms of e_1, e_2, e_3. Hint: Use Buchberger's algorithm to find a Gröbner basis for $I = \langle e_1 - y_1, e_2 - y_2, e_3 - y_3 \rangle$ and then use the division algorithm.

3. Consider the cyclic group C_3 generated by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

   (a) Find the elements of C_3.

   (b) Find finitely many invariants which generate $k[x, y]^{C_3}$ using the Reynolds operator.

   (c) Can you find fewer invariants thta generate $k[x, y]^{C_3}$? Hint: check if $f_i \in k[f_1, \ldots, f_m]$ where $f_1, \ldots, f_m$ are the invariants you found in the preceeding exercise.