# Symmetric Functions in Noncommuting Variables

Mercedes H. Rosas
http://www.ma.usb.ve/~mrosas
and
Bruce E. Sagan
http://www.math.msu.edu/~sagan

IT OFTEN HELPS TO THINK IN TERMS OF NONCOMMUTING VARIABLES

## EXAMPLE

$\mathcal{G}$ = THE CLASS OF PLANE TREES (IN $n+1$ vertices)

$\mathcal{Z} = \{\bullet\}$

$n=3$ (4 vertices)

$$\mathcal{G} = \mathcal{Z} \times \mathrm{Sequence}(\mathcal{G})$$

$\frac{1}{1-G(z)}$

$$G(z) = \frac{z}{1-G(z)}$$

Hence,

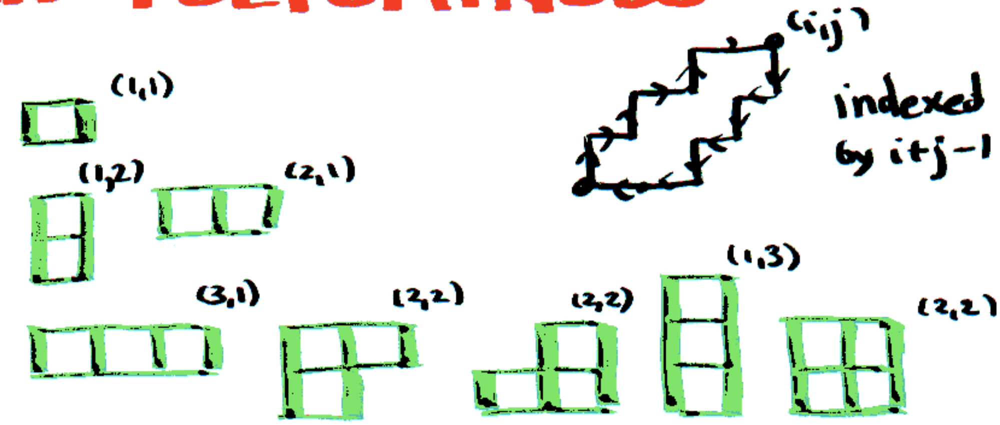
$$G(z) = \frac{1-\sqrt{1-4z}}{2}$$

# EXAMPLE 2 (PÓLYA-FLAJOLET)
# CATALAN POLYOMINOES

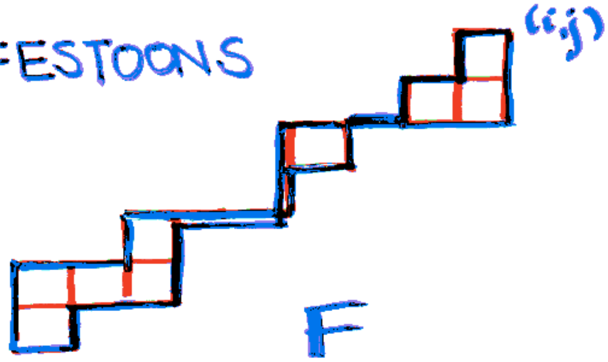
n=1

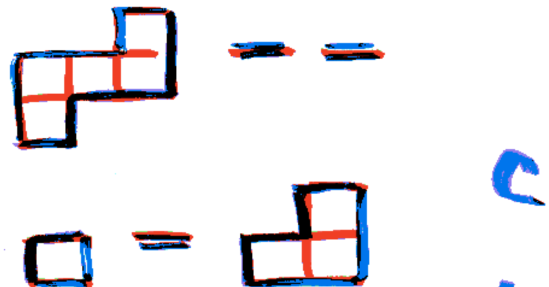
n=2

n=3




FESTOONS




irreducible parts

$$F = \frac{1}{1-C} \iff C = 1 - \frac{1}{F}$$

BLUE PATHS $(0,0) \to (k, n-k)$ $= \binom{n}{k} =$ PURPLE PATHS

FESTOONS 2n STEPS $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

$$\bar{F} = \sum_{n \geq 0} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}$$

WE HAVE TWO COPIES OF EACH CATALAN POLYOMINOE

$$F = \frac{1}{(1 - \underbrace{x - y}_{\text{degenerated cases}} - 2C(x,y))}$$

Catalan Polyominoes (two copies).

SET $x = y = z$ (make the variables commute)

$$F = \frac{1}{1 - 2z - 2C(z)}$$

the degenerated cases.

$$C(z) = \frac{1}{2}\left(1 - 2z - \sqrt{1-4z}\right)$$

$$= z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6 + O(z^7).$$

# Symmetric Functions

$\mathbb{Q}[x_1, x_2, \ldots] = \mathbb{Q}[x]$

$g \in \mathfrak{S}_n$ THE SYMMETRIC GROUP ACTS ON $f \in \mathbb{Q}[x]$

$g \in \mathfrak{S}_n \qquad g \cdot f(x_1, x_2, \ldots) = f(x_{g1}, x_{g2}, \ldots)$ ★

$(gi = i$ for $i > n)$

$f$ IS SYMMETRIC IF IT IS INVARIANT UNDER THE ACTION OF ALL $g \in \mathfrak{S}_n$ (and has bounded degree.)

SF IN NONCOMMUTING VARIABLES $\Pi(x)$.

$\mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle$

THE VARIABLES DO NOT COMMUTE.

THE ALGEBRA OF SYMMETRIC FUNCTIONS $\Pi(x)$ CONSISTS OF ALL FORMAL POWER SERIES

$f \in \mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle$

- INVARIANT UNDER $\star$.
- OF BOUNDED DEGREE.

# THE BASES OF $\Pi(x)$.

## SET PARTITION $\pi$ OF $[n]$.

$\pi = B_1 | B_2 | \cdots | B_\ell$ $\quad B_i$ BLOCKS

## THE MONOMIAL BASIS.

(in noncommuting variables.)

$$m_\pi = \sum_{i_1, i_2, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

positions

where $i_j = i_k$ iff $j$ and $k$ are in the same block of $\pi$.

$$m_{13|24} = x_1 x_2 x_1 x_2 + x_1 x_3 x_1 x_3 + \cdots + x_2 x_1 x_2 x_1 + \cdots$$

## THEY FORM A BASIS FOR $\Pi(x)$.

THE DIMENSION OF $(\Pi(x))_n$ IS $B_n$ (THE BELL NUMBERS)

number of set partitions of $[n]$.

the homogeneous piece of degree $n$.

SYMMETRIC FUNCTION

integer partitions

QUASI-SYMMETRIC FUNCTIONS

compositions

# THE POWER SUMS BASIS

(in noncommuting variables.)

$$p_\pi = \sum_{i_1, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

where $i_j = i_k$ if $j$ and $k$ are in the same block of $\pi$

both blocks have the same image.

$$p_{13|24} = x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 + x_1^4 + x_2^2 + \cdots$$

$$= m_{13|24} + m_{1234}$$

# THE ELEMENTARY BASIS

(in noncommuting variables.)

$$e_\pi = \sum_{i_1, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

where $i_j \neq i_k$ if $j$ and $k$ are in the same block of $\pi$.

$$e_{13|24} = x_1 x_1 x_2 x_2 + x_2 x_2 x_1 x_1 + x_1 x_2 x_2 x_3 + \cdots$$

$$= m_{12|34} + m_{14|23} + m_{14|2|3}$$

$$+ m_{1|23|4} + m_{1|2|34} + m_{1|2|3|4}.$$

THERE IS A NATURAL MAP

$$\left\{\begin{matrix}\text{SET}\\ \text{PARTITIONS}\end{matrix}\right\} \xrightarrow{\lambda} \left\{\begin{matrix}\text{INTEGER}\\ \text{PARTITIONS}\end{matrix}\right\}$$

$$\pi = B_1 | B_2 | \cdots | B_\ell \longrightarrow \lambda(\pi) = (|B_1|, \ldots |B_\ell|)$$

SIMILARLY, THERE IS THE FORGETFUL MAP.

$$\rho : \mathbb{Q}\langle\langle x \rangle\rangle \longrightarrow \mathbb{Q}[[x]]$$

IT MAKES THE VARIABLES TO COMMUTE.

NOTE THAT

$$\begin{array}{ccc} \mathbb{Q}\langle\langle x \rangle\rangle^{G_n} & \xrightarrow{\rho} & \mathbb{Q}[[x]]^{G_n} \\ \shortparallel & & \shortparallel \\ \Pi(x) & \xrightarrow{\rho} & \Lambda(x) \end{array}$$

SF IN NONCOMMUTING VARIABLES

SYMMETRIC FUNCTIONS

# THE MONOMIAL BASES

## AND THE FORGETFUL MAP.

$$\rho(m_\pi) = \pi^! \, m_{\lambda(\pi)}.$$

**proof:**

$B_1, B_2, \ldots B_k$ blocks of $\pi$ of size $b$.

$m_\pi$

- constant on the positions corresponding to each $B_i$. NO REPETITIONS.
- they are of the same size.

$k!$

- the variables in the $B_i$ can be switched with the variables in the $B_j$ position. in the projection.

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$$

$$\lambda^! = m_1! \, m_2! \ldots m_n! \quad (= \pi^!)$$

# CHANGE OF BASIS

i) $$p_\pi = \sum_{\sigma \geq \pi} m_\sigma$$ ← in the partition lattice.

ii) $$e_\pi = \sum_{\sigma \wedge \pi = \hat{0}} m_\sigma$$ ← in the partition lattice.

BY THE MÖBIUS INVERSION FORMULA.

i') $$m_\pi = \sum_{\sigma \geq \pi} \mu(\pi,\sigma)\, p_\sigma$$

ii') $$m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi,\sigma)}{\mu(\hat{0},\sigma)} \sum_{\tau \leq \sigma} \mu(\tau,\sigma)\, e_\tau$$

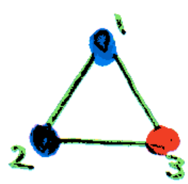
SIMILARLY, WE CAN COMPUTE THE SCALAR AND INNER PRODUCTS.

# STANLEY'S CHROMATIC SYMMETRIC FUNCTION

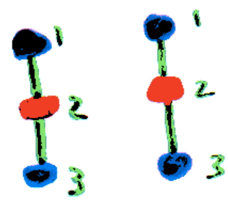
$\mathbf{x} = \{x_1, x_2, \dots\}$ commuting variables

$$X_G = \sum_{\kappa: V \to \mathbb{P}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

proper coloring, $|V| = n$

$G_1 =$ 

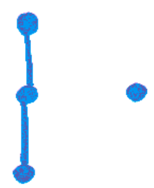

$$X_{G_1} = 3!\, m_{(1,1,1)} = \hat{m}_{(1,1,1)}$$

$G_2 =$ 


$$X_{G_2} = m_{(2,1)} + 3!\, m_{(1,1,1)}$$

# STANLEY-STEMBRIDGE'S CONJECTURE

$(P, \leq)$ POSET IS CALLED (3+1)-FREE IF IT HAS NO INDUCED SUBPOSET ISOMOR PHIC TO THE DISJOINT UNION OF A 1-CHAIN AND A 3-CHAIN



IF $P$ IS A (3+1)-FREE POSET AND ONE WRITES

$$X_{\underline{G(P)}} = \sum_{\lambda} c_\lambda e_\lambda \quad \text{e-positive}$$

THEN $c_\lambda \geq 0$ FOR ALL $\lambda$.

$G(P)$ = INCOMPARABILITY GRAPH

$V = P$

is an edge

$uv \in E(P) \Longleftrightarrow u$ and $v$ are incomparable

$X_{G(P)}$ DOES NOT SATISFY DELECTION-CONTRACTION LAW

# SAGAN AND GEBHARD

## CHROMATIC SF

(in noncommuting variables)

$$\hat{X}_G = \sum_{\kappa: V \to \mathbb{P}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

$\kappa: V \to \mathbb{P}$ ← proper coloring

WHERE THE $\{x_1, x_2, \ldots\}$ ARE NONCOMMUTING VARIABLES.

$\hat{X}_G$ SATISFIES THE DELECTION-CONTRACTION LAW.

P (3+1)- AND (2+2)-FREE

IT IS CALLED INDIFFERENCE GRAPH. (G(P))

THEY HAVE THE FOLLOWING STRUCTURE:

$\mathcal{I} = \{ I_1, \ldots, I_k \}$ FAMILY OF INTERVALS

$I_j \subseteq [n]$ FOR ALL $j$

$G(\mathcal{I})$ IS FORMED BY TAKING $V = [n]$ AND PUTTING A COMPLETE GRAPH ON THE VERTICES IN EACH INTERVAL

INDIFFERENCE-GRAPHS

THEOREM (GEBHARD-SAGAN)

$G(\mathcal{I})$ INDIFERENCE GRAPH SUCH THAT $|I_i \cap I_j| \leq 1 \ \forall i,j$ THEN $X_{G(\mathcal{I})}$ IS e-POSITIVE.

# THE SCHUR FUNCTIONS
(in noncommuting variables)

## MACMAHON SYMMETRIC FUNCTIONS

$X^{(1)} = \{x_1', x_2', \ldots\}$
$X^{(2)} = \{x_1'', x_2'', \ldots\}$
$\vdots$
$X^{(d)} = \{x_1^{(d)}, x_2^{(d)}, \ldots\}$

d-sets of commuting variables

$g \in G_n$ ACTS ON $f \in \mathbb{Q}[[X', X'', \ldots X^{(d)}]]$.
DIAGONALLY

$g\,f(x_1', x_1'', \ldots, x_2', x_2'', \ldots)$

$= f(x_{g1}', x_{g1}'', \ldots x_{g2}', x_{g2}'', \ldots)$

d = 2
diagonal harmonics

(of bounded degree) FOR ALL $g \in G_n$.

THE MONOMIAL

$x_1'^{a_1} x_1''^{b_1} \ldots x_1^{(d)c_1} x_2'^{a_2} x_2''^{b_2} \ldots x_2^{(d)c_2}$

HAS MULTIEXPONENT

$\vec{\lambda} = \{(a_1, b_1, \ldots, c_1), (a_2, b_2, \ldots, c_2), \ldots\}$

AND MULTIDEGREE

$\vec{n} = (a_1, b_1, \ldots, c_1) + (a_2, b_2, \ldots, c_2) + \cdots$

# THE MONOMIAL MACMAHON SYMMETRIC FUNCTION.

$m_{\vec{\lambda}}$ = THE SUM OF ALL MONOMIALS WITH MULTIEXPONENT $\vec{\lambda}$

multipartition

$$m_{\{(2,1),(3,0)\}} = \underbrace{x_1^{\prime 2}\, x_1^{\prime\prime}}_{(2,1)}\ \underbrace{x_2^{\prime 3}}_{(3,0)} + \underbrace{x_1^{\prime 3}}_{(3,0)}\ \underbrace{x_2^{\prime 2}\, x_2^{\prime\prime}}_{(2,1)} + \cdots$$

## THE ALGEBRA OF MACMAHON SYMMETRIC FUNCTIONS

$$\mathcal{M} = \mathcal{M}(x', x'', \ldots, x^{(d)})$$

$$= \operatorname{span}\{m_{\vec{\lambda}} : \text{all } \vec{\lambda}\}.$$

$$\mathcal{M}_{(1^d)} = \operatorname{span}\{m_{\vec{\lambda}} : \text{all } \vec{\lambda} \vdash \overbrace{(1,1,\ldots,1)}^{d}$$

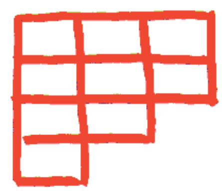
## THERE IS A VECTOR SPACE ISOMORPHISM

$$\bigoplus_{d \geq 0} \mathcal{M}_{(1^d)} \longrightarrow \Pi(x)$$

$$m_{\{(1,0,1,1,0),(0,1,0,0,1)\}} \longmapsto m_{134/25}$$

# SCHUR FUNCTIONS.

$\lambda = (3,3,2,1)$

$\lambda =$  $\lambda =$  FERRERS DIAGRAM

A YOUNG TABLEAU OF SHAPE $\lambda = (3,3,2,1)$

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \qquad \lambda(T) = \lambda \qquad \begin{array}{c} x_1^2 x_2^2 x_3^2 x_5^2 x_6 \\ \shortparallel \\ M_T \end{array}$$

THE SCHUR FUNCTION $S_\lambda$

$$S_\lambda = \sum_{\lambda(T)=\lambda} M_T$$

SHAPE OF THE TABLEAU

$\lambda = (2,1)$

$$T \quad \begin{array}{cc} 1 & 1 \\ 2 \end{array} \quad \begin{array}{cc} 1 & 2 \\ 2 \end{array}, \ldots, \begin{array}{cc} 1 & 2 \\ 3 \end{array}, \begin{array}{cc} 1 & 3 \\ 2 \end{array}, \ldots$$

$$S_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_3 + \cdots$$

# A PRIMED YOUNG TABLEAU $T'$

## OF SHAPE $\vec{\lambda}$ AND MULTIDEGREE $\vec{n} = (n_1, n_2 \ldots)$

IS OBTAINED FROM $T$

- BY PUTTING PRIMES ON $n_1$ ELEMENTS OF $T$.
- DOUBLE PRIMES ON $n_2$ ELEMENTS OF $T$, ...

THE CORRESPONDING MACMAHON SCHUR FUNCTION IS

$$S_{\lambda}^{\vec{n}} = \sum_{\lambda(T') = \lambda} M_{T'}, \qquad M_{T'} = \prod_{i^{(j)} \in T} x_i^{(j)}$$

$$S_{(2,1)}^{(1,1,1)} = \underset{\begin{smallmatrix} 1' & 1'' \\ 2''' & \end{smallmatrix}}{x_1' x_1'' x_2'''} + \underset{\begin{smallmatrix} 1'' & 1' \\ 2''' & \end{smallmatrix}}{x_1' x_1'' x_2'''} + \cdots$$

$$+ \underset{\begin{smallmatrix} 1''' & 1'' \\ 2' & \end{smallmatrix}}{x_1'' x_1''' x_2'} + \cdots$$

## THE JACOBI-TRUDI DETERMINANT

FOR THE PARTITION

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$

$$s_\lambda = \left| h_{\lambda_i - i + j} \right|$$

## THE JACOBI-TRUDI DETERMINANT

(noncommuting version)

GIVEN A PARTITION $\lambda$ AND A VECTOR $\vec{m}$ WITH $\lambda, \vec{m} \vdash m$

$$s_\lambda^{\vec{m}} = \det \left( \sum_{\vec{q} \vdash \lambda_i - i + j} h_{\vec{q}} \right)$$

WITH THE CONVENTION THAT IF THE PRODUCT OF TWO MONOMIALS IN THE DETERMINANT DOES NOT HAVE MULTIDEGREE $\vec{m}$, THEN THAT PRODUCT IS ZERO.

- THE SCHUR FUNCTIONS (in noncommuting variables) INDEXED BY AN INTEGER PARTITION ARE ORTHOGONAL:

$$\langle S_\lambda, S_\mu \rangle = (n!)^2 \cdot \delta_{\lambda,\mu}.$$

- THEY SATISFY AN ANALOG OF R-S-K ALGORITHM.

- AND OF CAUCHY'S IDENTITIES:

$$\sum_{m \geq 0} \sum_{\lambda, \vec{m}, \vec{p} \vdash m} S_\lambda^{\vec{m}}(x^{(1)}, \dots, x^{(n)}) \, S_\lambda^{\vec{p}}(y^{(1)}, \dots, y^{(n)})$$

$$= \prod_{i,j \geq 1} \frac{1}{1 - \sum_{k,\ell=1}^{n} x_i^{(k)} y_j^{(\ell)}}$$

BUT THERE ARE NOT A BASIS.

# SCHUR FUNCTIONS
## INDEXED BY SET PARTITIONS

- THERE IS NO ORTHONORMAL INTEGER BASIS.

WE WANT

I. $\sum_{\sigma \in \lambda} S_\sigma = S_\lambda$

- ORTHOGONALITY (AS MUCH AS POSSIBLE)

II. SYMMETRY $S_\pi$

- $\sigma \leq \pi$
  $\sigma' \leq \pi$
  $\lambda(\sigma) = \lambda(\sigma')$
  $[m_\sigma] S_\pi = [m_{\sigma'}] S_\pi$
- $\sigma \nleq \pi$
  $\sigma' \nleq \pi$
  $\lambda(\sigma) = \lambda(\sigma')$
  $[m_\sigma] S_\pi = [m_{\sigma'}] S_\pi$

$$S_{1|2|3} = m_{1|2|3}$$

$$S_{12|3} = 2!\, m_{12|3} + 2!\, \frac{1}{3}\, m_{1|2|3}$$

$$S_{13|2} = 2!\, m_{13|2} + 2!\, \frac{1}{3}\, m_{1|2|3}$$

$$S_{1|23} = 2!\, m_{1|23} + 2!\, \frac{1}{3}\, m_{1|2|3}$$

$$S_{123} = 3!\, m_{123} + 2!\, m_{12|3} + 2!\, m_{13|2} + 2!\, m_{1|23} + m_{1|2|3}$$

| | $S_{1\|2\|3}$ | $S_{12\|3}$ | $S_{13\|2}$ | $S_{1\|23}$ | $S_{21}$ | $S_{123}$ |
|---|---|---|---|---|---|---|
| $S_{1\|2\|3}$ | $(3!)^2$ | 0 | 0 | 0 | 0 | 0 |
| $S_{12\|3}$ | 0 | $4 \cdot 5$ | $-4$ | $-4$ | $4 \cdot 3$ | 0 |
| $S_{13\|2}$ | 0 | $-4$ | $4 \cdot 5$ | $-4$ | $4 \cdot 3$ | 0 |
| $S_{1\|23}$ | 0 | $-4$ | $-4$ | $4 \cdot 5$ | $4 \cdot 3$ | 0 |
| $S_{21}$ | 0 | $4 \cdot 3$ | $4 \cdot 3$ | $4 \cdot 3$ | $(3!)^2$ | 0 |
| $S_{\cancel{1\|2\|3}}$ $_{123}$ | 0 | 0 | 0 | 0 | 0 | $(3!)^2$ |

NOTE: $\langle S_1, S_1 \rangle = 1 \cdot 1$

$\langle S_{12}, S_{12} \rangle = 2 \cdot 2$ $\quad \langle S_{12|3}, S_{12|3} \rangle = 2^2 \cdot 5$

$\langle S_{12|3|4}, S_{12|3|4} \rangle = 2^4 \cdot 13$ $\quad \boxed{?\ \langle S_{12|3|4|5}, S_{12|3|4|5} \rangle = 2^8 \cdot 34}$