lyryx with Open Texts

LINEAR ALGEBRA
with APPLICATIONS

Lecture Notes
by Karen Seyffarth

Determinants and Diagonalization

lyryx

advancing learning

Champions of Access to Knowledge

OPEN TEXT

ONLINE ASSESSMENT

SUPPORT

INSTRUCTOR SUPPLEMENTS

Contact Lyryx Today!

info@lyryx.com

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Ilijas Farah, York University**

BE A CHAMPION OF OER!

Contribute suggestions for improvements, new content, or errata:
A new topic
A new example or problem
A new or better proof to an existing theorem
Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Consider the matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Observe that P is invertible (why?), and that

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}.

How can we do this efficiently?

Consider the matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Observe that P is invertible (why?), and that

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Furthermore,

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D,$$

where D is a diagonal matrix.

Example (continued)

This is significant, because

$$\begin{aligned} P^{-1}AP &= D \\ P(P^{-1}AP)P^{-1} &= PDP^{-1} \\ (PP^{-1})A(PP^{-1}) &= PDP^{-1} \\ IAI &= PDP^{-1} \\ A &= PDP^{-1}, \end{aligned}$$

Example (continued)

This is significant, because

$$\begin{aligned} P^{-1}AP &= D \\ P(P^{-1}AP)P^{-1} &= PDP^{-1} \\ (PP^{-1})A(PP^{-1}) &= PDP^{-1} \\ IAI &= PDP^{-1} \\ A &= PDP^{-1}, \end{aligned}$$

and so

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1} \cdots P)DP^{-1} \\ &= PDIDIDI \cdots IDP^{-1} \\ &= PD^{100}P^{-1}. \end{aligned}$$

Example (continued)

Now,

$$D^{100} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2 \cdot 2^{100} + 5^{100} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2^{101} + 5^{100} \end{bmatrix} \end{aligned}$$

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$, *then* $A^k = PD^kP^{-1}$ *for each* $k = 1, 2, 3, \ldots$

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$, *then* $A^k = PD^kP^{-1}$ *for each* $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

Theorem (Diagonalization and Matrix Powers)

If $A = PDP^{-1}$*, then* $A^k = PD^kP^{-1}$ *for each* $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

Problem

- *When is it possible to diagonalize a matrix?*
- *How do we find a diagonalizing matrix?*

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $\mathbf{x} \neq \mathbf{0}$ an n-vector. If $A\mathbf{x} = \lambda\mathbf{x}$, then λ is an eigenvalue of A, and $\mathbf{x}$ is an eigenvector of A corresponding to λ, or a λ-eigenvector.

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $\mathbf{x} \neq \mathbf{0}$ an n-vector. If $A\mathbf{x} = \lambda\mathbf{x}$, then λ is an eigenvalue of A, and $\mathbf{x}$ is an eigenvector of A corresponding to λ, or a λ-eigenvector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{x}.$$

This means that 3 is an eigenvalue of A, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 3 (or a 3-eigenvector of A).

Finding all Eigenvalues and Eigenvectors of a Matrix

Suppose that A is an $n \times n$ matrix, $\mathbf{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\mathbf{x} = \lambda\mathbf{x}$.

Finding all Eigenvalues and Eigenvectors of a Matrix

Suppose that A is an $n \times n$ matrix, $\mathbf{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\mathbf{x} = \lambda\mathbf{x}$.
Then

$$\begin{aligned} \lambda\mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ \lambda I\mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ (\lambda I - A)\mathbf{x} &= \mathbf{0} \end{aligned}$$

Finding all Eigenvalues and Eigenvectors of a Matrix

Suppose that A is an $n \times n$ matrix, $\mathbf{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\mathbf{x} = \lambda\mathbf{x}$.
Then

$$\begin{aligned} \lambda\mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ \lambda I\mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ (\lambda I - A)\mathbf{x} &= \mathbf{0} \end{aligned}$$

Since $\mathbf{x} \neq 0$, the matrix $\lambda I - A$ has no inverse, and thus

$$\det(\lambda I - A) = 0.$$

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(x) = \det(xI - A).$$

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(x) = \det(xI - A).$$

Example

The characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ is

$$\begin{aligned} c_A(x) &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10 \end{aligned}$$

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. *The eigenvalues of A are the roots of $c_A(x)$.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nontrivial solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$.*

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. *The eigenvalues of A are the roots of $c_A(x)$.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nontrivial solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$.*

Example (continued)

For $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$, we have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n \times n$ matrix.

1. *The eigenvalues of A are the roots of $c_A(x)$.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nontrivial solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$.*

Example (continued)

For $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$, we have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.
To find the 2-eigenvectors of A, solve $(2I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

Example (continued)

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

To find the 5-eigenvectors of A, solve $(5I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

To find the 5-eigenvectors of A, solve $(5I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R}.$$

Definition

A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)\mathbf{x} = \mathbf{0}$, where λ is an eigenvalue of A.

Definition

A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)\mathbf{x} = \mathbf{0}$, where λ is an eigenvalue of A.

Example (continued)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are basic eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

corresponding to eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, respectively.

Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and find corresponding basic eigenvectors.

Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and find corresponding basic eigenvectors.

$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and find corresponding basic eigenvectors.

$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3) \begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix}$$

Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of A, and find corresponding basic eigenvectors.

$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3)\begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix}$$

$$c_A(x) = (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1).$$

Example (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Example (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \left[\begin{array}{r} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{array}\right] = t \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]$, $t \in \mathbb{R}$.

Example (continued)

Therefore, the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

Choosing $t = 2$ gives us $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as a basic eigenvector corresponding to $\lambda_1 = 3$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $s \in \mathbb{R}$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $s \in \mathbb{R}$.

Choosing $s = 1$ gives us $\mathbf{x}_2 = \left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$ as a basic eigenvector corresponding to $\lambda_2 = 2$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

Choosing $r = 1$ gives us $\mathbf{x}_3 = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$ as a basic eigenvector corresponding to $\lambda_3 = 1$.

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does the linear transformation affect the eigenvectors of the matrix?

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does the linear transformation affect the eigenvectors of the matrix?

Definition

Let V be a nonzero vector in $\mathbb{R}^2$. We denote by L_V the unique line in $\mathbb{R}^2$ that contains V and the origin.

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does the linear transformation affect the eigenvectors of the matrix?

Definition

Let V be a nonzero vector in $\mathbb{R}^2$. We denote by L_V the unique line in $\mathbb{R}^2$ that contains V and the origin.

Lemma

Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$ *be a nonzero vector in* $\mathbb{R}^2$. *Then* L_V *is the set of all scalar multiples of* V, *i.e.,*

$$L_V = \mathbb{R}V = \{tV \mid t \in \mathbb{R}\}.$$

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L,

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L, i.e., $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L,
i.e., $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
i.e., $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L,
i.e., $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
i.e., $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,
i.e., $\mathbf{x}$ is an eigenvector of A.

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be A-invariant if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L,
i.e., $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
i.e., $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,
i.e., $\mathbf{x}$ is an eigenvector of A.

Theorem (A-Invariance)

Let A be a 2×2 matrix and let $V \neq 0$ be a vector in $\mathbb{R}^2$. Then L_V is A-invariant if and only if V is an eigenvector of A.

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L,
i.e., $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
i.e., $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,
i.e., $\mathbf{x}$ is an eigenvector of A.

Theorem (A-Invariance)

Let A be a 2×2 matrix and let $V \neq 0$ be a vector in $\mathbb{R}^2$. Then L_V is A-invariant if and only if V is an eigenvector of A.

This theorem provides a geometrical method for finding the eigenvectors of a 2×2 matrix.

Example

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.

Example

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.
The matrix that induces Q_m is

$$A = \frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right].$$

Example

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.
The matrix that induces Q_m is

$$A = \frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right].$$

Claim. $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ m \end{array}\right]$ is an eigenvector of A corresponding to eigenvalue $\lambda = 1$.

Example

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.
The matrix that induces Q_m is

$$A = \frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right].$$

Claim. $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ m \end{array}\right]$ is an eigenvector of A corresponding to eigenvalue $\lambda = 1$.
The reason for this: $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ m \end{array}\right]$ lies in the line $y = mx$, and hence

$$Q_m \left[\begin{array}{c} 1 \\ m \end{array}\right] = \left[\begin{array}{c} 1 \\ m \end{array}\right], \text{ implying that } A \left[\begin{array}{c} 1 \\ m \end{array}\right] = 1 \left[\begin{array}{c} 1 \\ m \end{array}\right].$$

Example (continued)

More generally, any vector $\begin{bmatrix} k \\ km \end{bmatrix}$, $k \neq 0$, lies in the line $y = mx$ and is an eigenvector of A.
Another way of saying this is that the line $y = mx$ is A-invariant for the matrix

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Claim. A has no real eigenvectors unless θ is an integer multiple of π, i.e., $\pm\pi, \pm 2\pi, \pm 3\pi$, etc.

Example

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Claim. A has no real eigenvectors unless θ is an integer multiple of π, i.e., $\pm\pi, \pm 2\pi, \pm 3\pi$, etc.

The reason for this: a line L in $\mathbb{R}^2$ is A invariant if and only if θ is an integer multiple of π.

Diagonalization

Notation.

An $n \times n$ diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

is written $\text{diag}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

Diagonalization

Notation.

An $n \times n$ diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

is written $\text{diag}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

Recall that if A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix so that $P^{-1}AP$ is diagonal, then P is called a diagonalizing matrix of A, and A is diagonalizable.

Theorem (Matrix Diagonalization)

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ so that*

$$P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right]$$

   is invertible.

Theorem (Matrix Diagonalization)

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ so that*

$$P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right]$$

   is invertible.

2. *If P is invertible, then*

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

   where λ_i is the eigenvalue of A corresponding to the eigenvector $\mathbf{x}_i$, i.e., $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$.

Example

$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$ has eigenvalues and corresponding basic eigenvectors

$$\lambda_1 = 3 \text{ and } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Example

$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$ has eigenvalues and corresponding basic eigenvectors

$$\lambda_1 = 3 \text{ and } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Example

$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$ has eigenvalues and corresponding basic eigenvectors

$$\lambda_1 = 3 \text{ and } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Then P is invertible (check this!), so by the above Theorem,

$$P^{-1}AP = \text{diag}(3, 2, 1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note

It is not always possible to find n eigenvectors so that P is invertible.

Note

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$.

Note

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$. Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

Note

It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$. Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

A has only one eigenvalue, $\lambda_1 = 2$, with multiplicity three.

To find the 2-eigenvectors of A, solve the system $(2I - A)\mathbf{x} = \mathbf{0}$.

Example (continued)

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$\mathbf{x} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}.$$

Example (continued)

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$\mathbf{x} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}.$$

Since the system has only **two** basic solutions, there are only two basic eigenvectors, implying that the matrix A is **not diagonalizable**.

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = -3$ of multiplicity one.

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \left[\begin{array}{c} s \\ t \\ 0 \end{array}\right]$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$$

Example (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \left[\begin{array}{c} -\frac{1}{4}t \\ 0 \\ t \end{array}\right]$, $t \in \mathbb{R}$ so a basic eigenvector corresponding to $\lambda_2 = -3$ is

$$\left[\begin{array}{r} -1 \\ 0 \\ 4 \end{array}\right]$$

Example (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Example (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible,

Example (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible, and

$$P^{-1}AP = \text{diag}(-3, 1, 1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem (Matrix Diagonalization Test)

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, i.e., the solution to $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has m parameters.

Theorem (Matrix Diagonalization Test)

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, i.e., the solution to $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has m parameters.

A special case of this is

Theorem (Distinct Eigenvalues and Diagonalization)

An $n \times n$ matrix with distinct eigenvalues is diagonalizable.

Example

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Example

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

First,

$$c_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2),$$

so A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = 2$ (of multiplicity one).

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\mathbf{x} = \left[\begin{array}{c} s \\ 0 \\ 0 \end{array}\right]$, $s \in \mathbb{R}$.

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\mathbf{x} = \left[\begin{array}{c} s \\ 0 \\ 0 \end{array}\right]$, $s \in \mathbb{R}$.

Since $\lambda_1 = 1$ has multiplicity two, but has only one basic eigenvector, A is not diagonalizable.

Problem

Let $A = \left[\begin{array}{rrr} 8 & 5 & 8 \\ 0 & -1 & 0 \\ -4 & -5 & -4 \end{array} \right]$.

- *Show that* 4 *is an eigenvalue of* A*, and find a corresponding basic eigenvector.*
- *Verify that* $\left[\begin{array}{rrr} 1 & -1 & -1 \end{array} \right]$ *is an eigenvector os* A*, and find its corresponding eigenvalue.*

Linear Dynamical Systems

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector V_0;

Linear Dynamical Systems

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector V_0;
- a matrix recursion defining $V_1, V_2, V_3, \ldots$ by $V_{k+1} = AV_k$; i.e.,

Linear Dynamical Systems

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector V_0;
- a matrix recursion defining $V_1, V_2, V_3, \ldots$ by $V_{k+1} = AV_k$; i.e.,

$$\begin{aligned} V_1 &= AV_0 \\ V_2 &= AV_1 = A(AV_0) = A^2 V_0 \\ V_3 &= AV_2 = A(A^2 V_0) = A^3 V_0 \\ \vdots &\quad \vdots \quad \vdots \\ V_k &= A^k V_0. \end{aligned}$$

Linear Dynamical Systems

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector V_0;
- a matrix recursion defining $V_1, V_2, V_3, \ldots$ by $V_{k+1} = AV_k$; i.e.,

$$\begin{array}{rcl} V_1 & = & AV_0 \\ V_2 & = & AV_1 = A(AV_0) = A^2 V_0 \\ V_3 & = & AV_2 = A(A^2 V_0) = A^3 V_0 \\ \vdots & \vdots & \vdots \\ V_k & = & A^k V_0. \end{array}$$

Linear dynamical systems are used, for example, to model the evolution of populations over time.

If A is diagonalizable, then

$$P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.

If A is diagonalizable, then

$$P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.

Thus $A = PDP^{-1}$, and $A^k = PD^kP^{-1}$. Therefore,

$$V_k = A^kV_0 = PD^kP^{-1}V_0.$$

Example

Consider the linear dynamical system $V_{k+1} = AV_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for V_k.

Example

Consider the linear dynamical system $V_{k+1} = AV_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for V_k.

First, $c_A(x) = (x-2)(x+1)$, so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and thus is diagonalizable.

Example

Consider the linear dynamical system $V_{k+1} = AV_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for V_k.

First, $c_A(x) = (x-2)(x+1)$, so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and thus is diagonalizable.

Solve $(2I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\mathbf{x} = \begin{bmatrix} s \\ s \end{bmatrix}$, $s \in \mathbb{R}$, and basic solution $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example (continued)

Solve $(-I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{rr|r} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\mathbf{x} = \left[\begin{array}{c} 0 \\ t \end{array}\right]$, $t \in \mathbb{R}$, and basic solution $\mathbf{x}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$.

Example (continued)

Solve $(-I - A)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{rr|r} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has general solution $\mathbf{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$, and basic solution $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a diagonalizing matrix for A,

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Example (continued)

Therefore,

$$\begin{aligned} V_k &= A^k V_0 \\ &= PD^k P^{-1} V_0 \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2^k & 0 \\ 2^k & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix} \end{aligned}$$

Remark

Often, instead of finding an exact formula for V_k, it suffices to estimate V_k as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Remark

Often, instead of finding an exact formula for V_k, it suffices to estimate V_k as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Suppose that

$$V_k = PD^k P^{-1} V_0,$$

and assume that A has a dominant eigenvalue, λ_1, with corresponding basic eigenvector $\mathbf{x}_1$ as the first column of P.
For convenience, write $P^{-1}V_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Then

$$\begin{aligned}
V_k &= PD^kP^{-1}V_0 \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array}\right] \left[\begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right] \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \\
&= \lambda_1^k\left(b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right)
\end{aligned}$$

Then

$$
\begin{aligned}
V_k &= PD^kP^{-1}V_0 \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array}\right] \left[\begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right] \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \\
&= \lambda_1^k\left(b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Then

$$
\begin{aligned}
V_k &= PD^kP^{-1}V_0 \\
&= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \\
&= \lambda_1^k \left(b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k \mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k \mathbf{x}_n \right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Therefore, for large values of k, $V_k \approx \lambda_1^k b_1\mathbf{x}_1$.

Example

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate V_k for large values of k.

Example

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate V_k for large values of k.

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

Example

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate V_k for large values of k.

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

As before $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_1 = 2$, and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_2 = -1$, giving us

Example

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate V_k for large values of k.

In our previous example, we found that A has eigenvalues 2 and -1. This means that $\lambda_1 = 2$ is a dominant eigenvalue; let $\lambda_2 = -1$.

As before $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_1 = 2$, and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basic eigenvector for $\lambda_2 = -1$, giving us

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Example (continued)

$$P^{-1}V_0 = \left[\begin{array}{rr} 1 & 0 \\ -1 & 1 \end{array}\right]\left[\begin{array}{r} 1 \\ -1 \end{array}\right] = \left[\begin{array}{r} 1 \\ -2 \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]$$

Example (continued)

$$P^{-1}V_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$V_k \approx \lambda_1^k b_1 \mathbf{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}$$

Example (continued)

$$P^{-1}V_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$V_k \approx \lambda_1^k b_1 \mathbf{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}$$

Let's compare this to the formula for V_k that we obtained earlier:

$$V_k = \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix}$$