Symmetric functions =

polynomials in variables h_r $\deg(h_r) = r$

$$Sym = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$

Spanned by monomials in the h_r

$$\deg(h_{i_1} h_{i_2} h_{i_3} \cdots h_{i_d}) = i_1 + i_2 + \cdots + i_d$$

Example: monomials of degree 3

$$h_1 h_1 h_1, \quad h_2 h_1, \quad h_1 h_2, \quad h_3$$

but $h_2 h_1 = h_1 h_2$

basis $h_1 h_1 h_1, \quad h_2 h_1, \quad h_3$

Monomials are all of the form

$h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$ and to make sure that we take a unique set of representatives

choose $h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$

define $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$

where λ is a partition of the integer n.

that is $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ and
$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$

Example: a basis for the symmetric functions of degree 4 is

$h_4, h_{31}, h_{22}, h_{211}, h_{1111}$

Sym is a vector space because we can take linear combinations of basis elements and we consider that as an element of Sym too.

e.g. $7h_{42} - 6h_6 + \frac{1}{2}h_3 + 4h_1 - 14$

It is infinite dimensional because there are an infinite # of basis elements...

$1, h_1, h_2, h_{11}, h_3, h_{21}, h_{111}, h_4, h_{31}, h_{22}, \ldots$

But what is a little special about this vector space is that we can break it into finite components

Sym_r = linear span of h_λ for λ partition of r

then $$\mathrm{Sym} = \bigoplus_{r \geq 0} \mathrm{Sym}_r$$

The symmetric functions forms an <u>algebra</u>
(this just means that it is a vector space + mutiplication)
and the product comes from the mutiplication of polynomials:

e.g. $$(3h_1 + h_2)(h_{21} + 2h_3) = 3h_{211} + 6h_{31} + h_{221} + 2h_{32}$$

the multiplication is "easy" and we describe what happens on a basis

$$h_\lambda \cdot h_\mu = h_{\mathrm{sort}(\lambda \cup \mu)}$$
$$= h_{\mathrm{sort}(\lambda_1, \lambda_2, \ldots, \lambda_\ell, \mu_1, \mu_2, \ldots, \mu_m)}$$

e.g. $$h_{32} \cdot h_{2111} = h_{322111}$$
$$h_1 \cdot h_{2211} \cdot h_{32} \cdot h_{321} = h_{332222111 1}$$