The Schur functions - A new basis of Sym

$$\mathrm{Sym} = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$

Take a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Define

$$S_\lambda := \det \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \cdots & h_{\lambda_2+\ell-2} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & h_{\lambda_3} & \cdots & h_{\lambda_3+\ell-3} \\ & & & \ddots & \\ h_{\lambda_\ell-\ell+1} & h_{\lambda_\ell-\ell+2} & \cdots & h_{\lambda_\ell-1} & h_{\lambda_\ell} \end{vmatrix}$$

where we take as convention $h_0 = 1$ and $h_{-k} = 0$ for $k \geq 0$.

For example

$$\lambda = (4) \qquad s_4 = \det \begin{vmatrix} h_4 \end{vmatrix} = h_4$$

$$\lambda = (32) \qquad s_{32} = \det \begin{vmatrix} h_3 & h_4 \\ h_1 & h_2 \end{vmatrix} = h_3 h_2 - h_4 h_1$$

$$= h_{32} - h_{41}$$

$$\lambda = (221) \qquad s_{221} = \det \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ h_{-1} & h_0 & h_1 \end{vmatrix}$$

$$= \det \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$$

$$= h_2(h_{21} - h_3) - h_1(h_{31} - h_4)$$

$$= h_{221} - h_{32} - h_{311} + h_{41}$$

The Schur functions indexed by partitions form a basis for the vector space Sym.

This is because they are "triangularly" related to the homogeneous basis.

That is

$$s_\lambda = h_\lambda + \text{terms which are lexicographically larger.}$$

e.g.

$$\begin{aligned} s_{111} &= h_{111} - 2h_{21} + h_3 \\ s_{21} &= \phantom{h_{111} - 2} h_{21} - h_3 \\ s_3 &= \phantom{h_{111} - 2h_{21} +} h_3 \end{aligned}$$

$$\begin{bmatrix} s_{111} \\ s_{21} \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{111} \\ h_{21} \\ h_3 \end{bmatrix}$$

Since upper triangular matrices can be inverted, ~~we~~ we can write h_λ as a linear combination of the s_μ too.

$$\begin{bmatrix} h_{111} \\ h_{21} \\ h_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} s_{111} \\ s_{21} \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{111} \\ s_{21} \\ s_3 \end{bmatrix}$$

The transition matrix between the h-basis and the Schur basis is always positive. This is an interesting fact about Schur functions. They are the "fundamental" basis of Sym because they arise with lots of interesting combinatorial and algebraic properties.