For an arbitrary partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell),$$

$$S_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+\ell-2} \\ \vdots & & \ddots & \vdots \\ h_{\lambda_\ell-\ell+1} & h_{\lambda_\ell-\ell+2} & \cdots & h_{\lambda_\ell} \end{vmatrix}$$

With $h_0 = 1$ and $h_{-k} = 0$ for all k, $k = 1, 2, 3, \ldots$

Let $\quad F = 3h_{31} + h_{22} - h_{1111}$

$$S_{1111} = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix}$$

$$= h_1 \begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} - \begin{vmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix}$$

$$= h_1 \left(h_1 \begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix} - \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix} \right) - \left(h_2 \begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix} - \begin{vmatrix} h_3 & h_4 \\ 1 & h_1 \end{vmatrix} \right)$$

$$= h_1 \left[h_1 (h_{11} - h_2) - (h_{21} - h_3) \right] - \left[h_2 (h_{11} - h_2) - (h_3 - h_4) \right]$$

$$S_{1111} = h_{1111} - 3h_{211} + h_{22} + 2h_{31} - h_4 .$$

$$S_{1111} = h_{1111} - 3h_{211} + h_{22} + 2h_{31} - h_4$$

$$S_{211} = h_{211} - h_{22} - h_{31} + h_4$$

$$S_{22} = h_{22} - h_{31} + 0$$

$$S_{31} = h_{31} - h_4$$

$$S_4 = h_4$$

$$\begin{bmatrix} S_{1111} \\ S_{211} \\ S_{22} \\ S_{31} \\ S_4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{1111} \\ h_{211} \\ h_{22} \\ h_{31} \\ h_4 \end{bmatrix}$$

$$\begin{bmatrix} h_{1111} \\ h_{211} \\ h_{22} \\ h_{31} \\ h_4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} s_{1111} \\ s_{211} \\ s_{22} \\ s_{31} \\ s_4 \end{bmatrix}$$

$$\begin{bmatrix} h_{1111} \\ h_{211} \\ h_{22} \\ h_{31} \\ h_4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{1111} \\ s_{211} \\ s_{22} \\ s_{31} \\ s_4 \end{bmatrix}$$

So:

$$h_{1111} = s_{1111} + 3s_{211} + 2s_{22} + 3s_{31} + s_4$$

$$h_{211} = s_{211} + s_{22} + 2s_{31} + s_4$$

$$h_{22} = s_{22} + s_{31} + s_4$$

$$h_{31} = s_{31} + s_4$$

$$h_4 = s_4 .$$

Recall: $F = 3h_{31} + h_{22} - h_{1111}$

Substitute:

$$F = 3(s_{31} + s_4) + (s_{22} + s_{31} + s_4) - (s_{1111} + 3s_{211} + 2s_{22} + 3s_{31} + s_4)$$

$$= 3s_{31} + 3s_4 + s_{22} + s_{31} + s_4 - s_{1111} - 3s_{211} - 2s_{22} - 3s_{31} - s_4 .$$

$$F = -s_{1111} - 3s_{211} - s_{22} + s_{31} + 3s_4 .$$

Another method:

$$F = 3h_{31} + h_{22} - h_{1111}$$

Let's put this in lexicographic order:

$$F = -h_{1111} + h_{22} + 3h_{31}$$

Since $\quad S_{1111} = h_{1111} - 3h_{211} + h_{22} + 2h_{31} - h_4$

$\therefore \quad h_{1111} = S_{1111} + 3h_{211} - h_{22} - 2h_{31} + h_4.$

Substitute into F:

$$F = -(S_{1111} + 3h_{211} - h_{22} - 2h_{31} + h_4) + h_{22} + 3h_{31}$$

$$= -S_{1111} - 3h_{211} + 2h_{22} + 5h_{31} - h_4.$$

Now we look for a Schur basis containing a term h_{211}.

$$S_{211} = h_{211} - h_{22} - h_{31} + h_4$$

so $$h_{211} = S_{211} + h_{22} + h_{31} - h_4.$$

Substitute into F:

$$F = -S_{1111} - 3(S_{211} + h_{22} + h_{31} - h_4) + 2h_{22} + 5h_{31} - h_4.$$

$$= -S_{1111} - 3S_{211} - h_{22} + 2h_{31} + 2h_4$$

$$S_{22} = h_{22} - h_{31}$$

so $$h_{22} = S_{22} + h_{31}.$$

Substitute into F: $F = -S_{1111} - 3S_{211} - (S_{22} + h_{31}) + 2h_{31} + 2h_4.$

$$F = -S_{1111} - 3S_{211} - S_{22} + h_{31} + 2h_4.$$

$$S_{31} = h_{31} - h_4$$

so $h_{31} = S_{31} + h_4$.

substitute into F:

$$F = -S_{1111} - 3S_{211} - S_{22} + S_{31} + h_4 + 2h_4 .$$

$$= -S_{1111} - 3S_{211} - S_{22} + S_{31} + 3h_4 .$$

$$S_4 = h_4 .$$

substitute into F:

$$F = -S_{1111} - 3S_{211} - S_{22} + S_{31} + 3S_4 .$$