QUIZ 2 SOLUTIONS : MATH 1200- PROBLEMS, CONJECTURES AND PROOFS

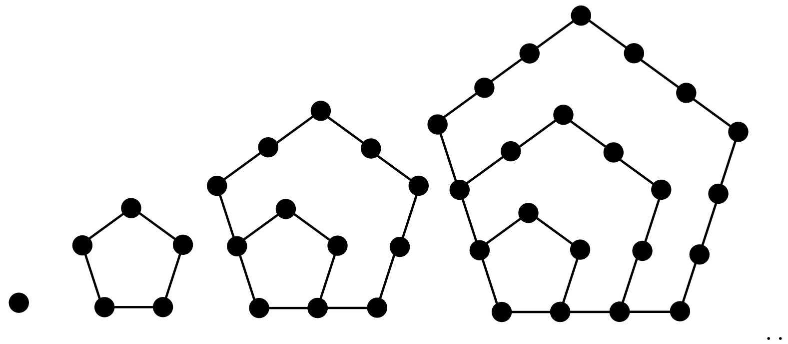
NOVEMBER 3, 2008

You have 1 hour 15 minutes to complete the following quiz. You may use any notes or books but please do not use any calculators.

(1) (3 points) Use induction to show that for all $n \geq 0$ and all $k \geq 0$ that
$$\sum_{r=0}^{n} \frac{1}{(rk+1)((r+1)k+1)} = \frac{n+1}{(n+1)k+1}.$$
Solution: If $n = 0$, then the left hand side of the equation is
$$\sum_{r=0}^{0} \frac{1}{(rk+1)((r+1)k+1)} = \frac{1}{k+1}$$
the right hand side of the equation at $n = 1$ is
$$\frac{0+1}{(0+1)k+1} = \frac{1}{k+1}$$
and hence they are equal (for all k and not just $k \geq 0$).

Now assume that $\sum_{r=0}^{n} \frac{1}{(rk+1)((r+1)k+1)} = \frac{n+1}{(n+1)k+1}$ for a fixed n, then
$$\begin{aligned}
&\sum_{r=0}^{n+1} \frac{1}{(rk+1)((r+1)k+1)} \\
&= \left(\frac{1}{1\cdot(k+1)} + \frac{1}{(k+1)(2k+1)} + \cdots + \frac{1}{(nk+1)((n+1)k+1)} \right) + \frac{1}{((n+1)k+1)((n+2)k+1)} \\
&= \frac{n+1}{(n+1)k+1} + \frac{1}{((n+1)k+1)((n+2)k+1)} \\
&= \frac{(n+1)((n+2)k+1)+1}{((n+1)k+1)((n+2)k+1)} \\
&= \frac{(n+1)(n+2)k+(n+2)}{((n+1)k+1)((n+2)k+1)} \\
&= \frac{(n+2)((n+1)k+1)}{((n+1)k+1)((n+2)k+1)} \\
&= \frac{n+2}{((n+2)k+1)}
\end{aligned}$$
Therefore the equation holds for n replaced by $n+1$. Since we have shown that if it is true for n, then it is true for $n+1$, and we know it is true for the base case of $n = 0$ so by the principle of mathematical induction it is true for all $n \geq 0$.

(2) (4 points) The n^{th} pentagonal a_n is defined as the number of dots in the n^{th} diagram of the following sequence diagrams.



So we have that $a_1 = 1$, $a_2 = 5$, $a_3 = 12$, $a_4 = 22$ and the next term in the sequence is $a_5 = 35$. Use induction to explain why $a_n = \frac{3n^2-n}{2}$.
Solution: Notice that the number of dots in the $n+1^{st}$ picture is equal to the number of dots in the n^{th} picture plus three sides of the diagram with $n+1$ dots each and those sides have two dots overlapping, this means $a_{n+1} = a_n + 3(n+1) - 2$. e.g. the second picture is equal to the first picture plus $3 \cdot 2 - 2 = 4$ dots, the third picture is equal to the second picture plus $3 \cdot 3 - 2 = 7$ dots, the fourth picture is equal to the third picture plus $3 \cdot 4 - 2 = 10$ dots, the $n+1^{st}$ picture is equal to the n^{th} picture plus $3 \cdot (n+1) - 2 = 3n + 1$ dots.
We check that $a_1 = 1 = \frac{3 \cdot 1^2 - 1}{2}$, hence the statement that $a_n = \frac{3n^2-n}{2}$ is true for $n = 1$. Assume that it is true for a given n, that is, assume that the n^{th} pentagonal number is $\frac{3n^2-n}{2}$. Then we have

$$\begin{aligned} a_{n+1} &= a_n + 3n + 1 \\ &= \frac{3n^2 - n}{2} + 3n + 1 \\ &= \frac{3n^2 - n + 6n + 2}{2} \\ &= \frac{3(n^2 + 2n + 1) - n - 1}{2} \\ &= \frac{3(n+1)^2 - (n+1)}{2} \ . \end{aligned}$$

Therefore we have shown that if the n^{th} pentagonal number is $\frac{3n^2-n}{2}$, then the $n+1^{st}$ pentagonal number is $\frac{3(n+1)^2-(n+1)}{2}$, since we also know that it is true for the base case of $n = 1$ then by the principle of mathematical induction it is true for all n.

(3) (a) (1 point) Explain why there are 10 solutions to the equation

$$x_1 + x_2 + x_3 = 3$$

with x_1, x_2, x_3 are all integers which are greater than or equal to 0.
Solution: So if three non-negative integers add up to 3 then either two are 0 and one is 3, or one is 0 one is 1 and one is 2, or all three are 1. That is the possible solutions to this are:

$$x_1 = 3, x_2 = 0, x_3 = 0$$
$$x_1 = 0, x_2 = 3, x_3 = 0$$
$$x_1 = 0, x_2 = 0, x_3 = 3$$
$$x_1 = 0, x_2 = 1, x_3 = 2$$
$$x_1 = 0, x_2 = 2, x_3 = 2$$
$$x_1 = 1, x_2 = 0, x_3 = 2$$
$$x_1 = 1, x_2 = 2, x_3 = 0$$

$$x_1 = 2, x_2 = 1, x_3 = 0$$
$$x_1 = 2, x_2 = 0, x_3 = 1$$
$$x_1 = 1, x_2 = 1, x_3 = 1$$

(b) (1 point) Explain why there are 66 solutions to the equation

$$x_1 + x_2 + x_3 = 10$$

with x_1, x_2, x_3 are all integers which are greater than or equal to 0.
Solution:
If $x_1 = 0$, then $x_2 + x_3 = 10$ and there are 11 ways this could happen ($x_2 = 0$ through 10).
If $x_1 = 1$, then $x_2 + x_3 = 9$ and there are 10 ways this could happen ($x_2 = 0$ through 9).
If $x_1 = 2$, then $x_2 + x_3 = 8$ and there are 9 ways this could happen ($x_2 = 0$ through 8).
If $x_1 = 3$, then $x_2 + x_3 = 7$ and there are 8 ways this could happen ($x_2 = 0$ through 7).
If $x_1 = 4$, then $x_2 + x_3 = 6$ and there are 7 ways this could happen ($x_2 = 0$ through 6).
If $x_1 = 5$, then $x_2 + x_3 = 5$ and there are 6 ways this could happen ($x_2 = 0$ through 5).
If $x_1 = 6$, then $x_2 + x_3 = 4$ and there are 5 ways this could happen ($x_2 = 0$ through 4).
If $x_1 = 7$, then $x_2 + x_3 = 3$ and there are 4 ways this could happen ($x_2 = 0$ through 3).
If $x_1 = 8$, then $x_2 + x_3 = 2$ and there are 3 ways this could happen ($x_2 = 0$ through 2).
If $x_1 = 9$, then $x_2 + x_3 = 1$ and there are 2 ways this could happen ($x_2 = 0$ through 1).
If $x_1 = 10$, then $x_2 + x_3 = 0$ and there is 1 way this could happen ($x_2 = 0$ and $x_3 = 0$).
Therefore there are $1+2+3+4+5+6+7+8+9+10+11 = 66$ solutions to $x_1+x_2+x_3 = 10$.

(c) (2 points) Find/guess at a formula for the number of solutions to the equation

$$x_1 + x_2 + x_3 = n$$

with x_1, x_2, x_3 are all integers which are greater than or equal to 0.
Solution:
$x_1 + x_2 + x_3 = 0$ has 1 solution.
$x_1 + x_2 + x_3 = 1$ has $3 = 1 + 2$ solutions.
$x_1 + x_2 + x_3 = 2$ has $6 = 1 + 2 + 3$ solutions
$x_1 + x_2 + x_3 = 3$ has $10 = 1 + 2 + 3 + 4$ solutions.
hmmmm...I guess that $x_1 + x_2 + x_3 = n$ has $1 + 2 + 3 + \ldots + (n + 1) = \frac{(n+1)(n+2)}{2}$ solutions.
Does this work for $n = 10$? why yes, $1 + 2 + 3 + \ldots + 11 = 66 =$ the number of solutions to $x_1 + x_2 + x_3 = 10$.

(d) (3 points) Give a proof (explain why) the formula that you found in part (c) for the number of solutions to this equation is correct.
For $x_1 + x_2 + x_3 = n$, x_1 will be a number between 0 and n. If $x_1 = r$, then $x_2 + x_3 = n - r$ and there are $n - r + 1$ solutions in this particular case. Since r can be any number between 0 and n, then by the addition principle the total number of solutions is $(n - 0 + 1) + (n - 1 + 1) + (n - 2 + 1) + \cdots + (n - n + 1) = 1 + 2 + 3 + \cdots + n + (n + 1)$.