DIFFERENCES OF SEQUENCES

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I had noticed that some of you were using some results about differences of sequences in order to solve problems that need to be made more precise. We had some discussion in class and I wanted to write down precisely what we had learned and fill in any details that I left off in the class discussion.

Say that we have a sequence of numbers (for convenience we start the index for this sequence at 0),

$$a_0, a_1, a_2, a_3, \ldots$$

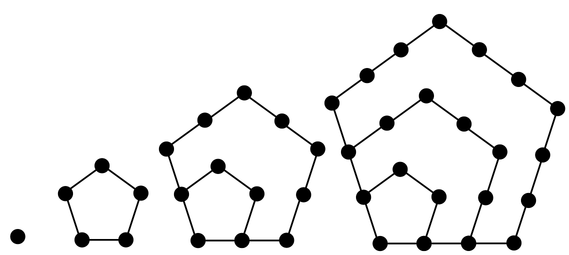
The *first differences* of this sequence is defined to be the sequence

$$a_0^{(1)} = a_1 - a_0, a_1^{(1)} = a_2 - a_1, a_2^{(1)} = a_3 - a_2, a_3^{(1)} = a_4 - a_3, \ldots.$$

The *second differences* of this sequence is defined to be the first differences of the first differences $a_i^{(2)} = a_{i+1}^{(1)} - a_i^{(1)}$. In general, the k^{th} *differences* $a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, \ldots$ are defined to be the first differences of the $k-1^{st}$ differences assuming that we have already defined the $k-1^{st}$ differences so that $a_i^{(k)} = a_{i+1}^{(k-1)} - a_i^{(k-1)}$.

We say that a sequence $a_0, a_1, a_2, a_3, \ldots$ is constant if $a_i = a_0$ for all $i \geq 0$.

For example, the pentagonal numbers (which we have spoken about in class) are defined to be the number of dots within nested diagrams of pentagons:



You should draw the next diagram yourself and convince yourself that there are 35 dots in the picture. How many are in the picture after that?

If we assume that the first diagram is empty (i.e. with no dots) then this sequence and its first and second differences begin as follows.

sequence	0		1		5		12		22		35	...
first differences		1		4		7		10		13	...	
second differences			3		3		3		3	...		

In the example above, $a_0^{(1)} = 1$, $a_1^{(1)} = 4$, $a_2^{(1)} = 7$, $a_3^{(1)} = 10$, $a_4^{(1)} = 13$ are the first five terms of the first differences and $a_0^{(2)} = a_1^{(2)} = a_2^{(2)} = a_3^{(2)} = 3$ are the first 4 terms of the first differences (remark that we have not shown that $a_i^{(2)} = 3$ for all i, just that it is true for $0 \leq i \leq 3$ so we do not know for sure that it is constant without additional information).

We say that a polynomial in the variable x is of degree n if the highest degree of the exponent of x that appears in the polynomial is n. That is that there exists a coefficients $a_n \neq 0$ and $a_{n-1}, a_{n-2}, \ldots, a_0$ such that

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

We say that a sequence $a_0, a_1, a_2, \ldots$ is given by a polynomial $p(x)$ if

$$a_0 = p(0), a_1 = p(1), a_2 = p(2), \ etc.$$

and in general $a_n = p(n)$ for all $n \geq 0$.

The main result that we would like to prove is the following:

Theorem 1. *The k^{th} differences of a sequence are constant if and only if there is a polynomial $p(x)$ of degree k such that $a_n = p(n)$ for all $n \geq 0$.*

We will show an even stronger result by giving a formula for the polynomial which gives the sequence. Before we do I will need to introduce a little notation.

For a polynomial $p(x)$, define $\Delta p(x) = p(x+1) - p(x)$. Remark that if $a_n = p(n)$ for all $n \geq 0$ (that is, if the sequence a_n is given by a polynomial $p(x)$), then the first differences of this sequence is $a_n^{(1)} = a_{n+1} - a_n = p(n+1) - p(n) = \Delta p(n)$.

Notice that when we apply Δ to x^n (which is a polynomial of degree n) we have

$$\Delta(x^n) = (x+1)^n - x^n = nx^{n-1} + \binom{n}{2} x^{n-2} + \cdots + nx + 1 \tag{1}$$

which is a polynomial of degree $n-1$. In general we will see that if $p(x)$ is a polynomial of degree n then $\Delta p(x)$ will be a polynomial of degree at most $n-1$ (in fact, it is of degree exactly $n-1$). You may notice that Δ is similar to the derivative operation and indeed this they share many of the same properties.

Notice that Δ is a linear operation, that is,

$$\begin{aligned}\Delta(cf(x) + dg(x)) &= cf(x+1) + dg(x+1) - (cf(x) + dg(x))\\ &= (cf(x+1) - cf(x)) + (dg(x+1) - dg(x))\\ &= c\Delta f(x) + d\Delta g(x)\end{aligned}$$

We know that if $p(x)$ is a polynomial of degree n and has highest x degree $a_n x^n$, then $\Delta p(x)$ will have a term with highest x degree $na_n x^{n-1}$.

Now consider the following family of polynomials which act nicely with the operator Δ. Let $g_0(x) = 1$ and for r an integer greater than 0, we set

$$g_r(x) = \frac{1}{r!}x(x-1)(x-2)\cdots(x-r+1).$$

So for example, we have that $g_1(x) = x$, $g_2(x) = \frac{1}{2}x(x-1)$, $g_3(x) = \frac{1}{6}x(x-1)(x-2)$, $g_4(x) = \frac{1}{24}x(x-1)(x-2)(x-3)$. Clearly, $g_r(x)$ is a polynomial of degree r.

For example, $\Delta g_1(x) = (x+1) - x = 1 = g_0(x)$, $\Delta g_2(x) = \frac{1}{2}(x+1)x - \frac{1}{2}x(x-1) = x = g_1(x)$, and in general for $r \geq 1$,

$$\begin{aligned}
\Delta g_r(x) &= g_r(x+1) - g_r(x) \\
&= \frac{1}{r!}(x+1)x(x-1)\cdots(x-r+2) - \frac{1}{r!}x(x-1)(x-2)\cdots(x-r+1) \\
&= \frac{1}{r!}x(x-1)\cdots(x-r+2)((x+1)-(x-r+1)) \\
&= \frac{1}{r!}x(x-1)\cdots(x-r+2)r \\
&= \frac{1}{(r-1)!}x(x-1)\cdots(x-r+2) \\
&= g_{r-1}(x)
\end{aligned}$$

What we shall show is that if the k^{th} differences are constant, then

$$p(x) = a_k g_k(x) + a_0^{(k-1)} g_{k-1}(x) + a_{k-2}^{(k-2)} g_2(x) + \cdots + a_0^{(1)} g_1(x) + a_0 \tag{2}$$

is a polynomial of degree k and

$$p(n) = a_n$$

for all $n \geq 0$.

In order to prove our main result we shall require the following lemma. I really just stated this in class and didn't prove it.

Lemma 2. *Let $a_0, a_1, a_2, \ldots$ be a sequence and $b_0, b_1, b_2, \ldots$ be the first differences so that $b_i = a_{i+1} - a_i$ for all $i \geq 0$. Let $q(x)$ be a polynomial such that $q(0) = a_0$ and $\Delta q(x)$ is a polynomial such that the first differences are given by $\Delta q(x)$. Then the sequence $a_0, a_1, a_2, \ldots$ is given by the polynomial $q(x)$.*

Proof. We will show this result by induction on n that $q(n) = a_n$.

We have that $q(0) = a_0$ as one of the hypotheses and that $a_{r+1} - a_r = \Delta q(r)$ for all $r \geq 0$.

Assume that $q(n) = a_n$ for some fixed n, then

$$a_{n+1} = a_n + (a_{n+1} - a_n) = a_n + b_n = q(n) + \Delta q(n) = q(n) + q(n+1) - q(n) = q(n+1)$$

. Hence $q(n) = a_n$ implies that $q(n+1) = a_{n+1}$.

By the principle of mathematical induction $q(n) = a_n$ for all $n \geq 0$ so the sequence $a_0, a_1, a_2, \ldots$ is given by the polynomial $q(x)$. □

Theorem 3. *If a sequence has the k^{th} differences which are constant, then it will be given by equation* (2)*. Moreover, any sequence whose values are given by a polynomial of degree k will have the k^{th} differences which are constant.*

Proof. First, lets prove the second part of this theorem (because it is easier). Namely, if a sequence is given by a polynomial of degree k, then the k^{th} differences of the sequence will be constant. We will show this by induction on k.

If a sequence is given by a polynomial of degree 0, then this polynomial is itself constant (that is what it means for the degree of a polynomial to be 0). Therefore the sequence itself is constant so the 0^{th} differences (the sequence itself) is constant.

A sequence given by a polynomial of degree 1 is linear and of the form $ax + b$ hence the sequence is of the form $b, a + b, 2a + b, 3a + b, \ldots$. Therefore each of the first differences of this sequence are $a = a(k + 1) + b - (ak + b)$ and hence are constant.

Assume that for a fixed k, if a sequence of numbers is given by a polynomial of degree k then the k^{th} differences of this sequence are constant. Then if a sequence is given by a polynomial of degree $k + 1$, say $p(x)$, then the first differences of this sequence will be given by $\Delta p(x)$ which we know (by the discussion above) will be a polynomial of degree k. Therefore by our inductive hypothesis, the k^{th} differences of this sequence will be constant and the k^{th} differences of this sequence *are* exactly the $k + 1^{st}$ differences of our sequence which is given by a polynomial of degree $k + 1$. We conclude, that "if sequences given by a polynomial of degree k have their k^{th} differences constant, then sequences given by a polynomial of degree $k + 1$ will have their $(k + 1)^{st}$ differences constant."

By the principle of mathematical induction we conclude that sequences given by a polynomial of degree k have their k^{th} differences constant for all $k \geq 0$.

Now we wish to prove the more interesting result that if we know what the first term of all of the differences are (i.e. we know what the values of $a_0, a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(k)}$ are and we know that the k^{th} differences $a_i^{(k)}$ are all constant (are independent of i), then the sequence is given by the following polynomial of degree k

$$p(x) = a_k g_k(x) + a_0^{(k-1)} g_{k-1}(x) + a_{k-2}^{(k-2)} g_2(x) + \cdots + a_0^{(1)} g_1(x) + a_0 \tag{3}$$

where

$$g_r(x) = \frac{1}{r!} x(x - 1)(x - 2) \cdots (x - r + 1).$$

Again we prove this result by induction on the value of k. Certainly if the sequence itself is constant then it is given by the constant polynomial $p(x) = a_0$ because $a_i = a_0 = p(i)$ for all $i \geq 0$.

Now assume that if the k^{th} differences of a given sequence are constant and the first values of those different k^{th} differences are $a_0, a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(k)}$, then the sequence will be given by equation (3).

Let $b_0, b_1, b_2, \ldots$ be a sequence whose $k+1^{st}$ differences are constant then let $a_0, a_1, a_2, \ldots$ are the first differences of this sequence and we will reuse the notation for the differences of this sequence from above. Since we know that that the k^{th} differences of $a_0, a_1, a_2, \ldots$ are constant because they are the $k+1^{st}$ differences of the sequence $b_0, b_1, b_2, \ldots$, then we know that the sequence $a_0, a_1, a_2, \ldots$ is given by the polynomial

$$p(x) = a_k g_k(x) + a_0^{(k-1)} g_{k-1}(x) + a_0^{(k-2)} g_{k-2}(x) + \cdots + a_0^{(1)} g_1(x) + a_0$$

by our inductive hypothesis. Now let

$$q(x) = a_k g_{k+1}(x) + a_0^{(k-1)} g_k(x) + a_0^{(k-2)} g_{k-1}(x) + \cdots + a_0^{(1)} g_2(x) + a_0 g_1(x) + b_0.$$

Clearly we have that $q(0) = b_0$ and we also have that since $\Delta g_r(x) = g_{r-1}(x)$ that

$$\begin{aligned}\Delta q(x) &= a_k \Delta g_{k+1}(x) + a_0^{(k-1)} \Delta g_k(x) + a_0^{(k-2)} \Delta g_{k-1}(x) + \cdots + a_0^{(1)} \Delta g_2(x) + a_0 \Delta g_1(x) \\ &= a_k g_k(x) + a_0^{(k-1)} g_{k-1}(x) + a_0^{(k-2)} g_{k-2}(x) + \cdots + a_0^{(1)} g_1(x) + a_0 \\ &= p(x)\ .\end{aligned}$$

By our lemma we conclude that since $q(0) = b_0$ and $\Delta q(x)$ is a polynomial which gives the first differences, then $q(x)$ is a polynomial which gives the sequence $b_0, b_1, b_2, \ldots$.

By the principle of mathematical induction, if $a_0, a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(k)}$ are the first entries of the k differences and the k^{th} differences are constant, then the sequence $a_0, a_1, a_2, \ldots$ is given by the polynomial

$$p(x) = a_k g_k(x) + a_0^{(k-1)} g_{k-1}(x) + a_0^{(k-2)} g_{k-2}(x) + \cdots + a_0^{(1)} g_1(x) + a_0$$

for all $n \geq 0$. □

Lets apply this result to the example that we looked at above:

sequence	0		1		5		12		22		35		...
first differences		1		4		7		10		13		...	
second differences			3		3		3		3		...		

If we know that the second differences are constant, then the sequence $0, 1, 5, 12, 22, 35, \ldots$ will be given by the formula

$$p(x) = 3g_2(x) + 1g_1(x) + 0g_0(x) = 3\frac{1}{2}x(x-1) + x = \frac{3x^2 - x}{2}\ .$$

Sure enough we see that $\frac{30^2-0}{2} = 0$, $\frac{31^2-1}{2} = 1$, $\frac{32^2-2}{2} = \frac{10}{2} = 5$, $\frac{33^2-3}{2} = \frac{24}{2} = 12$, $\frac{34^2-4}{2} = \frac{44}{2} = 22$, $\frac{35^2-5}{2} = \frac{70}{2} = 35$. But does this work beyond the 6^{th} term in the sequence?

The answer is in fact, yes. But we need one more piece of information. Namely that the 2^{nd} differences are constant. In this case our sequence is coming from a picture, namely, the number of dots in concentric pentagons (see the picture on the first page). At each step we add $3n+1$ more dots than were in the previous picture (when $n=0$ we are talking about the difference between the picture with no dots to the one with one dot, when $n=1$ we are looking at the difference of the pentagon and the picture with one dot, etc.). This is the important observation to make because it says that the first differences of this sequence are given by a polynomial of degree 1, namely $p(x) = 3x+1$. Therefore by our Theorem 3 we know that the first differences of the first differences (the second differences of our sequence) will be constant.

Perhaps a larger example is in order. Say that we wanted to find a polynomial formula for $1^3 + 2^3 + 3^3 + \cdots + n^3$. We know that when $n = 0$ we have the empty sum (which will be 0), when $n = 1$ we have 1^3, when $n = 2$ we have $1^3 + 2^3 = 9$, etc. Notice that the difference between $1^3+2^3+3^3+\cdots+n^3$ and $1^3+2^3+3^3+\cdots+(n+1)^3$ is just $(n+1)^3$ so it is given by a polynomial of degree 3. Since the first differences of our sequence are given by a polynomial of degree 3 then the fourth differences of our sequence will be constant. Lets compute the first four differences:

sequence	0		1		9		36		100		...
first		1		8		27		64		...	
second			7		19		37		...		
third				12		18		...			
fourth					6		...				

We don't need to calculate any more of the differences (unless we want to insure that our calculation is correct) because we already know that the fourth differences will all be 6. We can conclude that our sequence is given by the polynomial

$$\begin{aligned} &6g_4(x) + 12g_3(x) + 7g_2(x) + g_1(x) \\ &\quad = \frac{6}{24}x(x-1)(x-2)(x-3) + \frac{12}{6}x(x-1)(x-2)\frac{7}{2}x(x-1) + x \\ &\quad = \frac{x^2(x+1)^2}{4} \end{aligned}$$

This polynomial is a formula for our sequence!

Exercises:

(1) Find a formula for the hexagonal numbers. This sequence is given by the generalization of the pentagonal numbers except that we work with concentric hexagons (6 sided figures) rather than pentagons. Draw the picture. Figure out what you do to each picture to get the next one (how many points do you add at each step?) Calculate the first few terms of each of the differences and apply Theorem 3 to find a formula.

(2) Use the method described above to find a formula for the sequence

$$1^3 + 4^3 + 7^3 + \cdots + (3n-2)^3$$

(note that n represents the number of terms in this sum so that when $n = 0$ there are no terms in the sum).