The range 71–80 is special because 75 contributes two factors of 5 and 80 contributes one factor of 5. The total contribution is three zeros.

The range 81–90 contributes two factors of 5 in the usual fashion, and thus adds two zeros.

The range 91–100 contains 95 and 100. The first of these contributes one factor of five and the second contributes two. Thus three zeros are added.

Taking all of our analyses into account, we have six ranges of numbers that each contribute two zeros and four ranges that each contribute three zeros. This gives a total of 24 zeros that will appear at the end of 100! □

This example already exhibits several important features of successful problem solving:

- We identified the essential feature on which the problem hinges (that a trailing zero comes from multiplication by 10).
- We began by analyzing a special case (i.e., the product $10 \cdot 9 \cdot 8 \cdots 3 \cdot 2 \cdot 1$).
- We determined how to pass from the special case to the full problem.

It is not always true that examining a special case, or a smaller case, will lead to a solution of the problem at hand. But it will get you started. This will be one of our many devices for attacking a problem.

Looking back at our solution of the first problem, we see that we could have been more clever. The numbers from 1 to 100 contain $100 \div 5 = 20$ multiples of 5. Four of these multiples of 5 are in fact multiples of 25, hence contribute two 5's. That gives a total of 24 factors of 5 in 100! Pairing each of these with an even number gives a factor of ten, and hence a zero. We conclude that there are 24 zeros at the end of 100!

Here is another example of specialization:

PROBLEM 1.2.2 *A math class has 12 students. At the beginning of each class hour, each student shakes hands with each of the other students. How many handshakes take place?*

Solution: We begin with a special case and build up to the case of 12 students.

Suppose that there are just 2 students. Then only one handshake is possible.

Now suppose that a new student walks in the door. He/she must shake hands with each of the students that is already in the room. So that makes two more handshakes. The total number of handshakes is $1+2=3$.

If a fourth student walks in the door, then he/she must shake hands with each of the students already in the room. The total number of handshakes is then $1+2+3=6$.

The pattern is now clear: the addition of a fifth student would result in $1+2+3+4$ handshakes. When we get up to twelve students, we will have required

$$1+2+3+\cdots+9+10+11=66$$

handshakes.

That solves the problem. □

Many times the solution or analysis of one problem will suggest others. Here is another problem that Problem 1.2.2 suggests:

PROBLEM 1.2.3 *Assume that k is a positive integer. What is the sum of the integers*

$$S=1+2+3+\cdots+(k-1)+k\,?$$

Before we present a solution, we conduct some preliminary discussion of this problem.

We think of S as function: set

$$S(k)=1+2+3+\cdots+(k-1)+k.$$

What sort of function might this be? If a function $f(k)$ increases by a fixed amount, say 3, each time that k is increased by 1, then f must be a linear function. Indeed, f must have the form $f(k) = 3k+b$.

Likewise, if the function g increases by a linear function of k each time that k is increased by 1, then we might suspect that g is quadratic. (For those who know calculus, think of the concept of derivative: the derivative of a quadratic function is linear.) For instance, if $g(k) = k^2$ then $g(k+1) - g(k) = 2k + 1$, and that difference is linear.

These considerations motivate our attack on the present problem:

Solution: A useful method for analyzing a sum is to rewrite each term so that some cancellations are introduced. Notice that

$$\begin{aligned} 2^2 - 1^2 &= 3 = 2\cdot 1 + 1 \\ 3^2 - 2^2 &= 5 = 2\cdot 2 + 1 \\ 4^2 - 3^2 &= 7 = 2\cdot 3 + 1 \\ &\cdots \\ k^2 - (k-1)^2 &= 2\cdot(k-1) + 1 \\ (k+1)^2 - k^2 &= 2\cdot k + 1 \end{aligned}$$

Now we add the columns:

$$\begin{aligned} &[2^2 - 1^2] + [3^2 - 2^2] + [4^2 - 3^2] + \cdots + [(k+1)^2 - k^2] \\ &= [2\cdot 1 + 1] + [2\cdot 2 + 1] + [2\cdot 3 + 1] + \cdots + [2\cdot k + 1]. \end{aligned}$$

The left hand side "telescopes" (that is, all but the first and last terms cancel) and the right side may be factored. The result is

$$(k+1)^2 - 1^2 = 2[1 + 2 + 3 + \cdots + k] + \underbrace{[1 + 1 + 1 + \cdots + 1]}_{k \text{ times}}$$

or

$$k^2 + 2k = 2\cdot S + k.$$

Recall here that S is that sum that we wish to calculate.

Solving for S, we find that

$$S = \frac{k^2 + k}{2}. \qquad \square$$

The formula derived in the last problem is often attributed to Carl Friedrich Gauss (1777–1855), although there is evidence that it was known much earlier.

CHALLENGE PROBLEM 1.2.4 *Imitate the method used in the last problem to find a formula for the sum*

$$1^2 + 2^2 + 3^2 + \cdots + k^2$$

when k is a positive integer.

The solution of our last problem sheds some light on the preceding "handshake" problem. For if a class contains k students and the class period begins with everyone shaking everyone else's hand, then our solution of Problem 1.2.2 shows that the total number of handshakes that occurs is $1 + 2 + 3 + \cdots + (k - 1)$. Now Problem 1.2.3 teaches us that this last sum equals $[(k-1)^2 + (k-1)]/2 = [k^2 - k]/2$.

Here is another question that one might ask about the handshake problem:

PROBLEM 1.2.5 *Refer again to the situation in Problem 1.2.2, but suppose now that there are k students in the class. If k is even then will the number of handshakes that takes place be even or odd? If k is odd then will the number of handshakes that takes place be even or odd?*

Solution: If there are 2 students (an even value for k) then the total number of handshakes is 1, an odd number. If we add one student, that adds two handshakes: the number of students is 3 (odd) and the number of handshakes is 3 (odd). If we add yet another student then there are 3 additional handshakes. Thus the total number of handshakes is 6 (even) while the total number of students is 4 (even).

In fact if we draw up a chart then a pattern begins to emerge: