## SOULUTIONS TO HOMEWORK ASSIGNMENT NO. 2

Your assignment should include complete sentences and explanations and not just a few equations or numbers. A solution will not receive full credit unless you explain what your answer represents and where it came from. You may discuss the homework with other students in the class, but please write your own solutions. The first question is adapted from "What is the Name of This Book?" by Raymond Smullyan.

(1) A bank was robbed and Inspector Craig and Sergeant McPherson were on the case trying to establish the guilt or innocence of four suspects Alice, Bob, Carol and Dave. The nefarious characters are the only people who could be involved in these bank robberies and at least one of them is guilty. In each case the Inspector and Sergeant establish certain facts.

Write an argument in words to establish the guilt or innocence of Alice, Bob and Carol and Dave. Note that the clues provided may not be sufficient to determine the guilt and innocence of all of the suspects, but should be sufficient to establish the guilt of at least one person.

Say that we establish that:

(1) If Alice was guilty, then she had exactly one other accomplice.
(2) Bob and Carol were both together at the time of the crime.
(3) If exactly two are guilty then Alice is one of them.
(4) Bob and Dave never work together.
(5) If both Bob and Carol were not involved then Dave is guilty.

Translate each of the clues to a truth valued sentence using the connectives *and*, *or*, *not* and *if ... then* and the propositions: $A$ representing the statement "Alice is guilty," $B$ representing the statement "Bob is guilty," $C$ representing "Carol is guilty," and $D$ representing "Dave is guilty." Create a truth table establishing the truth values of the clues in terms of the truth values of $A$, $B$, $C$ and $D$.

The translation of the logical statements.

(1) The shortest translation that I could come up with is:
$A \Rightarrow ((B \text{ } xor \text{ } C \text{ } xor \text{ } D) \wedge \sim (B \wedge C \wedge D))$
but the one that is most direct is
$A \Rightarrow ((B \wedge \sim C \wedge \sim D) \vee (\sim B \wedge C \wedge \sim D) \vee (\sim B \wedge \sim C \wedge D))$
(2) $B \equiv C$
(3) $((A \wedge B \wedge \sim C \wedge \sim D) \vee (A \wedge \sim B \wedge C \wedge \sim D) \vee (A \wedge \sim B \wedge \sim C \wedge D) \vee (\sim A \wedge B \wedge C \wedge \sim D) \vee (\sim A \wedge \sim B \wedge C \wedge D) \vee (\sim A \wedge B \wedge \sim C \wedge D)) \Rightarrow A$
(4) $\sim (B \wedge D)$
(5) $(\sim B \wedge \sim C) \Rightarrow D$

Now the following table is the truth table for the 5 statements above. The header $S(i)$ indicate the column for the truth value of statement $i$ where $1 \leq i \leq 5$.

| row | $A$ | $B$ | $C$ | $D$ | $S(1)$ | $S(2)$ | $S(3)$ | $S(4)$ | $S(5)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | F | T | T | F | T |
| 2 | T | T | T | F | F | T | T | T | T |
| 3 | T | T | F | T | F | F | T | F | T |
| 4 | T | T | F | F | T | F | T | T | T |
| 5 | T | F | T | T | F | F | T | T | T |
| 6 | T | F | T | F | T | F | T | T | T |
| 7 | T | F | F | T | T | T | T | T | T |
| 8 | T | F | F | F | F | T | T | T | F |
| 9 | F | T | T | T | T | T | T | F | T |
| 10 | F | T | T | F | T | T | F | T | T |
| 11 | F | T | F | T | T | F | F | F | T |
| 12 | F | T | F | F | T | F | T | T | T |
| 13 | F | F | T | T | T | F | F | T | T |
| 14 | F | F | T | F | T | F | T | T | T |
| 15 | F | F | F | T | T | T | T | T | T |
| 16 | F | F | F | F | T | T | T | T | F |

Now in this table, only row 7 and 15 have all true statements. We conclude that Dave is guilty, Bob and Carol are innocent and Alice could either be guilty or innocent.

(2) The following questions are about subsets of the integers $\{1, 2, 3, \ldots, 10\}$. The numbers that are in the subset are sometimes referred to as the elements of the set and the number of elements is the size of the subset.

Give an example of one subset that makes the following statements true and one subset that makes the statement false (if possible) and explain why your subset makes the statement true or false. If the sentence is always true or always false for all subsets, then explain why.

(a) The subset $S$ does not contain 1 but does contain 10, or contains both 1 and 10 or the maximum value in $S$ is less than 7.

$S = \{1\}$ makes the statement true because the maximum value in the set is less than 7.
$S = \{1, 8\}$ makes the statement false because it contains 1, doesn't contain 10, and the maximum value in the set is greater than 7.

(b) The subset $S$ contains 1 and it does not contain 9, and for every $x$ in $S$, $x + 4$ is also an element in $S$.

This statement is always false because the largest value that can be in the set is 10, but if there was a largest value, say $x$, then $x + 4$ cannot be in the set.

(c) If $S$ has more than 5 elements or 1 is an element of $S$, then 10 is not an element of $S$.

$S = \{1\}$ makes the statement true because 1 is an element of $S$ and 10 is not an element of $S$.

$S = \{1, 10\}$ makes the statement false because 1 is an element of $S$ and 10 is an element of $S$.

(d) For every $x$ in $S$, if $x$ is prime, then $x + 1$ is in $S$.

$S = \{\}$ makes the statement true because there are no prime numbers in the set.
$S = \{3\}$ makes the statement false because 3 is in the set, it is prime, but 4 is not in the set.

(e) If $S$ contains 1 and 10 and the size of $S$ is more than 7, then the largest value in $S$ less than 10 is at least 5.

This statement is always true because if the largest value less than 10 is $1, 2, 3$ or 4 then $S$ is a subset of $\{1, 2, 3, 4, 10\}$ and there are at most 5 elements in $S$.

(f) There is an $x$ in $S$ such that either $x$ is odd and prime or $3x$ is in $S$.

$S = \{\}$ makes the statement not true because there is no element in the set that could satisfy the condition.
$S = \{3\}$ makes the statement true because 3 is an odd prime

(3) Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3, 4, 5\}$

(a) Let $S = \{(b, a) : b \in B, a \in A, a+b \text{ is odd }\}$ and let $T$ be the non-empty proper subsets of $A$. What are $|S|$ and $|T|$? List the elements of $S$ and $T$.

$$S = \{(1, 2), (2, 1), (2, 3), (3, 2), (4, 1), (4, 3)\}$$

so there are 6 elements in $S$.

$$T = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

and so there are 6 elements in $T$.

(b) What is $|A \times B \times C|$?

$|A \times B \times C| = 3 \cdot 4 \cdot 5 = 60$. This is because $A \times B \times C$ is equal to the number of ways of picking one element from $A$, one element from $B$ and one element from $C$.

(c) Let $V = \{(a, b) : (a, |b|) \in A \times C\}$. Let $U = \{(a, b) \in V : ak = b \text{ for some } k \in \mathbb{Z}\}$. List the elements of $U$.

The notation $(a, \pm b)$ will represent the two points $(a, b)$ and $(a, -b)$ since $U$ and $V$ have a lot of elements we would like to use as compact notation as possible. $V = \{(1, \pm 1), (1, \pm 2), (1, \pm 3), (1, \pm 4), (1, \pm 5), (2, \pm 1), (2, \pm 2), (2, \pm 3), (2, \pm 4), (2, \pm 5), (3, \pm 1), (3, \pm 2), (3, \pm 3), (3, \pm 4), (3, \pm 5)\}$ so $V$ has 30 elements in it. The set $U$ is $\{(1, \pm 1), (1, \pm 2), (1, \pm 3), (1, \pm 4), (1, \pm 5), (2, \pm 2), (2, \pm 4), (3, \pm 3)\}$ and so $U$ has 16 elements.