

## MORE PROBLEMS

MARCH 27, 2018

- (1) Let  $\mathbb{N}$  represent the set of positive integers. Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$f(1) = 2, \text{ and } f(xy) = f(x)f(y) - f(x+y) + 1 \text{ for all } x, y \in \mathbb{N}.$$

**Hint:** Start by determining  $f(2), f(3), f(4), \dots$ . Make a conjecture and then use Mathematical Induction to prove your conjecture is correct.

- (2) Let  $S$  be a square region (in the plane) of side length 2 inches. Show that among any nine points in  $S$  there are three which are the vertices of a triangle of area  $\leq \frac{1}{2}$  square inch.

- (3) This exercise provides a proof that  $\sqrt{3} + \sqrt{2}$  is an irrational number. You may take as given (no proof required) that  $\sqrt{3}$  and  $\sqrt{2}$  are irrational numbers.

(a) Define what it means for a real number to be rational, and for a real number to be irrational.

(b) Prove that the sum and that the product of two rational numbers is rational.

(c) Verify that if  $\sqrt{3} + \sqrt{2}$  is rational so is  $\sqrt{3} - \sqrt{2}$ .

**Hint:** What is their product?

(d) Verify that if  $\sqrt{3} + \sqrt{2}$  is rational, so is  $\sqrt{3}$ .

(e) Given that  $\sqrt{3}$  is not rational, what can you conclude about  $\sqrt{3} + \sqrt{2}$ ? Explain your argument.

(f) Generalize. If  $x$  and  $y$  are irrational, what condition on  $x^2 - y^2$  ensures that  $x + y$  be irrational?

- (4) Consider sequences of 1s and 0s which we shall refer to as binary words. A word is called palindromic if it reads the same forwards as backwards. For example the word 0110110 is palindromic while the word 001110 is not. If  $u$  and  $v$  are words then  $uv$  is defined to be the concatenation of those words (put the two words next to each other). For example if  $u = 0110110$  and  $v = 001110$ , then  $uv = 0110110001110$ .

(a) Show that if  $u$  and  $v$  are two palindromic words then it is not generally true that  $uv$  (the concatenation of the two words) is palindromic.

(b) Explain why if  $u$  and  $v$  are two palindromic words, then  $uvu$  is also palindromic.

(c) Is there a palindromic word with fifteen 0s and twenty three 1s? Explain.

- (5) Let  $f(n) = \sum_{r=1}^n \frac{1}{4r^2-1}$ . Compute  $f(n)$  for  $n = 1, 2, 3, 4$ . Conjecture a formula and prove it by induction.