SOME QUESTIONS ON PROOFS BY COUNTING

(1) Prove the following identities in two different ways by writing a story which counts something in two different ways (recall the two ways of coloring rooms in a house). You should also verify that the identities are correct using algebra (but this is not what I am asking here).
  (a) $n^2 = n + n(n-1)$
  (b) $n(n+1) = n^2 + n$
  (c) $n(n+1)(n+2) = n^3 + 3n^2 + 2n$
(2) Recall the example of coloring balls in a bin. Create a story which explains the following identities by counting the same thing in two different ways:
  (a) $k\binom{n}{k} = n\binom{n-1}{k-1}$
  (b) $n\binom{n}{k} = n\binom{n-1}{k-1} + (n-k)\binom{n}{k}$
  (c) $nk\binom{n}{k} = n\binom{n-1}{k-1} + n(n-1)\binom{n-2}{k-2} + n(n-1)\binom{n-2}{k-1}$
(3) Prove the following (a) induction and (b) by counting a set of objects in two different ways.
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1}$$
(4) Prove the following (a) induction and (b) by counting a set of objects in two different ways.
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \cdots + \binom{n-k-1}{0}$$