A COMBINATORIAL PROOF

On February 14 and 25th I gave an argument which explained the following identity

$$\binom{n}{k}^2 = \binom{n}{0}\binom{n}{k}\binom{n-k}{k} + \binom{n}{1}\binom{n-1}{k-1}\binom{n-k}{k-1} + \cdots + \binom{n}{k}\binom{n-k}{0}\binom{n-k}{0}$$

I did not write down the proof of this argument on the backboard because it involved a lot of writing. This document is to fill in those blanks that I did not write down, but instead stated orally.

We will use that $\binom{n}{k}$ is the number of subsets of $\{1, 2, \ldots, n\}$ of a k element set. I will state it here as the number of ways of choosing a subset of k balls from n numbered balls.

Proof: Let n and k be positive integers (with $n \geq k$). Imagine that we have n numbered white balls in a bin (and lots of ink) and then we reach in to the bin and pull out k of those and color them red and then put the balls back in and reach in and pick out k of the balls again and color them blue. In the end we will have some number of white balls, red balls, blue balls and purple balls (those which were colored both red and blue). Since there are $\binom{n}{k}$ ways of choosing k balls the first time and another $\binom{n}{k}$ ways of choosing the k balls the second time, in total there are $\binom{n}{k}^2$ ways of coloring balls this way.

Now lets count the same outcomes in a different way. The number of ways of coloring balls will have some number of purple balls and the minimum number of purple balls is 0 (where none of the red and blue balls were the same ones picked) and the maximum number of purple balls is equal to k (where all of the red balls were then picked to also be painted blue). By the addition principle, this means that the number of ways of coloring the balls will be
the number of ways of picking 0 purple balls, k red balls and k blue balls
+the number of ways of picking 1 purple balls, $k-1$ red balls and $k-1$ blue balls
+the number of ways of picking 2 purple balls, $k-2$ red balls and $k-2$ blue balls
+ ... +the number of ways of picking k purple balls, 0 red balls and 0 blue balls.

Say that there are d purple balls, $k-d$ red balls and $k-d$ blue balls. There are $\binom{n}{d}$ ways of choosing d of the n white balls to be purple and there are $\binom{n-d}{k-d}$ ways of picking $k-d$ of the remaining $n-d$ balls to be red and from the remaining $n-d-(k-d) = n-k$ balls there are $\binom{n-k}{k-d}$ ways of choosing $k-d$ of them to be blue. By the multiplication principle, in total there are $\binom{n}{d}\binom{n-d}{k-d}\binom{n-k}{k-d}$ ways of choosing a coloring of the balls such that d are purple, $k-d$ are red and $k-d$ are blue.

By the addition principle, since the number of purple balls is between $d = 0$ and $d = k$, the total number of ways of coloring balls purple, red and blue is equal to

$$\binom{n}{0}\binom{n}{k}\binom{n-k}{k}+\binom{n}{1}\binom{n-1}{k-1}\binom{n-k}{k-1}+\binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{k-2}+\cdots+\binom{n}{k}\binom{n-k}{0}\binom{n-k}{0}.$$

Since these outcomes are the same, I have just argued that

$$\binom{n}{k}^2=\binom{n}{0}\binom{n}{k}\binom{n-k}{k}+\binom{n}{1}\binom{n-1}{k-1}\binom{n-k}{k-1}+\binom{n}{2}\binom{n-2}{k-2}\binom{n-k}{k-2}+\cdots+\binom{n}{k}\binom{n-k}{0}\binom{n-k}{0}$$

Remarks: If you were to write this in summation notation, it is shorter and more compact as:

$$\binom{n}{k}^2=\sum_{r=0}^{k}\binom{n}{r}\binom{n-r}{k-r}\binom{n-k}{k-r}.$$

Someone asked me after class “how did you know to use three colors?” - The real answer is “I know from experience...” but let me make some comments about why how this idea of red/blue/white/purple balls comes about. If I had just been given the left hand side and the right hand side of the equation, then I would have asked myself how can I multiply two things together and break it into a sum of products of three things. Coming up with a story about coloring balls to explain it comes from seeing arguments like this lots of times. In this case, I actually came up with what the identity was just by starting with the idea that $\binom{n}{k}^2$ represented coloring white balls red and blue. The rest follows from figuring out what happens when you mix colors and knowing how to count the end result in a different way.