SOME PRACTICE PROBLEMS

(1) Prove that if a is not divisible by 3 then neither is a^2.
(2) Using the first part, prove that $\sqrt{15}$ is not rational.
(3) Using the second part, prove by contradiction that $\sqrt{3} + \sqrt{5}$ is not rational.
(4) Prove that if z, w and y are complex numbers then $x(w + y) = xw + xy$.
(5) Prove that if z is a complex number then $(z + \overline{z})/2$ is equal to the real part of z.
(6) Prove that for any $n \geq 2$ the sum of all of the n^{th} roots of unity is a real number.
(7) Prove that $\sum_{j=1}^{n} \frac{1}{j(j+1)(j+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all $n \geq 1$.
(8) Prove that $17n^3 + 103n$ is divisible by 6 for all integers n.
(9) Prove that if $x > 0$ is any fixed real number then $(1 + x)^n > 1 + nx$ for all $n \geq 2$.
(10) A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$ and this is often represented by $(a(n))_{n=1}^{\infty}$ namely $a(n)$ is the value of the function at n. Express the following statements about sequences using quantifiers, without any negation symbol in front of a quantifier:
   (a) The sequence $(a(n))_{n=1}^{\infty}$ is constant.
   (b) The sequence $(a(n))_{n=1}^{\infty}$ is not constant.
   (c) The sequence $(a(n))_{n=1}^{\infty}$ is eventually constant.
   (d) The sequence $((a(n))_{n=1}^{\infty}$ is not eventually constant.
   (e) The sequence $(a(n))_{n=1}^{\infty}$ is increasing.
   (f) The sequence $(a(n))_{n=1}^{\infty}$ is not increasing.
   (g) Forevery $\epsilon > 0$ there is some $M \in \mathbb{N}$ such that if $n > M$ then $|a(n)| < \epsilon$ or, in other words, $\lim_{n\to\infty} a(n) = 0$.
   (h) $lim_{n\to\infty} a(n) \neq 0$.
   (i) The sequence $(a(n))_{n=1}^{\infty}$ is bounded.
(11) Prove that if $hcf(a, b) = d$ then $hcf(a/d, b/d) = 1$.
(12) Prove that if $hcf(a, b) = d$ and k and b are coprime then $hcf(ka, b) = d$
(13) Prove that if m/n and and j/k are fractions represented in lowest common terms and $m/n + j/k$ is an integer then $n = k$.
(14) Prove that $(-1)^2 \equiv 1 \ (mod\ m)$ for all $m \geq 2$.
(15) Prove that $(-2)^2 \equiv 4 \ (mod\ m)$ for all $m \geq 2$.
(16) Calculate 5^{-1} modulo 13.
(17) Prove that

$$\binom{n+m}{k} = \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0}$$

where $\binom{a}{b}$ is interpreted to be 0 if $b > a$.

(18) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ that have derivatives of all orders. Let $h^{(k)}$ denote the k^{th} derivative of any function. Prove using the product rule for derivatives, the fact that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ and induction that

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

(19) The Fibonacci numbers are defined recursively by $F_{n+2} = F_{n+1} + F_n$. Prove that the number of subsets of $\{1, 2, 3, \ldots, n\}$ containing no two successive integers is F_n.

(20) Prove that

$$n2^{n-1} = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n}$$