A SAMPLE PROOF BY INDUCTION

I start off by proving the following inequality that I will need for my induction step. I am separating this in the form of a Lemma because it requires a lot of work to show why it is true. The argument for this identity will seem rather strange, but the way that I figured out is on the last page of this document.

Lemma 1. *For $n \geq 2$,*

$$\frac{2}{n^2}\left(1 - \frac{1}{\sqrt{n+1}}\right) < \frac{2}{(n+1)^2} \tag{1}$$

Proof. Note that for $n \geq 2$, $n^2 - n - 1 > 0$. Therefore multiplying both sides of this equation by $0 < 4n(n+1)$, we have $0 < 4(n+1)n(n^2 - n - 1) = 4n^4 - 8n^2 - 4n$.

Next add $16n^3 + 32n^2 + 20n + 4 = (4n+2)^2(n+1)$ to both sides of the equation. Therefore,

$$(4n+2)^2(n+1) = 16n^3 + 32n^2 + 20n + 4 < 4n^4 + 16n^3 + 24n^2 + 16n + 4 = 4(n+1)^4 \ .$$

Since both of these expressions are positive for $n > 2$, then we can take the square root of both sides of the equation and the inequality will still hold, so

$$(4n+2)\sqrt{n+1} < 2(n+1)^2 \ .$$

If we add $2n^2\sqrt{n+1} - 2(n+1)^2$ to both sides of the equation, then

$$2(n+1)^2\left(\sqrt{n+1} - 1\right) < 2n^2\sqrt{n+1}$$

Next divide both sides of the equation by $(n+1)^2 n^2 \sqrt{n+1}$ so that we have

$$\frac{2(n+1)^2\left(\sqrt{n+1} - 1\right)}{(n+1)^2 n^2 \sqrt{n+1}} < \frac{2n^2\sqrt{n+1}}{(n+1)^2 n^2 \sqrt{n+1}} \ .$$

and this is equivalent to equation (1). □

But now here is the problem that I (tried, but didn't have time to) show in class:

Proposition 2. *For all $n \geq 2$,*

$$\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\cdots\left(1 - \frac{1}{\sqrt{n}}\right) < \frac{2}{n^2} \tag{2}$$

Proof. Note that $\left(1 - \frac{1}{\sqrt{2}}\right) \doteq 0.2928$ and $\frac{2}{2^2} = .5$. So as a base case for our proof by induction, we know that equation (2) is true for $n = 2$.

Now assume that

$$\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\cdots\left(1 - \frac{1}{\sqrt{n}}\right) < \frac{2}{n^2} \tag{3}$$

is true for some fixed n, then

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\cdots\left(1-\frac{1}{\sqrt{n}}\right)\left(1-\frac{1}{\sqrt{n+1}}\right) < \frac{2}{n^2}\left(1-\frac{1}{\sqrt{n+1}}\right) < \frac{2}{(n+1)^2}$$

by Lemma 1.

Therefore, by the principle of mathematical induction,

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\cdots\left(1-\frac{1}{\sqrt{n}}\right) < \frac{2}{n^2} \tag{4}$$

is true for all $n \geq 2$. □

Here is my scratch calculation that I used to prove Lemma 1. I did something a little different than I did during class on Thursday because I was a little more careful with how I showed the relation and calculation simplified quite a bit.

$$\text{Want to show: } \frac{2}{n^2}\left(1-\frac{1}{\sqrt{n+1}}\right) \;?<?\; \frac{2}{(n+1)^2}$$

$$n^2(n+1)^2\sqrt{n+1}\frac{2}{n^2}\left(1-\frac{1}{\sqrt{n+1}}\right) \;?<?\; n^2(n+1)^2\sqrt{n+1}\frac{2}{(n+1)^2}$$

$$2(n+1)^2\left(\sqrt{n+1}-1\right) \;?<?\; 2n^2\sqrt{n+1}$$

$$(2n^2+4n+2)\left(\sqrt{n+1}-1\right) \;?<?\; 2n^2\sqrt{n+1}$$

$$(4n+2)\sqrt{n+1} \;?<?\; (2n^2+4n+2)$$

$$(4n+2)^2(n+1) \;?<?\; (2n^2+4n+2)^2$$

$$16n^3+32n^2+20n+4 \;?<?\; 4n^4+16n^3+24n^2+16n+4$$

$$0 \;?<?\; 4n^4-8n^2-4n$$

$$0 \;?<?\; 4(n+1)n(n^2-n-1)$$

and this last statement looks like something that I know is true for $n \geq 2$.

This is something I would do on a scratch sheet of paper and then throw away, but I include it here because you can then see where the proof of Lemma 1 comes from (otherwise it is completely mysterious).