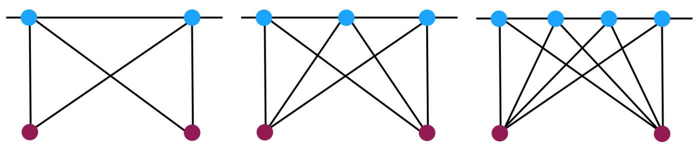
HOMEWORK #2 - SOLUTION

Let n be an integer greater than or equal to 2. The figures below are line drawings made up of n (where in the figures below, n is 2, then 3, then 4) points along a line and two points off the line. The points off the line are connected to the points on the line.

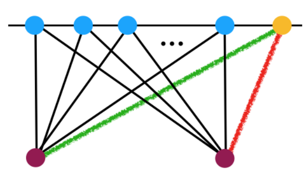
Assume that the sequence of figures continues and that there are n points in the n^{th} figure in the sequence. How many regions will be cut out in the n^{th} figure?



Solution:

As we see in the diagrams above, the image with 2 points on the line has 3 regions, the image with 3 points on the line has 7 regions, the image with 4 points on the line has 12 regions. We conjecture that the number of regions when there are n points on the line is equal to $p(n) := \frac{n^2+3n-4}{2}$.

The figure that contains $n+1$ points can be created from the diagram with n points by adding one point on the line and then connecting that point with two segments to each of the points off the line. See the diagram below where the yellow point and the red and green segments are added to make the next diagram.



Adding the red segment connecting the (yellow) $n+1$st point adds one new region that was not in the previous diagram. The green segment crosses n lines and hence creates $n+1$ regions which were not in the previous diagram.

Therefore if we let $b_n =$ "the number of regions in the diagram when there are n points on the line," then the description of what we wrote above shows $b_{n+1} = b_n + 1 + (n+1)$ and for $n \geq 2$, $b_n = 3 + 4 + \cdots + (n+1)$. Note that $p(n) - p(n-1) = \frac{n^2+3n-4}{2} - \frac{(n-1)^2+3(n-1)-4}{2} = n+1$ and $p(1) = 0$, so by telescoping sums $p(n) - p(1) = 3 + 4 + \cdots + (n+1) = b_n$.

Notice in this solution that

(1) My solution contains mostly words, and just a few well placed equations and calculations.
(2) The picture I drew clearly showed the information I described in the text.
(3) I used two symbols b_n and $p(n)$ to help make my explanation simpler and these were clearly defined.