HOMEWORK #4

ASSIGNED: FEBRUARY 12, 2020; DUE: MARCH 4, 2020

(1) Recall on February 6 in class we discussed

$$e^0 + e^{2\pi i/n} + e^{4\pi i/n} + \cdots + e^{2(n-1)\pi i/n} = 0$$

and in order to explain why it was true we needed to show that the sum of the real parts equals 0 and the sum of the imaginary parts is equal to 0.

(a) In class I showed the following identity for n even using the fact that $\sin(2\pi - x) = -\sin(x)$:

$$\sin(0) + \sin(2\pi/n) + \sin(4\pi/n) + \cdots + \sin(2(n-1)\pi/n) = 0$$

Do the same thing for n odd (make sure it is clear, at least to yourself, why the argument is slightly different for n even and n odd).

(b) Using the identity $\cos(x) = -\cos(x + \pi)$, show that

$$\cos(0) + \cos(2\pi/n) + \cos(4\pi/n) + \cdots + \cos(2(n-1)\pi/n) = 0$$

for n even.

(c) Why does the same proof not work for n odd ? Show and explain what goes wrong for the example of $n = 3$.

(2) Prove the following identity for $n \geq 1$ by induction on n.

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = \frac{2^{n+1} - n - 2}{2^n} .$$

Prove the same identity using the method of telescoping sums.

(3) Let $g_1(n) := \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$.

(a) Conjecture and prove a formula for $g_1(n)$ for $n \geq 1$.

(b) Now let $g_2(n) := \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}$. Conjecture and prove a formula for $g_2(n)$ for $n \geq 1$.

(c) Define an expression $f_k(n)$ for any $k \geq 1$ so that the formula for $f_k(n)$ agrees $f_1(n) = g_1(n)$ and $f_2(n) = g_2(n)$. Conjecture an expression for the value of this sum and prove it by induction.