MATH 1200: Review for Midterm

1. Prove that for all $n \geq 1$,
$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} .$$

2. Prove or disprove the following statements:
   (a) For all odd integers n that are not divisible by 3, $n^2 - 1$ is divisible by 6.
   (b) For all real numbers a, if a^2 is rational, then a is rational.
   (c) There exists a rational number q such that q^2 is irrational.
   (d) There exists a complex number z such that $|z| + |z-1| \leq 1$.

3. Prove that for all integers $n \geq 1$,
$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + (n+1)2^n = (n+2)2^{n+1} - 2^{n+2} + 1$$

4. Find all complex numbers z such that $|z| = 1$ and $|z + 2i| = 1$.

5. Express the following complex numbers in the form $a + bi$ where a and b are real numbers.

   (a) $\frac{1-2i}{3-i}$ (b) $(2-i)(1+i)$ (c) $(1+3i)^3$

6. Express the following complex numbers in the form $re^{i\theta}$ where r is a positive real number and $0 \leq \theta < 2\pi$.

   (a) $\sqrt{3} - i$ (b) $\frac{2+2i}{1-\sqrt{3}i}$ (c) $(1-i)^3$

7. Find the roots of the following polynomials:

   (a) $z^2 = \sqrt{3} - i$ (b) $z^2 + 3z + 1 = 0$ (c) $z^2 + iz + 1 = 0$ (d) $z^2 + 2z + i = 0$

8. Prove that for all integers $n \geq 1$,
$$\left(1 - \frac{1^2}{2^2}\right)\left(1 - \frac{2^2}{3^2}\right)\left(1 - \frac{3^2}{4^2}\right) \cdots \left(1 - \frac{n^2}{(n+1)^2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{((n+1)!)^2} .$$

9. Let a, b and d be integers. Prove or disprove the following
   (a) If d divides ab, then d divides a or d divides b.
   (b) If d does not divide ab, then d does not divide a and d does not divide b.
   (c) If ab divides d, then a divides d and b divides d.