HOMEWORK #1 SOLUTIONS - MATH 3260

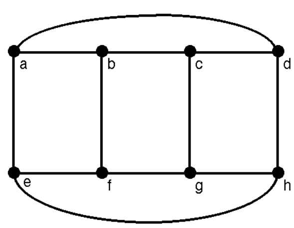
ASSIGNED: JANUARAY 20, 2003 DUE: JANUARY 29, 2002 AT 2:30PM

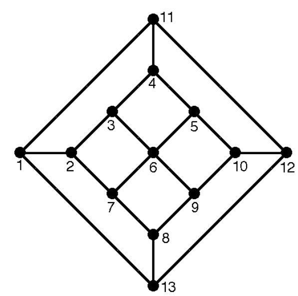
Note for all of these problems we use repeatedly the fact that the number of edges in a graph is the sum of the degrees of all of the vertices divided by 2.

(1) Explain clearly what is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3. Explain why this is the maximum value.

If all vertices of a graph are of degree 3 or more then there will be at least $ceiling(3n/2)$ edges in the graph where n is the number of vertices. This means that if there are 19 edges that n must be strictly less than 13 (because with 13 vertices there must be at least 20 edges). We can have a graph with 12 vertices and 19 edges (draw an example) and so this must be the maximum number of vertices possible.

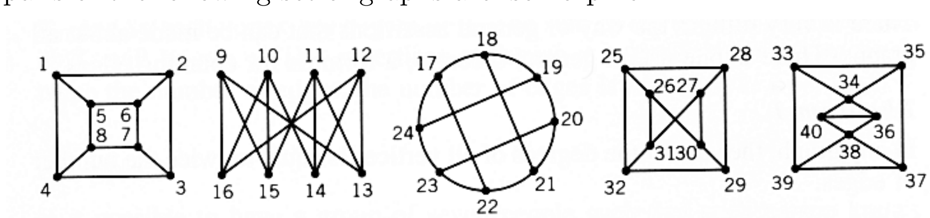
(2) Determine whether the following graphs are bipartite. If so, give the partition of the vertices into two sets.





The first one is (take the sets $\{a, f, c, h\}$ and $\{b, e, g, d\}$ as the two sets) and the second one isn't. The second one can't be because if it is bipartite then the partition of vertices will contain 1 in one set, 2 will be a second set, 7 in the first set, 8 must be in the second set, 13 must be in the first set and this leads to a contradiction since there is an edge between 13 and 1 and they are in the same bipartite set.

(3) Which pairs of the following set of graphs are isomorphic.



Give explicitly the isomorphisms between the pairs of graphs.

The first three of these graphs are isomorphic, the last two are not. The first three are bipartite graphs, the third one contains a cycle of length 5 and so cannot be bipartite (for example $25 \to 26 \to 30 \to 29 \to 28 \to 25$). The last graph has a couple of vertices of degree 4 and so cannot be isomorphic to the others. The vertices $[1,2,3,4,5,6,7,8]$ correspond to $[9,16,12,13,15,10,14,11]$ correspond to $[17,18,19,24,22,21,20,23]$ and the edges $[\{1,2\},\{1,5\},\{1,4\},\{2,3\},\{2,6\},\{3,7\},\{3,4\},\{4,8\},\{5,6\},\{5,8\},\{6,7\},\{7,8\}]$ correspond to $[\{9,16\}$,$\{9,15\}$, $\{9,13\}$, $\{16,12\}$, $\{16,10\}$, $\{12,14\}$, $\{12,13\}$, $\{13,11\}$, $\{15,10\}$, $\{15,11\}$, $\{10,14\}$, $\{14,11\}]$ correspond to $[\{17,18\}$, $\{17,22\}$, $\{17,24\}$, $\{18,19\}$, $\{18,21\}$, $\{19,20\}$, $\{19,24\}$, $\{24,23\}$, $\{22,21\}$, $\{22,23\}$, $\{21,20\}$, $\{20,23\}]$. (Note: this answer is not unique).

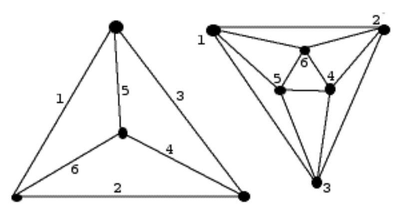
(4) The **line graph** $L(G)$ of a simple graph G is the graph whose vertices are in one-to-one correspondence with the edges of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.

(a) Show that $L(K_3)$ and $L(K_{1,3})$ are isomorphic.

K_3 and $K_{1,3}$ both contain 3 edges and all the edges are adjacent at at least one vertex so we have that both $L(K_3)$ and $L(K_{1,3})$ are isomorphic to K_3.

(b) Show that the line graph of the tetrahedron graph is isomorphic to the octahedron graph.

Looking at the graph for the tetrahedron we see that there are 6 edges and every edge is adjacent to 4 other edges and not adjacent to 1 other edge. The octahedron graph has 6 vertices and every vertex is adjacent to 4 other vertices and not adjacent to exactly 1 vertex. If we label the edges of the tetrahedron graph and the vertices of the octahedron graph we see the line graph of the tetrahedron is isomorphic to the octahedron graph.



(c) Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.

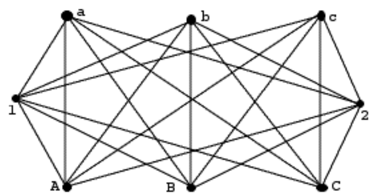
Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G and let d_i be the degree of the vertex v_i. An edge $\{v_i, v_j\}$ will be adjacent to $d_i - 1 + d_j - 1$ edges. Since there are d_i edges that contain v_i in G, the sum of the degrees of the vertices in $L(G)$ will be $\sum_{i=1}^{n} d_i(d_i - 1)$ and so the number of edges in $L(G)$ is $\sum_i \frac{d_i(d_i-1)}{2}$.

(d) Prove that if G is regular of degree k, then $L(G)$ is regular of degree $2k - 2$.

If G is regular of degree k, then every vertex is incident to k edges. Every edge is incident to 2 vertices and at one vertex the edge will be adjacent to $k-1$ other edges and at the other vertex the edge will be adjacent to $k-1$ other vertices. Therefore $L(G)$ will have each vertex adjacent to $2k-2$ other vertices and hence is regular of degree $2k-2$.

(5) The **complete tripartite graph** $K_{r,s,t}$ consists of three sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw the graphs $K_{2,2,2}$, $K_{3,3,2}$ and find the number of edges of $K_{3,4,5}$.

$K_{2,2,2}$ is the octahedron graph. I have drawn it in question $4(b)$ already and the partition of the vertices is $\{1,4\},\{2,5\}$ and $\{3,6\}$. The graph $K_{3,3,2}$ is shown below and it has 21 edges, the partitions of the vertices into three sets is $\{a,b,c\},\{A,B,C\}$ and $\{1,2\}$.



$K_{3,4,5}$ will have $(3\cdot 9+4\cdot 8+5\cdot 7)/2=47$ edges.