FINAL - TAKE HOME - MATH 4160

ASSIGNED: NOVEMBER 29, 2012 DUE: DECEMBER 14, 2012 AT 2:30PM

Write your solutions neatly and clearly. Provide full explanations and justify all of your answers. DO NOT DISCUSS THESE PROBLEMS WITH OTHERS. You must do this work alone and I will ask you to sign the statement below which states that you have not discussed these problems with others or received help on these problems. Please attach a copy of the last page of this to the front of the exam that you submit to me.

Note that in certain circumstances I am giving you the answer, and it is your job *explain* it. This means that you should write grammatically correct sentences, tell me why two things are equal, and make your calculations clear and easy to follow.

If you have any questions about the problems you may e-mail me at `zabrocki@mathstat.yorku.ca`.

Part I : 'odd Stirling numbers of the second kind' : The following problems are analogous to many results we have done, both in lecture and in homework. They should help you recall some of the techniques that you used to solve similar problems in the case of the usual Stirling numbers. There is one subtle difference is that the coefficients are recursively dependent on other coefficients where the indices differ by 2. What this means is that you should be very careful that you have considered that the coefficients are defined both when the indices are even and also when the indices are odd.

(1) Define $\langle x \rangle_1 = x$, $\langle x \rangle_2 = x^2$ and for $k \geq 3$,

$$\langle x \rangle_k = x \prod_{i=1}^{k-1} (x + k - 2i)$$

Prove that for $k > 0$, $\langle x \rangle_{k+2} = (x^2 - k^2)\langle x \rangle_k$. Make a table of expansions of the polynomials $\langle x \rangle_k$ for $1 \leq k \leq 7$. Find the expansion of x^6 and x^7 into $\langle x \rangle_n$ (for example $\langle x \rangle_3 + \langle x \rangle_1 = x(x-1)(x+1) + x = x^3$ and $\langle x \rangle_4 + 4\langle x \rangle_2 = x(x+2)x(x-2) + 4x^2 = x^4$).

(2) Define the coefficients $S^o(n,k)$ by the recursion
- $S^o(0,0) = 1$, and if $k \neq 0$, $S^o(0,k) = 0$.
- $S^o(1,1) = 1$ and if $k \neq 1$, $S^o(1,k) = 0$.
- for $n \geq 2$, $S^o(n,k) = k^2 S^o(n-2,k) + S^o(n-2,k-2)$.

Create a table of values of $S^o(n,k)$ for $0 \leq n, k \leq 7$. Show that for $n \geq 1$, that $S^o(n,k) = 0$ if $n \not\equiv k \ (mod\ 2)$ or $k > n$ or $k \leq 0$.

(3) Prove that for $n \geq 1$,

$$x^n = \sum_{k=1}^{n} S^o(n,k) \langle x \rangle_k \ .$$

(4) Prove for $n \in \mathbb{Z}$, $n > 1$ and $k > 0$, (for $n = 1$ the second term on the left hand side should be replaced by $\prod_{i=1}^{k}(k+1-2i)$).

$$\frac{\langle n+1 \rangle_{k+1}}{n+1} - \frac{\langle n-1 \rangle_{k+1}}{n-1} = 2k \frac{\langle n \rangle_k}{n} \ .$$

(5) Prove for $n > 0$ and n odd and $k > 0$,

$$\frac{\langle 1 \rangle_k}{1} + \frac{\langle 3 \rangle_k}{3} + \frac{\langle 5 \rangle_k}{5} + \cdots + \frac{\langle n \rangle_k}{n} = \frac{1}{2k} \left(\frac{\langle n+1 \rangle_{k+1}}{(n+1)} - \prod_{i=1}^{k} (k+1-2i) \right)$$

and for n even,

$$\frac{\langle 2 \rangle_k}{2} + \frac{\langle 4 \rangle_k}{4} + \frac{\langle 6 \rangle_k}{6} + \cdots + \frac{\langle n \rangle_k}{n} = \frac{1}{2k} \left(\frac{\langle n+1 \rangle_{k+1}}{(n+1)} - \langle 1 \rangle_{k+1} \right) \ .$$

(6) Define

$$B^o(x,u) := 1 + \sum_{n \geq 1} \sum_{k=1}^{n} S^o(n,k) u^k \frac{x^n}{n!}.$$

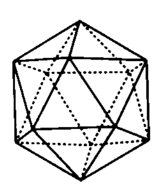
Show that $B^o(u,x) = e^{u \ sinh(x)}$ (see hint #1 below). Given that $\sum_{n \geq 0} B_n^{odd} \frac{x^n}{n!} = e^{sinh(x)}$ (a result from Homework #3), explain how this shows that $B_n^{odd} = \sum_{k=1}^{n} S^o(n,k)$.

(7) Prove that

$$S^o(n,k) = \frac{1}{k! 2^k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-2i)^n \ .$$

Part II: Burnside's Lemma and Polya's Theorem

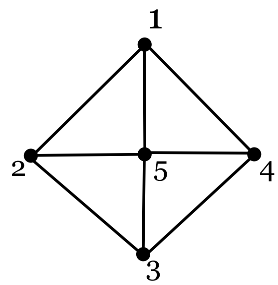
(8) Find the number of ways of coloring the faces of the icosahedron with k colors. The icosahedron has 20 triangular faces, 30 edges and 12 vertices (if you have already a dodecahedron, you may alternatively consider coloring the vertices of dodecahedron since these solids are dual).



(9) Find the number of ways of coloring the faces of the icosahedron with exactly 5 colors each used exactly 4 times.

(10) Give an argument which explains the number of ways of coloring the vertices of the following graph with k colors such that no two vertices connected with an edge is colored the same with the additional condition that two colorings are considered the same if they are equivalent under a motion of the graph which sends vertices to vertices and edges to edges.

Phrase your explanation in terms of an applications of the addition principle and the multiplication principle and use Burnside's Lemma. Explain precisely what you are computing. You may assume that you know that the group of motions of the graph is equal to $\{(1)(2)(3)(4)(5), (1234)(5), (1432)(5), (13)(24)(5), (13)(2)(4)(5), (1)(24)(3)(5), (14)(23)(5), (12)(34)(5)\}$.



Verify that this formula is correct with an example using $k = 3$ colors by exhaustively listing all colorings of the graph and showing that it agrees with your formula. (See hint #2 below).

Hint #1: One way to prove this is to show that $\frac{\partial^2}{\partial x^2} B^o(x,u) = u \frac{\partial}{\partial u}\left(u \frac{\partial}{\partial u} B^o(x,u)\right) + u^2 B^o(x,u)$

Hint #2: To gain some reasonable confidence that your formula is correct, let me tell you that there are 105 colorings of the graph using at most 5 colors where no two adjacent edges have the same color. Therefore, plugging $k = 5$ into your formula should give you 105.

WHEN YOU SUBMIT THIS EXAM : (1) attach a copy of this page on the front of the exam (2) please sign the following statement and fill out the information below (3) detail the references that you used (including websites, texts, notes, people, me, etc.) when completing the questions.

I attest that I have completed this exam myself without help from anyone else and I have not discussed the problems on this exam with other students in the class.

____

1 (5pts)	.
2 (5pts)	
3 (5pts)	
4 (5pts)	
5 (5pts)	
6 (5pts)	
7 (5pts)	
8 (5pts)	
9 (5pts)	
10 (5pts)	
total(50pts)	

This exam is open book, open notes, and other sources, but I expect you to not ask other people how to complete the assignment. Everyone should list books and websites that you consulted below. If you cannot sign the above statement truthfully, I would prefer if you just explain to me the situation rather than perjure yourself. Please detail below the sources you consulted, the that you have obtained on this exam or who you have discussed these problems with: