NOTES ON OCT 30, 2012

MIKE ZABROCKI

I started with asking who was willing to put up their solution for their problem of a generating function for a set of partitions of n. Rachel volunteered and this is the solution that we eventually came up with.

(4) the number of partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each.

Every partition of n with parts of size 1, 2 or 3 occurring at most 8 times each can be decomposed into ≤ 8 parts of size 1, ≤ 8 parts of size 2, ≤ 8 parts of size 3, therefore the generating function for partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each is equal to

$$\prod_{i=1}^{3} (\text{generating function for partitions of } n \text{ with parts of size } i \text{ occurring at most 8 times}) .$$

The generating function for partitions of n with parts of size i occurring at most 8 times is equal to

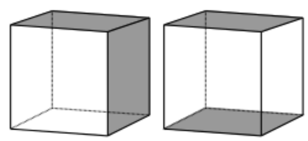
$$1 + x^i + x^{2i} + \cdots + x^{8i} = \frac{1 - x^{9i}}{1 - x^i}$$

and therefore the generating function for partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each is equal to

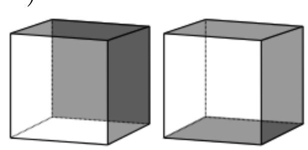
$$\frac{(1-x^9)(1-x^{18})(1-x^{27})}{(1-x)(1-x^2)(1-x^3)} .$$

I tried to add some details about the problem you were asked to do for homework. Note that the number of odd partitions of 4 is equal to 2 because only $(3,1)$ and $(1,1,1,1)$ are the only two partitions with odd parts of size 4. The problem that you are asked to compute for the problem in the homework is the exponential generating function for the odd *set* partitions. The odd partitions of 4 are different than the odd set partitions of $\{1,2,3,4\}$. There are 5 of odd set partitions (where all parts of odd size) of $\{1,2,3,4\}$ that are given by $\{\{1,2,3\},\{4\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1,3,4\},\{2\}\}$, $\{\{2,3,4\},\{1\}\}$, $\{\{1\},\{2\},\{3\},\{4\}\}$. For the recurrence on the coefficients, they satisfy $B_0^{odd} = 1$, $B_1^{odd} = 1$, $B_2^{odd} = \binom{1}{0} B_1^{odd} = 1$, $B_3^{odd} = \binom{2}{0} B_2^{odd} + \binom{2}{2} B_0^{odd} = 1 + 1 = 2$, $B_4^{odd} = \binom{3}{0} B_3^{odd} + \binom{3}{2} B_1^{odd} = 2 + 3 \cdot 1 = 5$. I am not asking you to show that B_n^{odd} is equal to the number of odd set partitions of n (you should be able to do this, but that is a different problem) but we see that this agrees for $n = 4$ and for $n = 1$ the only set partition is $\{\{1\}\}$, for $n = 2$ the only odd set partition is $\{\{1\}, \{2\}\}$, for $n = 3$ there are two set partitions $\{\{1, 2, 3\}\}$ and $\{\{1\}, \{2\}, \{3\}\}$.

I wanted to motivate what we are going to do with groups a bit so I posed the following problem. How many ways are there of coloring the faces of the cube with 2 black faces and 4 white faces? Immediately someone answered $\binom{6}{2}$ and while this is correct, I wrote this down as 'Answer 1,' because there is a way of thinking of this problems such that there is a different answer. Certainly if the faces of the cube are all numbered and all distinct then there are $\binom{6}{2}$ ways of coloring the faces, but if all the faces are identical and we are allowed to rotate the cube then there are 2 ways of coloring the cube, either the two black faces are next to each other or they are on opposite sides of the cube. This is my 'Answer 2.'
Answer 1: $\binom{6}{2}$
Answer 2: 2



I then asked the same question if we color the cube with three black faces and three white faces.
Answer 1: $\binom{6}{3}$
Answer 2: 2 (either all three black faces share two edges or only one of the faces is shares two black edges : see the diagram)

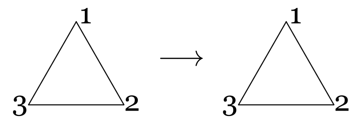


Hopefully these diagrams are clear enough to tell the difference between the two. I then suggested that we write down the generating function for the number of colorings with k black faces and $6 - k$ while faces.
Answer 1: $\binom{6}{0}B^0W^6 + \binom{6}{1}B^1W^5 + \binom{6}{2}B^2W^4 + \binom{6}{3}B^3W^3 + \binom{6}{4}B^4W^2 + \binom{6}{5}B^5W^1 + \binom{6}{6}B^6W^0$
Answer 2: $B^0W^6 + B^1W^5 + 2B^2W^4 + 2B^3W^3 + 2B^4W^2 + B^5W^1 + B^6W^0$

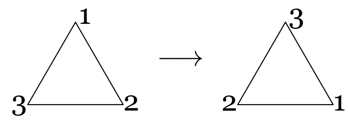
For answer 1 we should recognize that this is exactly $(W + B)^6$, but it isn't obvious what the second generating function formula is. What we will do in the next few weeks is develop the techniques which will give a formula for both of these generating functions such that they are both special cases. The reason I introduced the notion of a group last time is that we will use groups in our formula. The reason is that the set of motions of a shape form a group so I set up some examples and notation for looking motions of a shape.

I want to explain what the motions of a cube are. For this we need to set up some notation. Lets start with a much smaller example like the motions of a triangle and I want to indicate how it is an example of a group. Consider a triangle with labeled vertices and look at just the rotations of the triangle:

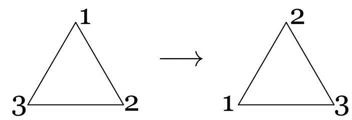
$identity = e :$



R_{120} :



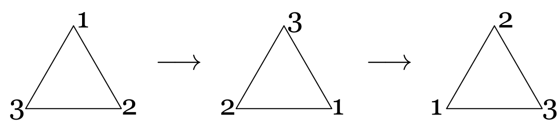
R_{240} :



The names that I have given to these operations are slightly misleading, because in a minute I am going to define them more precisely. The identity has the effect of doing nothing. The operation R_{120} takes the vertex 1 and sends it to 3, takes the vertex 3 and changes it to 2, takes the vertex labeled with 2 and changes it to 1. The operation R_{240} is the operation which takes the vertex 1 and changes it to a 2, takes the vertex 2 and changes it to a 3, takes the vertex 3 and changes it to a 1.

I noticed that if you do two operations of R_{120} then you obtain the same result as if you do one R_{240}.

R_{120} :



Simlarly, if you do two R_{240} operations then you get the same effect as a R_{120}. So what we do is define a binary operation which is composition of these operations and set $R_{120} \circ R_{120} = R_{240}$ and $R_{240} \circ R_{240} = R_{120}$ and $R_{120} \circ R_{240} = R_{240} \circ R_{120} = e$. I can make a 'multiplication table' for these operations as follows

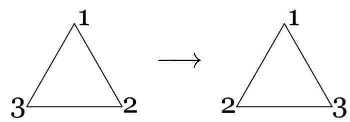
$\circ$	e	R_{120}	R_{240}
e	e	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	e
R_{240}	R_{240}	e	R_{120}

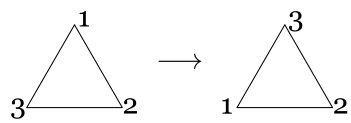
This is an example of a group. You can verify by checking on all the elements of the set $\{e, R_{120}, R_{240}\}$ that all of the conditions needed for this to be a group are satisfied with the operation of $\circ$ (see the definition from the notes on October 25). One thing that I plan to show at a later date is that in a multiplication table for a group, each element of the group appears exactly once in each row and each column.

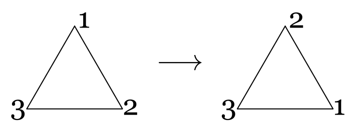
But this is not the only group that we can make with the motions of a triangle because I can also flip the

At this point I introduced notation which allowed me to use a shorthand for these operations and it is called *cycle notation*. When I write $R_{120} = (132)$, then I mean that "the vertex 1 is sent to 3, the vertex 3 is sent to 2, the vertex labeled by 2 is sent to 1 (the first entry in my cycle)." Using this same notation $R_{240} = (123)$ because as we said before "under the operation R_{240}, the vertex 1 is changed to the vertex 2, the vertex 2 is changed to the vertex 3 and the vertex 3 is changed to the vertex 1." Then, to give notation to the identity element I will say that $e = (1)(2)(3)$ because "the vertex 1 is 'changed' to the vertex 1, the vertex 2 is changed to the vertex 2, and the vertex 3 is changed to the vertex 3."

Then I introduced three more operations:

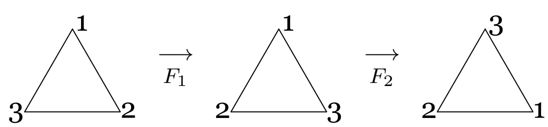
F_1 : 

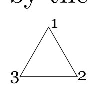
F_2 : 

F_3 : 

I want to represent them in my cycle notation as $F_1 = (1)(23)$ because "vertex 1 is fixed, vertex 2 is sent to vertex 3 and vertex 3 is sent to vertex 2." $F_2 = (2)(13)$ because "vertex 2 is fixed, vertex 1 is sent to vertex 3 and vertex 3 is sent to vertex 1." $F_3 = (12)(3)$ because "vertex 3 is fixed, vertex 1 is sent to vertex 2 and vertex 2 is sent to vertex 1."

If we do any of the F_1 operations twice then we get back to the original shape so $F_1 \circ F_1 = F_2 \circ F_2 = F_3 \circ F_3 = e$. If we do an F_1 operation followed by an F_2 then we have



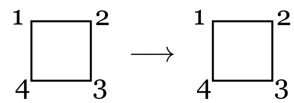
You should pay close attention to the second arrow and what is meant by that. This should resolve a question of what I mean by the operation of $\longrightarrow$. When I wrote the definition of F_2 and how it acts on the picture , it doesn't completely resolve what I mean when I act the operation of F_2 on another picture. I have to choose a convention because what I want it to mean is that F_2 leaves the vertex labelled by 2 alone and the vertex labeled by 1 is changed so that it is labelled by 3 and the vertex labelled by 3 is changed so that it is afterwards labelled by 1. The action of $F_2 \circ F_1($ triangle $) = F_2(F_1($ triangle $)) = R_{120}($ triangle $)$. Therefore we say that $F_2 \circ F_1 = R_{120}$.

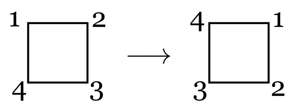
It turns out that the set $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$ also forms a group. You will have to check it has the following multiplication table by doing the individual compositions of the operations.

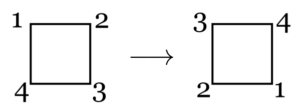
$\circ$	e	R_{120}	R_{240}	F_1	F_2	F_3
e	e	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	e	F_2	F_3	F_1
R_{240}	R_{240}	e	R_{120}	F_3	F_1	F_2
F_1	F_1	F_3	F_2	e	R_{240}	R_{120}
F_2	F_2	F_1	F_3	R_{120}	e	R_{240}
F_3	F_3	F_2	F_1	R_{240}	R_{120}	e

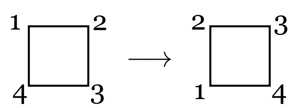
I didn't mention that $\{e, F_1\}$, $\{e, F_2\}$ and $\{e, F_3\}$ are also all groups of motions of the triangle. They all satisfy (0) if $x, y \in G$, then $x \circ y \in G$, (1) there is an e in G such that $e \circ x = x \circ e = x$ for all $x \in G$, (2) for each $x \in G$ there is an $x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = e$ and (3) for all $x, y, z \in G$, $x \circ (y \circ z) = (x \circ y) \circ z$.

I also said that we should next look at the operations that we can do on a square because this example is at least a little larger and we might be able to see some subtleties that we cannot see on the triangle. There are 4 'rotations' which I drew as:

$e = (1)(2)(3)(4) :$ 

$R_{90} = (1432) :$ 

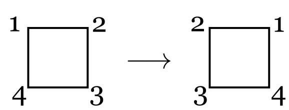
$R_{180} = (13)(24) :$ 

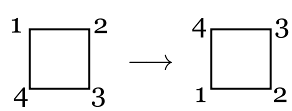
$R_{270} = (1234) :$ 

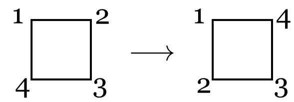
The 'multiplication table' for this group looks like

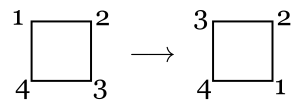
$\circ$	e	R_{90}	R_{180}	R_{270}
e	e	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	e
R_{180}	R_{180}	R_{270}	e	R_{90}
R_{270}	R_{270}	e	R_{90}	R_{180}

If we allow flipping this square then there are 4 more operations that involve flipping across the vertical, the horizontal and across either of the two diagonals.

$V = (12)(34) :$ 

$H = (14)(23) :$ 

$D_1 = (1)(24)(3)$: 

$D_2 = (13)(2)(4)$: 

I recommend that for practice that you build the 8×8 multiplication table for this group. It is good idea to try it to make sure that you understand the the operations and the notation that we have introduced here.