NOTES ON NOV 8, 2012

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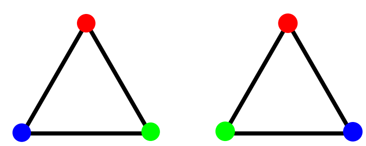
I started off with an exercise that I asked everyone to do while I put the group of motions of the cube on the board. How many ways are there of coloring the vertices of a triangle with the colors R, G, B such that two colorings are considered to be the same if one can be obtained from another by the action of an element of the group?

group	allowing repeated colors	exactly one of each color
$\{e\}$	3^3	6
$\{e, R_{120}, R_{240}\}$	11	2
$\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$	10	1

The first row we didn't really need any discussion to figure out. If we allow repeated colors and no two colorings are equivalent, then there are three choices for each vertex and hence 3^3 colorings. If we are allowed to use each color once then there are $3! = 6$ colorings.

If all three colors are different then under the group of rotations of the triangle the two colorings

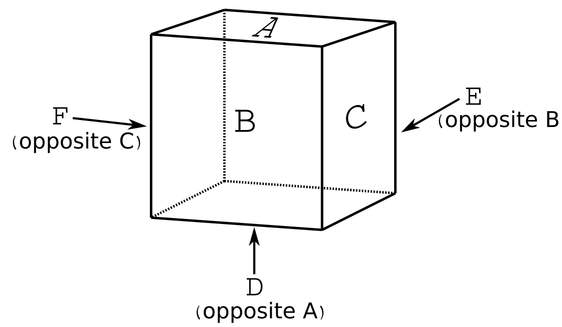


are both different, but all other colorings are equivalent to one of these two. If we are allowed to use each color more than once then there are three colorings where the three vertices are the same color, there are 6 colorings that where two of the first color and one of a second color, and the two colorings that are shown above. Therefore there are 11 different colorings in total.

If we consider the group of rotations and flips, then the two colorings that are shown above suddenly equivalent but the others are still unique and hence there is just one less coloring under this group (and so there are 10).

The reason I wanted to do this exercise is that we are working our way towards coming up with a formula for counting these things (we can do small examples like this by hand, but larger examples are difficult to enumerate).

Take a cube and label the faces with the letters A, B, C, D, E, F.



Last time I listed 16 elements of the group and I asked you to find the remaining 8. Those missing 8 were are the ones that are listed last below.

$e = (A)(B)(C)(D)(E)(F)$	$(C)(F)(ABDE)$	$(ABC)(DEF)$
$(A)(D)(BCEF)$	$(C)(F)(AD)(BE)$	$(ACB)(DFE)$
$(A)(D)(BE)(CF)$	$(C)(F)(AEDB)$	$(ABF)(DEC)$
$(A)(D)(BFEC)$	$(AB)(DE)(CF)$	$(AFB)(DCE)$
$(B)(E)(ACDF)$	$(AC)(DF)(BE)$	$(AEC)(DBF)$
$(B)(E)(AD)(CF)$	$(AE)(DB)(CF)$	$(ACE)(DFB)$
$(B)(E)(AFDC)$	$(AF)(DC)(BE)$	$(AEF)(DBC)$
	$(BC)(EF)(AD)$	$(AFE)(DCB)$
	$(BF)(EC)(AD)$	

We know that we have them all because we had a combinatorial argument that explained why there must be 24 motions of the group. But why is this a group?

(1) If you do a motion of the cube, and then you do a second motion of the cube you will have completed some motion of the cube. Therefore the set of motions is closed under the operation of composition.
(2) Doing nothing to the cube is the identity element of the group and is represented by the element $(A)(B)(C)(D)(E)(F)$.
(3) If you move a cube and then it is always possible to undo that movement and that will still be a motion of the cube. Therefore the inverse of every motion exists.

These three conditions are all that are required to for our set of motions with the operation of composition to form a group.

EXERCISE: Let $D1 =$ the diagonal between the vertex at corner of the faces ABC and the vertex at corner of the faces DEF, $D2 =$ the diagonal between the vertex at corner of the faces ABF and the vertex at corner of the faces DEC, $D3 =$ the diagonal between the vertex at corner of the faces AEC and the vertex at corner of the faces DBF, $D4 =$ the diagonal between the vertex at corner of the faces AEF and the vertex at corner of the faces DBC. Rewrite each of the motions of the cube as a permutation of the elements of $\{D1, D2, D3, D4\}$.

I wanted to use this group then to demonstrate the definitions of the orbit and stabilizer of an element x. I made a table and calculated an example of the group of motions of a cube acting on some sets of elements. I made a table and we calculated the orbit and the stabilizer of a few different elements. These motions of the cube act on the cube but you can think of these motions acting on the faces, edges, vertices or even combinations of these things.

object x	orbit O_x	stabilizer $stab(x)$
the face A	$\{A, B, C, D, E, F\}$	$\{(A)(B)(C)(D)(E)(F),$ $(A)(D)(BCEF),$ $(A)(D)(BE)(CF),$ $(A)(D)(BFEC)\}$
the edge adjoining AB	the edges adjoining $\{AB, AC$ $AE, AF, BC, BD, BF,$ $CD, CE, DE, DF, EF\}$	$\{(A)(B)(C)(D)(E)(F),$ $(AB)(DE)(CF)\}$
the vertex at the corner of ABC	vertices at $\{ABC, ABF, AEC,$ $AEF, DBC, DEC, DBF, DEF\}$	$\{(A)(B)(C)(D)(E)(F),$ $(ABC)(DEF),$ $(ACB)(DFE)\}$

The thing you should notice is the relationship between the number of elements in the orbit and the number of elements in the stabilizer. When the orbit has 6 elements, the stabilizer has 4. When the orbit has 12 elements, the stabilizer has 2 elements. When the orbit has 8 elements, the stabilizer has 3. This should lead you to make the following conjecture:

Theorem 1. *(the orbit stabilizer theorem) The product of the number of elements in the orbit and the number of elements in the stabilizer is equal to the number of elements in the group, or in equation form*

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

In order to show why this is true we need to develop a few results. The first is that the set $Stab(x)$ is a subgroup of G, that is it is a subset of G and is itself a group.

Lemma 2. *The set $Stab(x)$ is a group.*

Proof. We need to show that $Stab(x)$ is (1) closed under multiplication, that it (2) contains the identity and that it (3) contains the inverse of every element that is in the set. This all follows from the definition of group and group action. Let the multiplication in the group be denoted by $\circ$ and the action on the element x by $\bullet$.

(1) if $f, g \in Stab(x)$, then $f \bullet x = g \bullet x = x$ (by definition), hence $(f \circ g) \bullet x = f \bullet (g \bullet x) = f \bullet x = x$, therefore $f \circ g \in Stab(x)$.
(2) by the definition of group action $e \bullet x = x$, hence $e \in Stab(x)$.
(3) if $f \in Stab(x)$, then $f^{-1} \in G$ and $f^{-1} \bullet x = f^{-1} \bullet (f \bullet x) = (f^{-1} \circ f) \bullet x = e \bullet x = x$, hence $f^{-1} \in Stab(x)$.

□

In order to talk about some results we will need next I need the notion of a relation. I cover relations in Math 1200. A relation on a set X is a set of pairs $a \sim b$ where $a, b \in X$. Example of relations are things like a is greater than b, a is better than b, a is equal to b, a is taller than b, a older than b, a and b are second cousins (the set of things that a and b might be from differ wildly in those examples but you can guess from context what a and b might be, but you can have a relation on any set).

Example 1: On the set of integers $a \sim b$ if $a = b$
Example 2: On the set $\{1, 2, 3\}$, $1 \sim 2$, $2 \sim 3$
Example 3: on the set of integers $a < b$
Example 4: on the set of integers $a \leq b$

Definition 3. A relation $\sim$ on a set X is said to be reflexive if $a \sim a$ for all $a \in X$.

Definition 4. A relation $\sim$ on a set X is said to be symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in X$.

Definition 5. A relation $\sim$ on a set X is said to be transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in X$.

Any relation can have any one or none of these properties Examples 1 and 4 are reflexive, only Example 1 is symmetric, Example 1, 3 and 4 are transitive and Example 2 is not reflexive, symmetric nor transitive.

Definition 6. A relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Only Example 1 is an equivalence relations, but there are other equivalence relations which are not equals (e.g. two colorings are equivalent if one can be obtained from the other by a motion in the group G).

Proposition 7. *Let H be a subgroup of the group G. Define the relation on the group G so that for $a, b \in G$, $a \equiv b$ if there is an $h \in H$ such that $a = bh$. $\equiv$ is an equivalence relation.*

This proposition relies on the properties of the fact that H and G are groups to show that it is reflexive, symmetric and transitive. This relation on G depends on H so sometimes the relation is denoted $\equiv_H$ or $a \equiv b \ (mod\ H)$ to indicate what the subgroup is, but if it is clear then the reference to H is usually dropped.

Proof. We need to show that this relation is (1) reflexive, (2) symmetric and (3) transitive.

(1) Since $e \in H$ so $a = a \circ e$ implies $a \equiv a$.
(2) If $a \equiv b$, then $a = b \circ h$ so $b = a \circ h^{-1}$ and since H is a subgroup $h^{-1} \in H$, so $b \equiv a$.
(3) If $a \equiv b$ and $b \equiv c$, then $a = b \circ h$ and $b = c \circ h'$ so $a = (c \circ h') \circ h = c \circ (h \circ h')$ and so $a \equiv c$.

Therefore $\equiv$ is an equivalence relation. □

The reason I wanted to introduce equivalence relations (and this one in particular) is that an equivalence relation on a set of elements partitions that set of elements. Think, for example, the colorings of triangles that we started this class with. A coloring is equivalent to another one if there is a motion of the group that takes one to the other (an example of an equivalence relation). Now when I listed the number of colorings, I was saying that every coloring is equivalent to one of these that are counted in that first table. We partitioned the set into things that are equivalent to each other. We need to justify what we did in that example 'every element is equivalent to one of these representatives.'

Let a, b, c, d be elements of some set and let $c \sim d$ be an equivalence relation on that set. Define the equivalence class of a to be

$$C_a = \text{ the set of elements related to } a \ = \{c : c \sim a\}$$

To say that an element d which is related to a and related to b is to say that $d \in C_a$ and $d \in C_b$. To say that 'every element is equivalent to one of these representatives' I mean that I want to show that $C_a = C_b$.

Proposition 8. *Every two equivalence classes C_a and C_b either have no elements in common or they are equal.*