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In the previous class I had set up that we wanted to show the orbit-stablizer theorem. That is, we want to show,

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

Recall a couple of statements we have so far:

- if G acts on x, then $Stab(x) = \{g : g \bullet x = x\}$ is a subgroup of G
- For any subgroup H, $\equiv_H$ is an equivalence relation on G

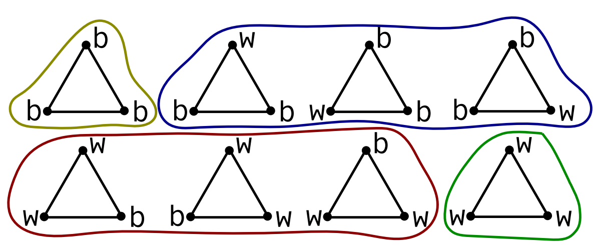
From last time I introduced the vocabulary and notation: orbit O_x, stabilizer $Stab(x)$, subgroup, reflexive relation, symmetric relation, transitive relation, equivalence relation, equivalence class C_a. I realize that this a vocabulary heavy period of the course, but these concepts are given names because they come up over and over in group theory.

I set up two other statements that I need to use to justify the orbit-stabilizer theorem.

(1) If C_a and C_b are the equivalence classes of a and b under some equivalence relation, then either C_a and C_b have no elements in common or the two sets C_a and C_b are equal.
(2) In particular, with the equivalence classes of the relation $\equiv_H$, all equivalence classes have the same number of elements.

Since I want to justify these statements, let me give a few examples of equivalence relations and equivalence classes so that we can convince ourselves that at least these statements are true on some small examples. Also I want to convince you that there is something important to show here and that statement number (2) is not always true.

Consider the set of colorings of the vertices of a triangle with B and W such that two colorings are equivalent if one can be obtained from another by rotation. That is, $coloring_1 \sim coloring_2$ if there is a $g \in \{e, R_{120}, R_{240}\}$ such that $g \bullet coloring_1 = coloring_2$. Lets try coloring the vertices of a triangle with b and w such that two colorings are distinct if they are the same under a rotation of the shape. Lets draw them:



I have drawn a loop around the colorings which are equivalent to each other under rotation. These groupings are called the orbits under the action of the group. We have a goal of counting the number of orbits in our set of colorings. We can see that there are two orbits with one element each and two orbits with 3 elements each.

Lets consider one more example, but this time with our equivalence relation $\equiv_H$. This time take our group G to be $G = \{e, R_{120}, R_{240}, F_1, F_2, F_3\}$ and the subgroup $H = \{e, R_{120}, R_{240}\}$ is used to define the equivalence relation $g_1 \equiv_H g_2$ if there exists an $h \in H$ such that $g_1 h = g_2$.

I note that in particular we have that $e \circ R_{120} = R_{120}$ so we know that $e \equiv_H R_{120}$. We also have that $R_{120} \circ R_{120} = R_{240}$ then $R_{120} \equiv_H R_{240}$. This also implies that $e \equiv_H R_{240}$ because we know that this relation is transitive. It is the case that the only elements of G which are equivalent to e are the elements of H because H is closed.

So then if we look at F_1, we find that $F_1 \circ R_{120} = (1)(23) \circ (132) = (12)(3) = F_3$. We also calculate that $F_3 \circ R_{120} = F_2$ and hence $C_{F_1} = \{F_1, F_2, F_3\}$.

One thing that is different about this example than the example with colorings of triangles is that there are two equivalence classes and they are both of the same size. It turns out with the equivalence relation $\equiv_H$ the equivalence classes are all the same size. Its hard to tell from a small example like this that the property continues.

Lemma 1. *Let $\sim$ be an equivalence relation and set $C_a = \{x : x \sim a\}$ (the set of things which are equivalent to a). If C_a and C_b have one element in common, then the sets are equal.*

In order to show why this is true, we need to show two sets are equal. The usual method for doing this is to show that $C_a \subseteq C_b$ and the reverse inclusion.

Proof. Say that C_a and C_b have an element d in common. That is, $d \sim a$ and $d \sim b$. Since $\sim$ is symmetric, $a \sim d$. Since $\sim$ is transitive and $a \sim d$ and $d \sim b$, then $a \sim b$. Let f be an element in C_a. By definition of C_a, $f \sim a$ and since $a \sim b$, then $f \sim b$, hence $f \sim b$ and $f \in C_b$. □

I then recalled that the statement $A \Rightarrow B$ is logically equivalent to *not A or B* (I even went so far as to draw the truth table for both of them to verify this). This means that the sentence

$$\text{If } C_a \text{ and } C_b \text{ have an element in common, then } C_a = C_b.$$

is equivalent to

$$\text{Either } C_a \text{ and } C_b \text{ don't have an element in common, or } C_a = C_b.$$

And this last statement is the same as (1).

Now since in an equivalence relation, every element is equivalent to some element because (at the very least) it is equivalent to itself. Hence every element is in some equivalence class and these equivalence classes are all disjoint so they form a partition of the set of elements.

Remark 2. You should note that a partition of a set also determines an equivalence relation by declaring that $a \sim b$ is if a and b are in the same part of the set partition. Therefore the number of set partitions on an n element set (the Bell numbers B_n given by the sequence $1, 1, 2, 5, 15, 52, \ldots$) is equal to the number of distinct equivalence relations on the set $\{1, 2, 3, \ldots, n\}$.

I also showed that every equivalence class of the equivalence relation $\equiv_H$ has the same number of elements. This is a special property and holds because H and G are groups. Let me rewrite the number of elements in the equivalence class of G. They are

$$C_g = \{g' : g'h = g \text{ for some } h \in H\} = \{g' : g' = gh^{-1} \text{ for some } h \in H\} = \{gh : h \in H\}$$

The reason that the third equality is true is because H is a group so running over all $h^{-1} \in H$ is the same as running over all $h \in H$. I then defined new notation for the set on the right hand side of the equality

$$gH := \{gh : h \in H\} \ .$$

These sets are called the (left) cosets of H.

What I want to show is that the equivalence classes of $\equiv_H$ are all the same size as the set H. Since the equivalence classes of $\equiv_H$ are all of the form gH for some g, then all I need to do is show that gH has the same size as H no matter what $g \in G$ is. In order to show that gH has the same size as H I need to find a bijection between the elements of H and the elements of gH.

Lemma 3. *The equivalence classes of $\equiv_H$ which partition the set G all have the same size. Since these equivalence classes are of the form gH for some g, they all have the same size as the subgroup $H = eH$.*

Proof. I want to define a bijection between H and gH. To do this I define the map ϕ_g which maps subsets $S \subseteq G$ to another subset $\phi_g(S) = \{gk : \text{ for } k \in S\}$. In particular, $\phi_g(H) = gH$. Because groups have so much structure, it will be the case that $\phi_g(H)$ and H have the same number of elements because ϕ_g is a bijection. How do we know? $\phi_{g^{-1}}(\phi_g(H)) = \phi_{g^{-1}}(gH) = \phi_{g^{-1}}(\{gh : h \in H\}) = \{g^{-1}gh : h \in H\} = H$ so there is a left inverse. The same calculation also shows that $\phi_g(\phi_{g^{-1}}(H)) = H$ so there is a right inverse, this means that ϕ_g is a bijection between H and gH and hence they have the same number of elements. □

I claim now that we have enough facts about sets, orbits, stabilizers, equivalence classes, groups, etc. to allow us to justify the orbit stabilizer theorem. We know that the stabilizer is a subgroup of G, therefore the equivalence relation $\equiv_{Stab(x)}$ partitions G and every equivalence class has the same number of elements. Conclusion:

$$|G| = |Stab(x)| \cdot \text{ the number of different equivalence classes of } \equiv_{Stab(x)}$$

But we want to show that $|G| = |Stab(x)| \cdot |O_x|$ so we just need to show that the number of different equivalence classes $= |O_x|$. In general, to show that two sets of objects have the same number of elements you show that there is a bijection between them. In this case we are looking for a bijection between the set of equivalence classes of $\equiv_{Stab(x)}$ and the elements of O_x. Remember that the equivalence classes of $\equiv_{Stab(x)}$ are the sets $gStab(x) = \{gh : h \in Stab(x)\}$.

Lemma 4. *the number of different equivalence classes of $\equiv_{Stab(x)}$ is equal to the number of elements in O_x.*

Proof. What we will do is define a bijection between the equivalence classes of $\equiv_{Stab(x)}$ (the cosets $gStab(x)$) and the elements of O_x. For a coset $g'Stab(x)$ of G, let $\psi(g'Stab(x))$ be defined as taking an element of $g \in g'Stab(x)$ and the result is $g \bullet x$. This maps a set $g'Stab(x)$ to an element of O_x. We need to show the following

(1) First, ψ must be well defined because there was some sort of arbitrary step that we did when we we took 'an element' from $g'Stab(x)$. How do we know that we get the same result each time?
(2) Second, we need to show that if you take two cosets $g'Stab(x)$ and $g''Stab(x)$ and if we find that $\phi(g'Stab(x)) = \phi(g''Stab(x))$, then $g'Stab(x) = g''Stab(x)$ (that is we need to know that this map is 1-1).
(3) Finally, we need to know that every element in the orbit of x, $y \in O_x$, there some coset $g'Stab(x)$ such that $\psi(g'Stab(x)) = y$ (that is that this map is onto).

If we have all three of these properties then we know that ψ is a well defined bijection between the cosets of $Stab(x)$ and the elements of O_x.

The first statement is true because if g_1 and g_2 are in $g'Stab(x)$, then $g_1 = g'h_1$ and $g_2 = g'h_2$ where $h_1 \bullet x = h_2 \bullet x = x$ so then

$$g_1 \bullet x = (g'h_1) \bullet x = g' \bullet (h_1 \bullet x) = g' \bullet x = g' \bullet (h_2 \bullet x) = (g'h_2) \bullet x = g_2 \bullet x \ .$$

This says that no matter which elements we take from $g'Stab(x)$ that we get the same value $g' \bullet x$.

The second statement is true because if $\phi(gStab(x)) = \phi(g'Stab(x))$ then $g \bullet x = g' \bullet x$ (because $g \in gStab(x)$ and $g' \in g'Stab(x)$ so by part (1) we know we can take these in particular) so

$$x = (g^{-1}g) \bullet x = g^{-1} \bullet (g \bullet x) = g^{-1} \bullet (g' \bullet x) = (g^{-1}g') \bullet x \ .$$

Therefore $g^{-1}g' \in Stab(x)$ and so $Stab(x) = \{g^{-1}g'h : h \in Stab(x)\}$ and

$$gStab(x) = \{gh : h \in Stab(x)\} = \{gg^{-1}g'h : h \in Stab(x)\} = \{g'h : h \in Stab(x)\} = g'Stab(x) \ .$$

The third statement is true because if $y \in O_x$ then there is some element $g \in G$ such that $y = gx$ (because that is what it means for y to be in the orbit of x). But then, $\psi(gStab(x)) = g \bullet x = y$. $\square$

Remark 5. The number of different equivalence classes of $\equiv_H$ (or the number of different left cosets of a subgroup H) is called the index of H in G. I wanted to avoid introducing one more name, definition, notation in this case because we don't really use it, but the name occurs frequently in group theory.