NOTES ON NOV 20, 2012

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This class I started by asking the question:

> How many ways are there of placing n colored beads made up of k colors around a necklace if you can slide these beads around the necklace, but not turn it over?

You can imagine a necklace with large beads that hang from the cord and that these beads can slide from one side of the necklace to the other. Two colored necklaces are equal if you can rotate the beads around the necklace so that the necklaces are the same.

I did this on an example of 8 beads. To apply Burnside's Lemma we need to recognize that there is a group of motions of rotations of the beads acting on the necklace. In the example of $n = 8$, the 8 group elements can be represented as the permutations in the table below. Let R_r be a rotation of r beads from the left side to the right side.

$g \in G$	cycle notation
$R_0 = R_8$	$(1)(2)(3)(4)(5)(6)(7)(8)$
R_1	(18765432)
R_2	$(1753)(2864)$
R_3	(16385274)
R_4	$(15)(26)(37)(48)$
R_5	(147258361)
R_6	$(1357)(2468)$
R_7	(12345678)

What I note is that there are four elements with one cycle of length 8 $\{R_1, R_3, R_5, R_7\}$, two elements with two cycles of length 4 $\{R_2, R_6\}$, one element with four cycles of length 2 $\{R_4\}$, and one element with eight cycles of length 1 $\{R_0 = R_8\}$. We conclude that the formula for making necklaces with 8 beads and k colors is equal to

$$\frac{1}{8}(k^8 + k^4 + 2k^2 + 4k)$$

because on each of the cycles we have k choices to use for the colors.

At this point I wrote down the formula that I know is true in general and I said that we should observe that it works in this case.

The number of ways of making an n bead necklace with k colored beads when you are not allowed to turn over the necklace but you can rotate the beads is equal to

$$\frac{1}{n}\sum_{d|n} \phi(d) k^{n/d}$$

where $\phi(d)$ is equal to the number of integers which is relatively prime to d (an integer n is relatively prime to d if $gcd(n, d) = 1$). In order to see why this might be true I made the following table.

d	integers between 1 and d which are relatively prime to d	motions which have n/d cycles of length d
8	$\{1, 3, 5, 7\}$	$\{R_1, R_3, R_5, R_7\}$
4	$\{1, 3\}$	$\{R_2, R_6\}$
2	$\{1\}$	$\{R_4\}$
1	$\{1\}$	$\{R_8\}$

What we want to do is show that this holds in general. If I define $\Phi(d)$ to be the integers between 1 and d which are relatively prime to d and $\Psi(d)$ be the indices i such that R_i is made up n/d cycles of length d. I claim that there is a bijection between $\Phi(d)$ and $\Psi(d)$. The bijection is simple $V_d(x) = \frac{n}{d}x$ is a map from the elements of $\Phi(d)$ to the elements of $\Psi(d)$ (notice in the table above to go from the set $\Phi(8) = \{1, 3, 5, 7\}$ to the set $\Psi(8) = \{i : R_i \in \{R_1, R_3, R_5, R_7\}\}$ we multiply each element by 1; to go from $\Phi(4) = \{1, 3\}$ to $\Psi(4) = \{i : R_i \in \{R_2, R_6\}\}$ we multiply each element by 2; to go from $\Phi(2) = \{1\}$ to $\Psi(2) = \{i : R_i \in \{R_4\}\}$ we multiply the element by 4; to go from $\Phi(1) = \{1\}$ to $\Psi(1) = \{i : R_i \in \{R_8\}\}$ we multiply the element by 8.

At this point there was some detail that I was missing (a basic fact about integers) and I totally blanked, so we moved on. I promised to come back to it and explain precisely why this was a bijection.

The next thing I wanted to do was give a formula for the number of elements with a given cycle structure. The reason that we might need to do this is because Burnside's Lemma and Polya's theorem requires that we sum over the group elements and the quantities $Fix(g)$ or the generating function $\sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$. There is a formula for the number of permutations with a given cycle structure and it uses the orbit-stabilizer theorem.

First we have to define an action on permutations. Assume that π is a permutation that when written in cycle notation has the following form:

$$\pi = (i_1 i_2 \cdots i_r)(j_1 j_2 \cdots j_s) \cdots (\ell_1 \ell_2 \cdots \ell_d)$$

then we computed $g \circ \pi \circ g^{-1}$. When we compute $g \circ \pi \circ g^{-1}$ on an element x, first we apply g^{-1} to x and get $x' = g^{-1}(x)$, then we apply π to x' to get $x'' = \pi(x')$ and then we apply g to x'' to get $g(x'')$ It doesn't matter which element we start our computation with, so for no other reason than it will work out nice, start with $g(i_1)$. In this case

$$\begin{array}{ccc} g(i_1) & \stackrel{g^{-1}}{\longrightarrow} & i_1 \\ i_1 & \stackrel{\pi}{\longrightarrow} & i_2 \\ i_2 & \stackrel{g}{\longrightarrow} & g(i_2) \end{array}$$

so we have that $g(i_1)$ is sent to $g(i_2)$.

$$g(i_2) \stackrel{g^{-1}}{\longrightarrow} i_2$$

$$i_2 \xrightarrow{\pi} i_3$$
$$i_3 \xrightarrow{g} g(i_3)$$

then we see that $g(i_2)$ is sent to $g(i_3)$. And $g(i_r)$ will be sent to $g(i_1)$ and in general $g(i_a)$ will be sent to $g(i_{a+1})$. This says that the first cycle of $g \circ \pi \circ g^{-1} = (g(i_1)g(i_2)\cdots g(i_r))\cdots$. Then with a similar argument $g \circ \pi \circ g^{-1}(g(j_a)) = g(j_{a+1})$ and the rest of the permutation $g \circ \pi \circ g^{-1}$ can be written in cycle notation as

$$g \circ \pi \circ g^{-1} = (g(i_1)\ g(i_2)\cdots g(i_r))(g(j_1)\ g(j_2)\cdots g(j_s))\cdots(g(\ell_1)\ g(\ell_2)\cdots g(\ell_d))$$

This was a tricky point, so I suggested that we figure this out on an example. We have done some compositions of permutations, but not a lot. Take as an experiment a permuation

$$\pi = (152)(3748)(6)$$

and $g = (14382)(56)(7)$ and compute $g \circ \pi \circ g^{-1}$ in two different ways. In one way, compute $g \circ \pi \circ g^{-1}(1)$, $g \circ \pi \circ g^{-1}(2)$, etc. and figure out the cycle structure. In another calculation, compute

$$(g(1)\ g(5)\ g(2))(g(3)\ g(7)\ g(4)\ g(8))(g(6)) = (461)(8732)(5)$$

and verify that you have the same permutation.

Now we can verify that g acting on π by $g \circ \pi \circ g^{-1}$ a group action. Remember that a group action has to satisfy two axioms, $e \bullet \pi = \pi$ and $g \bullet (h \bullet \pi) = (g \circ h) \bullet \pi$. In this case

$$e \bullet \pi = e \circ \pi \circ e = \pi$$

and

$$g \bullet (h \bullet \pi) = g \bullet (h \circ \pi \circ h^{-1}) = g \circ (h \circ \pi \circ h^{-1}) \circ g^{-1} = (gh) \circ \pi \circ (h^{-1} \circ g^{-1})$$

It is not too hard to verify that $(h^{-1} \circ g^{-1}) = (g \circ h)^{-1}$ since $(g \circ h) \circ (g \circ h)^{-1} = e$ and $(g \circ h) \circ (h^{-1} \circ g^{-1}) = e$. Hence $g \bullet (h \bullet \pi) = (g \circ h) \bullet \pi$. This group action is called 'conjugation.'

What this says is that the structure of the cycles of the permutations is preserved by the group action of conjugation. If you look closely it also says that for any two permutations with the same cycle structure, there is a permuation g which takes one to the other under the action of conjugation. For instance (152)(3748)(6) and (461)(8732)(5) have the same cycle structure and there is a permutation which takes one to the other, but any other permutatation with the same number of cycles of each length (say (123)(4567)(8)) is also in the orbit of these two permutations. In fact, the original question that I asked can now be rephrased as a question about the size of the orbit. Let me state it more precisely:

How many permutations have a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.?

Alternatively, how many permutations are in the orbit of a permutation with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. under the action of conjugation?

The answer is to use the orbit stabilizer theorem which says that now that we have the action of the group of permutations on π, if we divide $n!$ (the number of all permutations by the number of permutations g for which $g \bullet \pi = \pi$, then we will have the number of elements in the orbit of π.

Lets do this on an example of a permutation of 4 with two cycles of length 2.

The following permutations are all the same (12)(34), (21)(34), (12)(43), (21)(43), (34)(12), (43)(12), (34)(21), (43)(21) and each one of these permutations has a different g which sends (12)(34) to each of them, namely (1)(2)(3)(4), (12)(3)(4), (1)(2)(34), (12)(34), (13)(24), (1423), (1324), (14)(23). These permutations are the stabilizer of (12)(34) under the action of conjugation. This means that the orbit of (12)(34) is $4!/8 = 3$ and we know that there are three permutations with 2 cycles of length 2, namely (12)(34), (13)(24) and (14)(23).

What would happen if we had a_2 of cycles of length 2? say $(12)(34)(56)\cdots(2a_2-1, 2a_2)$? Well there are $a_2!$ ways of permuting the cycles and $(i, i+1)$ can be sent to $(j, j+1)$ or $(j+1, j)$ for each of the a_2 cycles so there are $2^{a_2}a_2!$ permutations g in the stabilizer of this permutation.

I then started rushing because I realized that I was more or less out of time. If there are a_3 cycles of length 3 then each of the a_3 cycles can be rearranged and $(i, i+1, i+2)$ can be sent either to $(j, j+1, j+2)$, $(j+1, j+2, j)$ or $(j+2, j, j+1)$ and all three of these cycles are exactly the same. Hence there are $a_3!3^{a_3}$ permutations in the orbit of $(123)(456)\cdots(3a_3-2, 3a_3-1, 3a_3)$.

In general I said that if there are a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. then there are

$$a_1!1^{a_1}a_2!2^{a_2}a_3!3^{a_3}\cdots = \prod_{i\geq 1} a_i!i^{a_i}$$

elements in the stabilizer by conjugation and

$$n!/(a_1!1^{a_1}a_2!2^{a_2}a_3!3^{a_3}\cdots)$$

permutations with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.